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Dynamical Features of a Mathematical Model on Smoking

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ABSTRACT

In this work, we present a non-linear mathematical model which analyzes the spread of smoking in a population. The total population is divided into five classes: potential smokers, occasional smokers, smokers, temporary quitters and permanent quitters. We study that the smoking quitters may become potential smokers again and then we discuss dynamical behavior of the model. Finally, we justify our work through numerical simulation.
KEYWORDS: smoking, threshold quantity, mathematical modeling.

INTRODUCTION

Tobacco pandemic is one of the largest public medicinal threats the world has ever faced as it puts to death up to half of its users. Smoking kills about six million people each year of whom more than five million are ex-smokers and smokers, and over 500,000 are nonsmokers revealed to second-hand smoke. Tobacco users who pass away prematurely deprive their families of earnings, lift the cost of fitness care, and hinder financial development. The World Health Organization forecasts that, by 2030, ten million persons will pass away every year due to tobacco associated illnesses. Numerous mathematical models have been developed in the last few years [1-12]. In 2000, Castillo-Garsow et al. [13] first time suggested a straightforward mathematical model for giving up smoking. They address a scheme with a total unchanging community which is split up into three classes: promise smokers, that is, persons who are not smoking yet but might become smokers in the future (P), smokers (S), and persons (former smokers) who have stop smoking lastingly (Q). Sharomi and Gumel evolved mathematical models by inserting gentle and string of connections categories [14]. In their work they offered the development and public wellbeing influence of smoking-related illnesses. Later on Zaman [15] expanded the work of Castillo-Garsow et al. [13] and deduced a mathematical model taking into account the occasional smoker's compartment in the giving up smoking model and offered its qualitative behavior. In this study we evolved such a smoking model that the smoking quitters may become potential smokers again and then we discuss the qualitative behavior of that model.

Model Formulation:
Let the total population size at time t be denoted by $N(t)$, we divide the total population into five subclasses: potential smokers (nonsmoker) $P(t)$, smokers $S(t)$, smokers who temporarily quit smoking $Q_t(t)$, smokers who permanently quit smoking $Q_P(t)$ and the occasional smokers $O(t)$, such that $N(t) = P(t) + O(t) + S(t) + Q_t(t) + Q_P(t)$. We describe the dynamics of smoking by the following five nonlinear differential equations:

$$\frac{dP}{dt} = \Lambda - \beta PS - \mu P$$
$$\frac{dO}{dt} = \beta PS - \alpha_1 O - \mu O$$
$$\frac{dS}{dt} = \alpha_1 O + \alpha_2 SQ_t - (\mu + \gamma) S \quad (1)$$

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$$\frac{dQ_t}{dt} = -\alpha_2 SQ_t - \mu Q_t + \gamma(1-\delta)S$$
$$\frac{dQ_p}{dt} = \delta\gamma S - \mu Q_p$$

In the above model
Λ is the recruitment rate in P,
β is the effective contact rate between S and P,
μ is the natural death rate,
α_1 is the rate at which occasional smokers become regular smokers,
α_2 is the contact rate between smokers and temporary quitters who revert back to smoking,
γ is the rate of quitting smoking,
$1-\delta$ is the fraction of smokers who temporary quit smoking (at the rate γ),
δ is the remaining fraction of smoking who permanently quit smoking.

Local Stability:
We illustrate the local stability of the disease free (smoking free) and the endemic equilibrium of the system (1) by the following theorems 1 and 2.
Theorem 1: The disease free equilibrium E_0 is locally asymptotically stable for $R_0 < 1$, otherwise unstable.
Proof: By linearizing the system (1) about disease free equilibrium E_0 we get the Jacobian matrix:

$$J(E_0) = \begin{bmatrix} -\beta s - \mu & 0 & -\beta P & 0 & 0 \\ \beta S & -\alpha_1 - \mu & \beta P & 0 & 0 \\ 0 & \alpha_1 & \alpha_2 Qt - (\mu + \gamma) & 0 & 0 \\ 0 & 0 & -\alpha_2 Qt + \gamma(1-\delta) & -\alpha_2 S - \mu & 0 \\ 0 & 0 & \delta\gamma & 0 & -\mu \end{bmatrix}$$

Now, we have analyzed the stability of DFE, and for this we calculate the characteristic equation of $J(E_0)$ as follows.

$$J(E_0) = \begin{bmatrix} -\mu & 0 & -\beta & 0 & 0 \\ 0 & \alpha_1 - \mu & \beta & 0 & 0 \\ 0 & \alpha_1 & -(\mu + \gamma) & 0 & 0 \\ 0 & 0 & \gamma(1-\delta) & -\mu & 0 \\ 0 & 0 & \delta\gamma & 0 & -\mu \end{bmatrix}$$

The characteristic equation of $J(E_0)$ is

$$Char\left(J(E_0)\right) = det\left(\lambda I - J(E_0)\right) = \begin{vmatrix} \lambda + \mu & 0 & \beta & 0 & 0 \\ 0 & \lambda + \alpha_1 + \mu & -\beta & 0 & 0 \\ 0 & 0 & \lambda + \mu + \gamma & 0 & 0 \\ 0 & 0 & -\gamma + \delta\gamma & \lambda + \mu & 0 \\ 0 & 0 & -\delta\gamma & 0 & \lambda + \mu \end{vmatrix}$$

$$= (\lambda + \mu)^3\{(\lambda + \mu + \delta)(\lambda + \alpha_1 + \mu)\}\left[1 - \frac{(\alpha_1\beta)}{(\lambda+\mu+\delta)(\lambda+\alpha_1+\mu)}\right]$$

$$= (\lambda + \mu)^3(\lambda + \mu + \delta)(\lambda + \alpha_1 + \mu)[1 - R_0].$$

Where

$$R_0 = \frac{(\alpha_1\beta)}{(\lambda + \mu + \delta)(\lambda + \alpha_1 + \mu)}$$

We see that all the eigen values are negative only for $R_0 < 1$. Thus the Disease Free Equilibrium E_0 is locally asymptotically stable for $R_0 < 1$, otherwise unstable.

Endemic Equilibrium:
To find the endemic equilibria of the system (1) where at least one of the infected components is non-zero, we need to take the following steps:
Let $E^* = (S^*, P^*, O^*, Q_t{}^*, Q_p{}^*)$ represents endemic equilibrium of the system (1). By solving the equations of the system (1) at steady state, we get

$$P^* = \frac{\Lambda}{(\beta S^* + \mu)} \qquad O^* = \frac{\beta\Lambda S^*}{(\beta S^* + \mu)(\alpha_1 + \mu)},$$

$$Q_t{}^* = \frac{\gamma(1-\delta)S^*}{(\alpha_2 S^* + \mu)} \qquad Q_p{}^* = \frac{\delta\gamma S^*}{\mu}$$

Let the following theorem analyzes the local stability of the endemic equilibrium when $R_0 > 1$.

Theorem 2: The endemic equilibria E^* is locally asymptotically stable if $R_0 > 1$, $\gamma + \mu > \alpha_2 Qt^* + \frac{\mu P^* \alpha_1{}^2 \beta^2}{M^*}$, and $\alpha_2 Qt^* < \gamma(1-\delta)$.

Proof: By linearizing the system (1) about an endemic equilibrium E^*, we get the Jacobian matrix:

$$J(E^*) = \begin{bmatrix} -\beta S* - \mu & 0 & -\beta P* & 0 & 0 \\ \beta S* & -\alpha_1 - \mu & \beta P* & 0 & 0 \\ 0 & \alpha_1 & \alpha_2 Qt* - (\mu + \gamma) & \alpha_2 S^* & 0 \\ 0 & 0 & -\alpha_2 Qt* + \gamma(1-\delta) & -\alpha_2 S^* - \mu & 0 \\ 0 & 0 & \delta\gamma & 0 & -\mu \end{bmatrix}$$

By some row operations, we have

$$J(E^*) = \begin{bmatrix} -\beta S* - \mu & 0 & -\beta P* & 0 & 0 \\ 0 & -\alpha_1 - \mu & M_1 & 0 & 0 \\ 0 & 0 & M_2 & \alpha_2 S* & 0 \\ 0 & 0 & 0 & M_3 & 0 \\ 0 & 0 & 0 & 0 & -\mu \end{bmatrix}$$

where

$$M_1 = \frac{\mu\beta P^*}{\beta S^* + \mu}, \quad M_2 = \alpha_2 Q_t{}^* - (\mu + \gamma) - \frac{M_1\alpha_1}{-\alpha - \mu}, \quad M_3 = \alpha_2 S^* - \mu - \frac{\alpha_2 S^*}{M_2}(\gamma(1-\delta) - \alpha_2 Q_t{}^*)$$

After some calculations, we see that all the Eigen values are negative if $R_0 > 1$, $\gamma + \mu > \alpha_2 Q_t{}^* + \frac{\mu P^* \alpha_1{}^2 \beta^2}{M^*}$, and $\alpha_2 Q_t{}^* < \gamma(1-\delta)$,
where $M^* = (\lambda + \mu + \sigma)(\lambda + \alpha_1 + \mu)(\beta S^* + \mu)(\alpha_1 + \mu)$.
Thus the endemic equilibria is locally asymptotically stable for the above conditions.

Global Dynamics:
To show that the system (1) is globally asymptotically stable, we use the Lyapunov function theory for both the disease-free (smoking free) and the endemic equilibrium. First we present the global stability of the disease-free equilibrium.

Theorem 3: For $R_0 < 1$, the infected-free equilibrium E_0 is globally asymptotically stable in the interior of Γ if $\alpha_1 \beta^2 PS + \mu + \gamma < M_4$, where $M_4 = \alpha_1 \beta^2 S Q_t + \frac{\mu \alpha_1 \beta^2 O}{(\lambda + \mu + \delta)(\lambda + \alpha_1 + \mu)}$.
Proof: To establish the global stability of the disease free equilibrium we construct the following Lyapunov function.

$$L(t) = \alpha_1 \beta O - S$$

Calculating the time derivative of "L", we have

$$\begin{aligned} L'(t) &= \alpha_1 \beta \frac{dO}{dt} - \frac{dS}{dt} \\ &= \alpha_1 \beta(\beta PS - \alpha_1 O - \mu O) - (\alpha_2 S Q_t - (\mu + \gamma)S) \\ &= \alpha_1 \beta^2 PS + (\mu + \gamma)S - \alpha_1{}^2 \beta O - \alpha_1 S Q_t - \mu\alpha\beta O(1 - R_0). \end{aligned}$$

Here the time derivative of Lyapunov function is negative if $R_0 < 1$ and $\alpha_1 \beta^2 PS + (\mu + \gamma)S < M_4$, and $L'(t) = 0$ if and only if $Q_t = O = S = 0$.
Hence E_0 is globally asymptotically stable.
Theorem 4: The endemic equilibrium E^* is globally asymptotically stable in the interior of Γ when $\mu(O + P) = -\alpha_1 O^* + \Lambda$.

Proof: To establish the global stability of endemic equilibrium we construct the following Lyapunov function

$$L(t) = P(t) + O(t)$$

Calculating the time derivative of "L"

$$\begin{aligned} L'(t) &= P'(t) + O'(t) \\ &= \Lambda - \beta PS - \mu P + \beta PS - \alpha_1 O - \mu O \\ &= -\alpha_1 (O - O^*) \\ &< 0 \end{aligned}$$

Thus the time derivative of Lyapunov function is negative and $L'(t) = 0$ for $O = O^*$. Hence by Lassalle's invariance principal E^* is globally asymptotically stable on Γ.

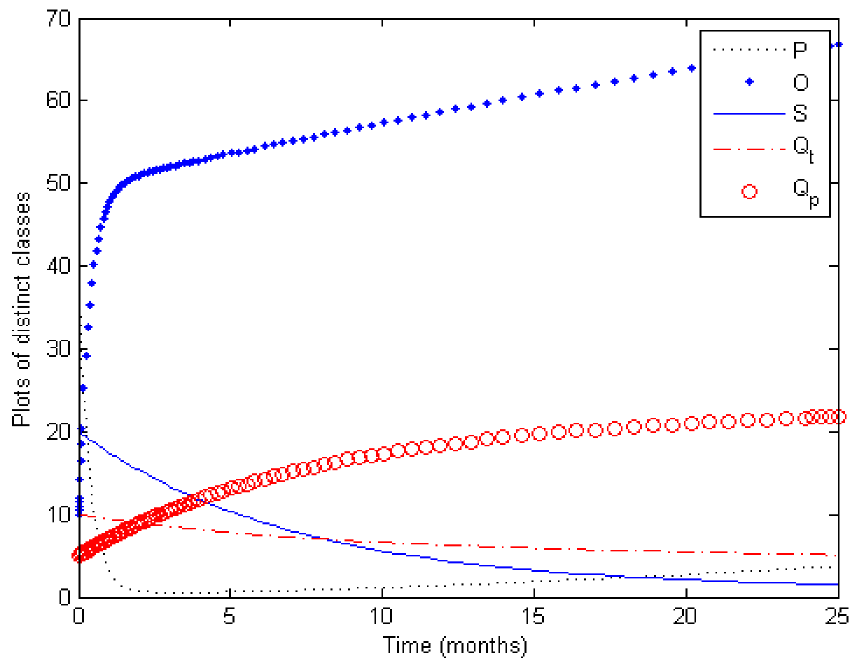

Figure 1. Dynamical behavior of the proposed model.

Numerical Results

In this section, we find the numerical solutions of the proposed smoking model (1) by choosing the base line for the potential smokers $P = 40$, occasional smokers $O = 10$, smokers $S = 20$, temporary quitters $Q_t = 10$ and permanent quitters $Q_P = 5$. The parameters and their values used are given as $\Lambda = 1$, $\beta = 0.14$, $\mu = 0.001$, $\alpha_1 = 0.002$, $\alpha_2 = 0.0025$ and $\sigma = 0.8$. Figure 1 shows the dynamical behavior of the proposed model.

Conclusion

The compartmental model of smoking is discussed in this work. Our model has two steady states, a nonsmoking steady state and endemically affected smoking steady state. By establishing the stability results we found both the nonsmoking and the endemic equilibria. We also presented that for $R_0 < 1$, the locally asymptotically stable nonsmoking equilibrium of the proposed model co-exists with a locally asymptotically stable endemic equilibrium. Then to present the global stability of both the nonsmoking and endemic states, we developed Lyapunov functions.

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