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EXISTENCE AND STABILITY OF *L4* OF THE R3BP WHEN THE SMALLER PRIMARY IS A HETEROGENEOUS TRIAXIAL RIGID BODY WITH *N* LAYERS

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ABSTRACT

We have investigated the Existence and Stability of the libration point *L4* in the R3BP, when the smaller primary is a heterogeneous axis symmetric body with *N* layers having different densities ρ_i and axes(a_i, b_i, c_i) $(i = 1, 2, 3, \ldots .N)$. It is observed that there exists five libration points L_i $(i = 1, 2, 3, 4, 5)$, three collinear along the line of X-axis and two non-collinear. We found that the non-collinear libration points are stable for $\mu < \mu_c$, a critical value.

KEYWORDS: Restricted three body problem, Libration points, Stability, Heterogeneous axis symmetric body.

1. INTRODUCTION

It is well known that the classical planar restricted three body problem possesses five libration points, two triangular and three collinear. The collinear libration points L_1, L_2, L_3 are unstable, while the two triangular libration points L_4, L_5 are stable for $\mu < \mu_c = 0.03852\ldots$, Szebehely (1967) [1]. In recent times many perturbing forces such as oblateness, radiation forces, Coriolis and centrifugal forces, variation of the masses of the primaries and of the infinitesimal mass etc., have been included in the study of the restricted three body problem. The stability in the Liapunov sense of the libration points has been studied by Vidyakin (1974) [2]. Moreover, Subba Rao et al. (1975) [3] have studied the stability of the libration points. A similar problem has been studied by El-Shaboury (1991) [4]. Again Khanna et. al (1999) [5] , too, have studied the problem when the smaller primary is a triaxial rigid body. Sharma (2001) [6] has studied the existence and stability of libration points of the restricted three body problem when primaries are triaxial rigid bodies. Sosnitskii et.al. (2008) [7] has also studied the stability of the triangular Lagrangian points in the Restricted Three-Body Problem. Vrbik (2013) [8] analyze the motion of a test particle of a planar, circular, restricted three-body problem in resonance, using the Kustaanheimo-Stiefel formalism. He showed that a good qualitative description of the motion can be reduced to three simple equations for semi-major axis, eccentricity and resonance angle. Studying these equations reveals the onset of chaos, and sheds a new light on its weak nature. Kumar et al. (2014) [9] also investigated the linear stability of equilibrium points in photogravitational restricted three body problem when primaries are triaxial rigid bodies and an oblate spheroid.

For the last so many years the literature of celestial mechanics is full of a number of research papers in the restricted three-body problem, where primaries are either point masses or spherical in shape. But, in general, the celestial bodies are axis-symmetric bodies. Therefore, we must take into account the shape of the bodies as well. The replacement of mass point by rigid-body is quite important because of its wide applications in practical problems.

In general, Scientists have taken the primaries as homogeneous point masses or spherical in shape but we will take the primaries as heterogeneous axis symmetric bodies with *N* layers.

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We have taken the idea of our problem from Esteban and Vazquez (2001) [10]. They have taken three layers in a stratified non-conformal heterogeneous oblate spheroidal system.
We wish to extend this study to the restricted three body problem when the smaller primary is heterogeneous axis-symmetric rigid body with N layers.
In the present paper our aim is to investigate the stability and stationary solutions of the restricted three body problem when we have taken one of the primaries with mass m_2 as heterogeneous triaxial rigid body with N layers having different densities ρ_i and axes (a_i, b_i, c_i) and $(i = 1, 2, 3, \ldots .N)$ respectively.
This paper consists of six sections. In the first section, we introduce the problem. In the second section, the equations of motion have been derived. In the third section, we have determined the value of the mean motion n of the primaries. In the fourth section, we have derived the coordinates of the libration points $L_i\left(i=1,2,3,4,5\right)$. In the fifth section, we have discussed the stability of libration point L_4. Finally, in the last sixth section, we have drawn the conclusion.

2. Equations of Motion

Let there be three masses $m_1, m_2, m_3;\ \left(m_1 \geq m_2\right)$. Let the bodies with masses m_1 and m_2 revolve with the angular velocity n (say) in circular orbits without rotation about their centre of mass *O*. Let there be an infinitesimal mass m_3 which is moving in the plane of motion of m_1 and m_2 and is being influenced by their motion but not influencing them. We consider the smaller primary with mass m_2 as heterogeneous triaxial rigid body with *N* layers having different densities ρ_i and axes (a_i, b_i, c_i) $(i = 1, 2, 3, \ldots .N)$ respectively. And its equatorial plane is coinciding with the plane of motion. Let the line joining m_1 and m_2 be taken as *X*-axis and *O* their center of mass as origin. Let the line passing through *O* and perpendicular to *OX,* and lying in the plane of motion of m_1 and m_2, be the *Y*-axis. Let us consider a synodic system of coordinates *O(xyz)*, initially coincident with the inertial system *O(XYZ),* rotating with the angular velocity *n* about *Z*-axis; (the *z*-axis is coincide with *Z*-axis). Let initially the principal axes of mass m_2 at B, the center of mass of the body, be parallel to the synodic axes *O(xyz)* and the axes of symmetry of the body with mass m_2 be perpendicular to the plane of motion. Since the rigid body is revolving without rotation about *O* with the same angular velocity as that of the synodic axes and the principal axes of m_2 are initially parallel to the synodic axes, the principal axes of m_2 will remains parallel to them throughout the motion.

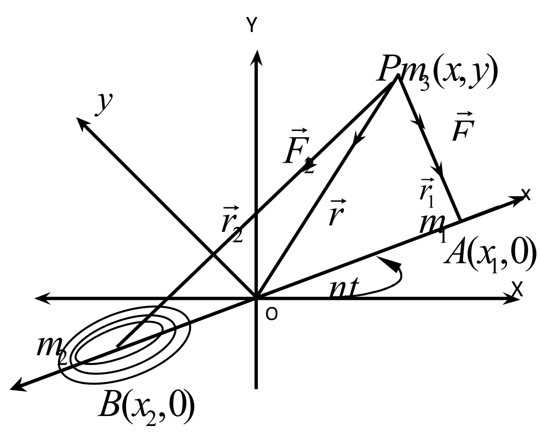

Figure 1. Configuration of the R3BP with the smaller primary as heterogeneous triaxial rigid body.

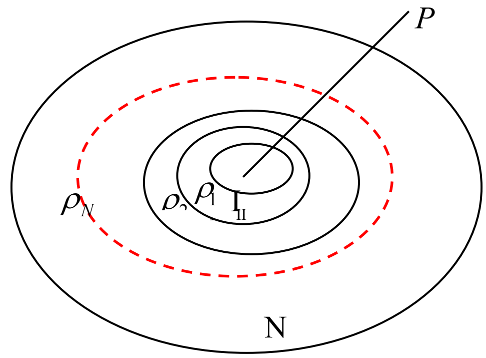

Figure 2. Heterogeneous triaxial rigid body with N Layers

The equation of motion of m_3 in the vector form is

$$m_3\left(\frac{\partial^2\vec{r}}{\partial t^2}+2\vec{\omega}\times\frac{\partial\vec{r}}{\partial t}+\frac{\partial\vec{\omega}}{\partial t}\times\vec{r}+\vec{\omega}\times\left(\vec{\omega}\times\vec{r}\right)\right)=F, \qquad [1]$$

where, $\vec{r}=\overline{OP}$, $\vec{\omega}=n\,k$ = angular velocity = constant, $\vec{F}$ = Total force acting on m_3.

Let the co-ordinates of P be (x y), then $\vec{r}=xi+yj$, and the L.H.S. of the Equation [1] will be

$m_3\left(\ddot{x}i+\ddot{y}j\right)-2n\left(\dot{y}i-\dot{x}j\right)-n^2\left(xi+yj\right)$.

Again we know that the gravitational potential of an irregular body, at a point P outside it, will be

$$V=\frac{-GM}{r}-\frac{G}{2r^3}\left(I_1+I_2+I_3-3I\right),$$

Here, we have calculated $I_1+I_2+I_3-3I$, $=\frac{M}{5}\left[2a^2-b^2-c^2-\frac{3}{r^2}\left(a^2-b^2\right)y^2\right]$, in the case of triaxial rigid body.

Therefore we can calculate the gravitational potential of the body of mass M at the point P as

$$V=-\frac{GM}{r}\left[1+\frac{1}{10r^2}\left\{2a^2-b^2-c^2-\frac{3}{r^2}\left(a^2-b^2\right)y^2\right\}\right].$$

In our case, the gravitational potential of the body of mass m_1, at the point P will be $V_1=-\frac{Gm_1}{r_1}$, and the gravitational potential of the body of mass m_2, at the point P will be V_2 = Potential at a point P outside a triaxial rigid body with N layers of densities ρ_i and axes $\left(a_i,\,b_i,\,c_i\right)$ $\left(i\;=\;1,\,2,\,3,\,4\,......N\right)$;

$$\rho_i\;<\;\rho_{i+1},\,a_i\;<\;a_{i+1},\,b_i\;<\;b_{i+1},\,c_i\;<\;c_{i+1},$$

$$=V_{NN}+V_{(N-1)N}+.......V_{2N}+V_{1N}\quad(say). \qquad [2]$$

where $V_{NN},V_{(N-1)N},.......V_{1N}$, are the potential of the axis symmetric bodies of densities ρ_N, $\rho_{(N-1)}$, ... ρ_1, for the regions N, N-1,…….. 1, respectively.

Now we have $V_{NN}\;=\;V'_{NN}\;-\;V'_{N(N-1)}$, (say).

where, V'_{NN} =Potential of the axis symmetric body of axes (a_N, b_N, c_N) with homogeneous density ρ_N throughout at $P = -\frac{4\pi\rho_N G}{3r_2} a_N b_N c_N \left[1 + \frac{1}{10r_2^2}\left(2a_N^2 - b_N^2 - c_N^2 - \frac{3}{r_2^2}\left(a_N^2 - b_N^2\right) y^2\right)\right]$.

and $V'_{N(N-1)}$ =Potential of the axis symmetric body of axes $\left(a_{(N-1)}, b_{(N-1)}, c_{(N-1)}\right)$ with homogeneous density ρ_N throughout at P ,

$$= -\frac{4\pi\rho_N G}{3r_2} a_{(N-1)} b_{(N-1)} c_{(N-1)} \left[1 + \frac{1}{10r_2^2}\left(2a_{N-1}^2 - b_{N-1}^2 - c_{N-1}^2 - \frac{3}{r_2^2}\left(a_{N-1}^2 - b_{N-1}^2\right) y^2\right)\right].$$

Thus,

$$V_{NN} = -\frac{4\pi\rho_N G}{3r_2}\left[a_N\ b_N\ c_N \left\{1 + \frac{1}{10r_2^{\ 2}} \times \left(2a_N^2 - b_N^2 - c_N^2 - \frac{3}{r_2^{\ 2}}\left(a_N^2 - b_N^2\right) y^2\right)\right\} - a_{(N-1)} b_{(N-1)} c_{(N-1)} \left\{1 + \frac{1}{10r_2^{\ 2}} \times \left(2a_{N-1}^2 - b_{N-1}^2 - c_{N-1}^2 - \frac{3}{r_2^{\ 2}}\left(a^2_{\ N-1} - b^2_{\ N-1}\right) y^2\right)\right\}\right] \quad [3]$$

Similarly, we have

$$V_{(N-1)N} = \frac{-4\pi\rho_{N-1} G}{3r_2}\left[a_{N-1}\ b_{N-1}\ c_{N-1} \left\{1 + \frac{1}{10r_2^{\ 2}} \times \left(2a_{N-1}^2 - b_{N-1}^2 - c_{N-1}^2 - \frac{3}{r_2^{\ 2}}\left(a_{N-1}^2 - b_{N-1}^2\right) y^2\right)\right\} - a_{N-2} b_{N-2} c_{N-2} \left\{1 + \frac{1}{10r_2^{\ 2}} \times \left(2a_{N-2}^2 - b_{N-2}^2 - c_{N-2}^2 - \frac{3}{r_2^{\ 2}}\left(a_{N-2}^2 - b_{N-2}^2\right) y^2\right)\right\}\right]. \quad [4]$$

And

$$V_{1N} = -\frac{4\pi\rho_1 G}{3r_2}\left[a_1 b_1 c_1 \left\{1 + \frac{1}{10r_2^2} \times \left(2a_1^2 - b_1^2 - c_1^2 - \frac{3}{r_2^2}\left(a_1^2 - b_1^2\right) y^2\right\}\right]. \quad [5]$$

Hence, from the Equations [2] , [3] , [4] and [5], we have

$$V_2 = -\frac{m_2 G}{r_2} - \frac{k_1 G}{2r_2^3} + \frac{3k_2 G y^2}{2r_2^5},$$

where,

$$m_2 = \frac{4\pi}{3}\sum_{i=1}^{N}\{\rho_i - \rho_{i+1}\}\ a_i\ b_i\ c_i,\ k_1 = \frac{4\pi}{3}\sum_{i=1}^{N}\left(\left(\rho_i - \rho_{i+1}\right) a_i b_i c_i \sigma_{i,1}\right),$$

$$k_2 = \frac{4\pi}{3}\sum_{i=1}^{N}\left(\left(\rho_i - \rho_{i+1}\right) a_i b_i c_i \sigma_{i,2}\right),$$

$$\sigma_{i,1} = \frac{\left(2a_i^2 - b_i^2 - c_i^2\right)}{5},\ \sigma_{i,2} = \frac{\left(a_i^2 - b_i^2\right)}{5},\ \rho_{N+1} \neq 0.$$

Hence the total potential at P due to m_1 and m_2 is given by

$$V = -\frac{m_1 G}{r_1} - \frac{m_2 G}{r_2} - \frac{k_1 G}{2r_2^3} + \frac{3k_2 G y^2}{2r_2^5}.$$

Now, we shall adopt the notation and terminology of Szebehely (1967). As a consequence, the distance between the primaries does not change and is taken equal to one; the sum of the masses of the primaries is also taken as one. The unit of time is chosen so as to make the gravitational constant unity. i.e.

$m_1 + m_2 = 1\ ; m_2 = \mu(say)\ \ , m_1 = 1 - \mu,\ d(m_1 m_2) = 1 = (a' + b'),\ \ G = 1,$

The equations of motion [1] become

$$\ddot{x} - 2n\dot{y} = \Omega_x,$$
$$\ddot{y} + 2n\dot{x} = \Omega_y, \qquad [6]$$

where, $\Omega = n\left(\frac{x^2 + y^2}{2}\right) + \frac{1-\mu}{r_1} + \frac{\mu}{r_2} + \frac{k_1'}{2r_2^3} - \frac{3k_2' y^2}{2r_2^5},$

$r_1^{\,2} = (x - \mu)^2 + y^2, r_2^{\,2} = (x - \mu + 1)^2 + y^2,$

$\mu = \frac{m_2}{m_1 + m_2} \le \frac{1}{2} \quad \Rightarrow m_2 = \mu\ ;\ m_1 = 1 - \mu$

$k_1' = \frac{4\pi}{3} \sum_{i=1}^{N} (\rho_i' - \rho_{i+1}')\, a_i' b_i' c_i' \sigma_{i,1}'\ ,\ k_2' = \frac{4\pi}{3} \sum_{i=1}^{N} (\rho_i' - \rho_{i+1}')\, a_i' b_i' c_i' \sigma_{i,2}'\ ,$

$a_i' = \frac{a_i}{R}\ ,\ b_i' = \frac{b_i}{R}\ ,\ c_i' = \frac{c_i}{R}\ ,\ \rho_{N+1}' \ne 0,\ \sigma_{i,1}' = \frac{\left(a_i'^2 - b_i'^2 - c_i'^2\right)}{5R^2}\ ,\ \sigma_{i,2}' = \frac{\left(a_i'^2 - b_i'^2\right)}{5R^2}\ ,$

$\rho_i' = \frac{\rho_i}{M}, M = m_1 + m_2,$ R = dimensional distance between the primaries.

3. The mean motion *n* of the primaries

The potential of the triaxial rigid body B at A is

$$-\left(\frac{m_2 G}{R} + \frac{k_1 G}{2R^3}\right),$$

R = dimensional distance between the primaries.

Let the distances of m_1 and m_2 from the centre of mass O be a' and b' respectively.

Since m_1 and m_2 are moving in circular orbits about O, we have

$$m_1 a' n^2 = \left(\frac{m_2 G}{R^2} + \frac{3k_1 G}{2R^4}\right) m_1,$$

and $\quad m_2 b' n^2 = \left(\frac{m_2 G}{R^2} + \frac{3k_1 G}{2R^4}\right) m_1,$

Adding these equations, we have

$$n^2 = \left(\frac{m_2 G}{(a'+b')^3} + \frac{3k_1 G}{2(a'+b')^5}\right)\left(\frac{m_1 + m_2}{m_2}\right).$$

Using the dimensionless variables, we get the mean motion as

$$n = 1 + \frac{3k_3}{4},$$

where $\quad k_3 = \frac{4\pi}{3\mu} \sum_{i=1}^{N} (\rho_i' - \rho_{i+1}')\, a_i' b_i' c_i' \sigma_{i,1}'.$

4. Libration Points

The libration points are the solutions of the equations

$\Omega_x = 0$ and $\Omega_y = 0$, i.e.

$$nx - \frac{(1-\mu)(x-\mu)}{r_1^3} - \frac{\mu(x-\mu+1)}{r_2^3} - \frac{3k_1'(x+1-\mu)}{2r_2^5} + \frac{15k_2'(x+1-\mu)y^2}{2r_2^7} = 0,$$

$$ny - \frac{(1-\mu)y}{r_1^3} - \frac{\mu y}{r_2^3} - \frac{3k_1'y}{2r_2^5} + \frac{15k_2'y^3}{2r_2^7} - \frac{3k_2'y}{r_2^5} = 0. \qquad [7]$$

Here taking $y \neq 0,$ in the Equation [7], we have

$$n - \frac{(1-\mu)}{r_1^3} - \frac{\mu}{r_2^3} - \frac{3k_1'}{2r_2^5} + \frac{15k_2'y^2}{2r_2^7} - \frac{3k_2'}{r_2^5} = 0.$$

If we take $k_1' = k_2' = 0$, then it will be the classical case of the restricted three body problem and $r_1 = r_2 = 1$ gives the required solution.

Now, solving these equations when $k_1' = k_2' \neq 0$ by taking $r_1 = 1+\pi_1,\ r_2 = 1+\pi_2,\ \pi_1, \pi_2 << 1$ and rejecting the second and higher order terms of π_1 and π_2 we get the co-ordinates (*x, y*) of the libration point

$$x = \mu - \frac{1}{2} + k_4, \quad y = \frac{\sqrt{3}}{2} + k_5,$$

Where $k_4 = \dfrac{3\left((1-\mu)4k_1' - (11-15\mu)k_2' - (1-\mu)4\mu k_3\right)}{24\mu(\mu-1) - (44+60\mu)k_1' + 143k_2' - 195\mu k_2' - 8k_3},$ [8]

$$k_5 = \frac{\sqrt{3}\left((1-\mu)4k_1' - (11-7\mu)k_2'\right)}{24\mu(\mu-1) - (44+60\mu)k_1' + 143k_2' - 195\mu k_2' - 8k_3}.$$

These are the co-ordinates of the libration point corresponding to L_4 in the classical case.

5. Stability of libration point L_4

The liberation point L_4 is given by $L_4(x,\ y) = \left(\mu - \frac{1}{2} + k_4,\ \frac{\sqrt{3}}{2} + k_5\right)$. Now, we write the variational equations by putting $x = L_x + \xi$ and $y = L_y + \eta$ in the equations of motion [6], where $(L_x,\ L_y)$ are the co-ordinates of $L_i\ (i = 1, 2, 3, 4, 5)$ and $\xi, \eta << 1$.

The variational equations are

$$\ddot{\xi} - 2n\dot{\eta} = \xi\,\Omega_{xx}^0 + \eta\,\Omega_{xy}^0,$$

$$\ddot{\eta} + 2n\dot{\xi} = \xi\,\Omega_{xy}^0 + \eta\,\Omega_{yy}^0.$$

Here '0' indicates that the derivatives are to be calculated at the libration point $(L_x,\ L_y)$ under consideration.

The characteristic equation is

$$\lambda^4 + \left\{4n^2 - \Omega_{xx}^0 - \Omega_{yy}^0\right\}\lambda^2 + \Omega_{xx}^0\Omega_{yy}^0 - \left(\Omega_{xy}^0\right)^2 = 0.$$

Now, at the libration point L_4

$$\Omega_{xx}^0 = \frac{3}{32}\left\{8 + 96\mu\left(k_5 - k_4\right) + 4k_1' - 45k_2' + 8k_3 + 64k_4 - 72k_5\right\},$$

$$\Omega_{yy}^0 = \frac{1}{32}\left\{72 + 264\mu\left(k_4 - k_5\right) + 132k_1' - 141k_2' + 24k_3 - 168k_4 - 320k_5\right\},$$

$$\Omega_{xy}^0 = \frac{-\sqrt{3}}{32}\left\{24 - 48\mu + 88\mu\left(k_4 + k_5\right) - 60k_1' + 195k_2' - 56k_4 - 32k_5\right\}.$$

Therefore the characteristic equation becomes

$$\lambda^4 + \frac{\lambda^2}{8}\left\{8 - 36k_1' + 69k_2' + 36k_3 - 6k_4 + 134k_5 + 6\mu\left(k_4 - k_5\right)\right\} + \frac{3}{32}\left\{-72\,\mu^2\right.$$
$$+72\mu + 132k_1' - 180\mu k_1' - 529k_2' + 585\mu k_2' + 24k_3 + 186k_4 - 450\mu k_4$$
$$+264\mu^2\left(k_4 + k_5\right) - 194k_5 - 78\mu k_5 = 0,$$

Thus,

$$\lambda^2 = -\frac{1}{2}\left(\frac{-1}{8}\left\{8 - 36k_1' + 69k_2' + 36k_3 - 6k_4 + 134k_5 + 6\mu\left(k_4 - k_5\right)\right\} \pm \frac{1}{2}\left(1\right.\right.$$
$$-27\mu + 27\mu^2 - 99\mu^2\left(k_4 + k_5\right) - \frac{117k_1'}{2} + \frac{135\mu k_1'}{2} + \frac{1425k_2'}{8} - \frac{1755\mu k_2'}{8} - \frac{285k_4}{4}$$
$$\left.+\frac{681\mu k_4}{4} + \frac{425k_5}{4} + \frac{111\mu k_5}{2}\right)^{\frac{1}{2}},$$

Now, the discriminant is zero, if

$$1 - 27\mu + 27\mu^2 - 99\mu^2\left(k_4 + k_5\right) - \frac{117k_1'}{2} + \frac{135\mu k_1'}{2} + \frac{1425k_2'}{8} - \frac{1755\mu k_2'}{8}$$
$$-\frac{285k_4}{4} + \frac{681\mu k_4}{4} + \frac{425k_5}{4} + \frac{111\mu k_5}{2} = 0,$$

$$\Rightarrow \quad \mu = \frac{1}{2\left(27 - 99k_4 - 99k_5\right)}\left\{-\frac{3}{8}\left(-72 + 180k_1' - 585k_2' + 454k_4 + 74k_5\right)\right.$$
$$\left.\pm\sqrt{621\left(1 + \frac{99k_1'}{23} - \frac{1095k_2'}{92} - \frac{245k_4}{138}\right)}\right\},$$

Since $\mu \le \frac{1}{2}$, rejecting the + sign, we get,

$$\mu = .0385 - \frac{5k_1}{4} - \frac{33k_1}{4\sqrt{69}} + \frac{65k_2}{16} + \frac{365k_2}{16\sqrt{69}}$$
$$- \frac{95k_4}{72} - \frac{767k_4}{72\sqrt{69}} + \frac{95k_5}{72} + \frac{1783k_5}{72\sqrt{69}},$$

It is clear from the nature of the solution $\xi = Ae^{\lambda t}$ *and* $\eta = Be^{\lambda t}$ that these will be periodic and bounded only if λ is purely imaginary. We must choose μ so that $\lambda^2 < 0$.

Hence

$$\left(1 - 27\mu + 27\mu^2 - 99\mu^2\left(k_4 + k_5\right) - \frac{117k_1'}{2} + \frac{135\mu k_1'}{2} + \frac{1425k_2'}{8} - \frac{1755\mu k_2'}{8} - \frac{285k_4}{4} + \frac{681\mu k_4}{4} + \frac{425k_5}{4} + \frac{111\mu k_5}{2}\right) \geq 0,$$

If stable, motion is to ensure.

$$\mu < .0385 - \frac{5k_1'}{4} - \frac{33k_1'}{4\sqrt{69}} + \frac{65k_2'}{16} + \frac{365k_2'}{16\sqrt{69}} - \frac{95k_4}{72} - \frac{767k_4}{72\sqrt{69}} + \frac{95k_5}{72} + \frac{1783k_5}{72\sqrt{69}} = \mu_c(say)$$

Therefore the stable motion around the point L_4 will take place if μ does not exceed μ_c.
Hence, we conclude that when $0 \leq \mu < \mu_c$, the motion is bounded and it is the super position of two harmonic oscillations with different frequencies. We have stability in the linear sense. In the range $\mu_c < \mu \leq 0.5$, the motion is unstable. At the value $\mu = \mu_c$ the solution contains secular terms.

6. Conclusion

We found that there exist five stationary solutions (called libration points) of the equations of motion. Three of them are collinear and two form triangles with co-ordinates (x, y) given in the equations of motion. Thus we have studied the existence and linear stability of the non- collinear libration point L_4 in the Restricted Three Body Problem when the smaller primary is a heterogeneous triaxial rigid body with *N* layers having different densities.
We found that non-collinear libration point L_4 is stable for $\mu < .0385 - \frac{5k_1}{4} - \frac{33k_1}{4\sqrt{69}} + \frac{65k_2}{16} + \frac{365k_2}{16\sqrt{69}} - \frac{95k_4}{72} - \frac{767k_4}{72\sqrt{69}} + \frac{95k_5}{72} + \frac{1783k_5}{72\sqrt{69}} = \mu_c$, unstable for $\mu_c < \mu \leq 0.5$ and for $\mu = \mu_c$, the solution contains secular terms.

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