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Classifications of Ordered Semigroups in Terms of Bipolar Fuzzy Bi-Ideals

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ABSTRACT

Bipolar fuzzy set is an extension of fuzzy set. In bipolar fuzzy set the range of the membership function is $[-1,1]$, whereas, in fuzzy set it is $[0,1]$. We employed the concept of bipolar fuzzy set theory in the structure of ordered semigroup and introduced a generalization of bipolar fuzzy bi-ideals. This generalized form of bipolar fuzzy bi-ideals is called $(\in,\in\vee q)$-bipolar fuzzy bi-ideals of ordered semigroups. We obtained some interesting characterization results of ordered semigroup in terms of this new concept.
KEYWORDS: Bipolar fuzzy bi-ideals, bipolar fuzzy point, (α,β)-bipolar fuzzy bi-ideals, positive t-cut, negative s-cut, (s,t)-cut.

1. INTRODUCTION

The list of abbreviations that are used frequently in this paper are given in see Table 1.

Table 1: NAMES AND ABBREVIATIONS

Name	Abbreviation
Bipolar fuzzy bi-ideals	BFBI
Bipolar fuzzy set	BFS
Fuzzy Bi-ideals	FBI
Generalised Fuzzy Bi-ideals	GFBI
Ordered Semigroups	OS
If and only if	iff
The set of all bipolar fuzzy subsets of S.	BF(S)
Ordered Bipolar Fuzzy Point	OBFP
Bipolar fuzzy bi-ideals	(α,β)-BFBI
$(\in,\in\vee q)$-bipolar fuzzy generalized bi-ideal	$(\in,\in\vee q)$-BFGBI
$(\in,\in)$-bipolar fuzzy bi-ideal	$(\in,\in)$-BFBI
$(\in,\in\vee q)$-bipolar fuzzy bi-ideal	$(\in,\in\vee q)$-BFBI
Bipolar fuzzy left (right) ideal	BFL (R) I
Bipolar fuzzy ideal	BFI

Zhang (Zhang, 1994, 1998) further generalized the theory of fuzzy set (Zadeh, 1965) and defined BFS. In BFS the range of the membership function is $[-1,1]$ rather than $[0,1]$ (as in case of fuzzy set). In BFS, we split the range $[-1,1]$ into $(0,1]$, 0 and $[-1,0)$. If the membership degree of an element x (say) belongs to the interval $(0,1]$, then it indicates that x to some extent satisfies the property, whereas, if the membership

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degree x is equal to 0, then we say that x is irrelevant to the corresponding property. On the other hand and the membership degree of x belongs to $[-1,0)$, then x satisfies the implicit counter property to a certain degree. Although BFS look like similar to intuitionistic fuzzy sets but they are totally different from each other as mentioned in (Lee, 2004).

Kuroki (Kuroki, 1979, 1981,1991) was the first to employed the theory of fuzzy set in the structure of semigroups. In (Bhakat and Das, 1992) a generalized form of fuzzy subgroup (Rosenfeld, 1971) is presented. In addition, the generalization of FBI in semigroups is given in (Kazanci and Yamak, 2008). Further, Jun et al (Jun et al., 2005) generalized fuzzy sub-algebra in BCK/BCI algebra and presented the idea of (α,β)-fuzzy sub-algebra, where α and β are in $\{\in, q, \in\vee q, \in\wedge q\}$ and $\alpha\neq\in\wedge q$. In this regard for further study in different branches of algebra see (Criste, 2010, Davvaz, 2010, Kazanci, 2008, Shabir, 2010, Ma, 2008).

A BFS in a set S is denoted by:

$f=\{(x,(f_n(x):\ S\to[-1,0], f_p(x):\ S\to[0,1]))\,|\,x\in S\}$, where f_n and f_p are called the negative membership and positive membership mappings respectively. If for an element x in S we have $f_p(x)\neq 0$, $f_n(x)=0$, then x has a positive satisfaction of a BFS f in S. On other hand if $f_n(x)\neq 0$, $f_p(x)=0$, then x satisfies the counter property of a BFS f in S up to some extent. There is a possibility for an element (s) x such that $f_p(x)\neq 0\neq f_n(x)$. For short write $f=(S;f_n,f_p)$ to represent a BFS.

This study aims to introduce a more generalized form of BFBI of OS and investigate several characterization of OS in terms of this new idea. We further discuss relations between $(\in,\in)$-BFBI, $(\in,\in\vee q)$-BFBI and a $(q,\in\vee q)$-BFBI.

2. PRELIMINARIES

Let S be a non-empty set, " $\cdot$ " denotes operation of multiplication and " $\leq$ " denotes partial order relation. If S satisfy the following conditions:

(1) $(S,\cdot)$ is a semigroup,

(2) $(S,\leq)$ is a partial ordered set,

(3) $(\forall a,b,x\in S)$ $(a\leq b\Rightarrow ax\leq bx$ and $xa\leq xb)$.

Then $(S,\cdot,\leq)$ is called an OS and is denoted by S.

If $\phi\neq A,B\subseteq S$ and $x,y,z\in S$, then we denote and define the following:

$$(A]:=\{t\in S\ :\ t\leq h\ \in A\},\ AB=\{ab\ :\ a\in A\ and\ b\in B\},$$
$$A_x=\{(y,z)\in S\times S\,|\,x\leq yz\},$$
$$A\subseteq B\Rightarrow A\subseteq(A],$$
$$(A]\subseteq(B],$$
$$(A](B]\subseteq(AB],$$
$$((A]]=(A].$$

The characteristic function of A is denoted by $\chi_A=(S,\chi_{nA},\chi_{pA})$ where χ_{nA} and χ_{pA} are defined as:

$$\chi_{nA}(x)=\begin{cases}-1, & \text{if}\quad x\in A,\\ 0, & \text{if}\quad x\notin A.\end{cases},\ \chi_{pA}(x)=\begin{cases}1, & \text{if}\quad x\in A,\\ 0, & \text{if}\quad x\notin A.\end{cases}$$

If $A^2\subseteq A$, then A is called a subsemigroup. A subsemigroup A of S is a bi-ideal if:

(1) $ASA\subseteq A$,

(2) $x\leq y\in A\Rightarrow x\in A$,

for all $x,y\in S$.

The negative s-cut is defined and denoted as $N(f;s):=\{x\in S\,|\,f_n(x)\leq s\}$ whereas the positive t-cut is represented by $P(f;t):=\{x\in S\,|\,f_p(x)\geq t\}$ for a BFS $f=(S;f_n,f_p)$. We represent and define the (s,t)-cut of f as $C(f;(s,t))=N(f;s)\cap P(f;t)$.

For two BFSs $f=(S;f_n,f_p)$ and $g=(S;g_n,g_p)$ and for all $x\in S$ we say that:

i. $f\preceq g$ iff $f_n(x)\geq g_n(x)$ and $f_p(x)\leq g_p(x)$,

ii. $f=g$ iff $f\preceq g$ and $g\preceq f$,

iii. $f\cap g=(S;f_n\vee g_n,\ f_p\wedge g_p)$,

iv. $f\cup g=(S;f_n\wedge g_n,\ f_p\vee g_p)$,

For all $x\in S$, the BFS $0=(S;0_n,0_p)$ and $1=(S;1_n,1_p)$ are defined as follows:

$0_n(x)=0=0_p(x)$, $1_n(x)=-1$ and $1_p(x)=1$.

The product of $f=(S;f_n,f_p)$ and $g=(S;g_n,g_p)$ is defined and denoted as:

$$f\circ g=(S;f_n\circ g_n,f_p\circ g_p),\ \text{where}$$

$$(f_n\circ g_n)(x)=\begin{cases}\wedge\{f_n(y)\vee g_n(z)\} & \text{if} \quad A_x\neq\phi,\\ 0 & \text{if} \quad A_x=\phi.\end{cases}$$

and

$$(f_p\circ g_p)(x)=\begin{cases}\vee\{f_p(y)\wedge g_p(z)\} & \text{if} \quad A_x\neq\phi,\\ 0 & \text{if} \quad A_x=\phi.\end{cases}$$

The multiplication "$\circ$" is well defined and associative and clearly $(BF(S),\circ,\preceq)$ is an ordered semigroup.

Definition 2.1 (Shabir, 2013): A BFS $f=(S;f_n,f_p)$ in S is called a BFL (R) I of S if:

1) $x\leq y\Rightarrow f_n(x)\leq f_n(y)$ and $f_p(x)\geq f_p(y))$

2) $f_n(xy)\leq f_n(y)$ and $f_p(xy)\geq f_p(y)$

($f_n(xy)\leq f_n(x)$ and $f_p(xy)\leq f_p(x)$)

for all $x,y\in S$.

A BFS is called BFI of S if it is both BFLI and BFRI.

Definition 2.2 (Shabir, 2013): A BFS f is called BFBI of S, if:

(1) $x\leq y\Rightarrow f_n(x)\leq f_n(y)$ and $f_p(x)\geq f_p(y)$

(2) $f_n(xy)\leq\vee\{f_n(x),f_n(y)\}$, and $f_p(xy)\geq\wedge\{f_p(x),f_p(y)\}$

(3) $f_n(xyz)\leq\vee\{f_n(x),f_n(z)\}$ and $f_p(xyz)\geq\wedge\{f_p(x),f_p(z)\}$

for all $x,y,z\in S$.

Theorem 2.3 Let f be a BFS in S. Then f is a BFBI iff $C(f;(s,t))\neq\phi$ is a bi-ideal of S for all $(s,t)\in[-1,0)\times(0,1]$.

Proposition 2.4 The following hold for $A,B\subseteq S$.

(i) $A\subseteq B$ if and only if $\chi_A\subseteq\chi_B$,

(ii) $\chi_A\cap\chi_B=\chi_{A\cap B}$,

(iii) $\chi_A\circ\chi_B=\chi_{(AB]}$.

Proof. (i) It is straightforward to prove and thus omitted.

(ii) First we consider $A\cap B=\phi$. Then clearly $\chi_A\cap\chi_B=\chi_{A\cap B}$. On the other hand if $A\cap B\neq\phi$, then we let $x\in A\cap B$ and therefore

$$\chi_{{}_nA\cap B}(x)=-1=(\chi_{{}_nA}\wedge\chi_{{}_nB})(x),$$

and

$$\chi_{{}_pA\cap B}(x)=1=(\chi_{{}_pA}\wedge\chi_{{}_pB})(x).$$

Hence $\chi_A\cap\chi_B=\chi_{A\cap B}$.

(iii) Now we prove $\chi_A\circ\chi_B=\chi_{(AB]}$. For this it is enough to show that $\chi_{{}_nA}\circ\chi_{{}_nB}=\chi_{{}_n(AB]}$ and $\chi_{{}_pA}\circ\chi_{pB}=\chi_{p(AB]}$. Let $x\in S$: If $x\in(AB]$, then $\chi_{{}_n(AB]}(x)=-1$, $\chi_{{}_p(AB]}(x)=1$ and $x\leq ab$ for some $a\in A$ and $b\in B$. Hence $(a,b)\in A_x$ and therefore $A_x\neq\phi$ and we have

$$(\chi_{{}_nA}\circ\chi_{{}_nB})(x)=\bigwedge_{(y,z)\in A_x}\{\chi_{{}_nA}(y)\vee\chi_{{}_nB}(z)\}$$
$$\leq\{\chi_{{}_nA}(a)\vee\chi_{{}_nB}(b)\}=-1,$$

and

$$(\chi_{{}_pA}\circ\chi_{{}_pB})(x)=\bigvee_{(y,z)\in A_x}\{\chi_{{}_pA}(y)\wedge\chi_{{}_pB}(z)\}$$
$$\geq\{\chi_{{}_pA}(a)\wedge\chi_{{}_pB}(b)\}=1.$$

It follows that $\chi_{{}_n(AB]}(x)=-1=(\chi_{{}_nA}\circ\chi_{{}_nB})(x)$ and $\chi_{{}_p(AB]}(x)=1=(\chi_{{}_pA}\circ\chi_{{}_pB})(x)$.

3. Generalized BFBI of OS

The idea of BFBI is generalized to introduce the notion of (α,β)-BFBI in OS in this section.

If $(s,t)\in[-1,0)\times(0,1]$, then a BFS $f=(S;f_n,f_p)$ of the form:

$$f_n(y):=\begin{cases}s, & \text{if} \quad y\in(x],\\ 0, & \text{if} \quad y\notin(x],\end{cases},\ f_p(y):=\begin{cases}t, & \text{if} \quad y\in(x],\\ 0, & \text{if} \quad y\notin(x].\end{cases}$$

is called an OBFP with x and (s,t) are the value and support of the OBFP and is denoted by $\frac{x}{(s,t)}$. An OBFP $\frac{x}{(s,t)}$ is said to belongs to f (denoted as $\frac{x}{(s,t)}\in f$) if $f_n(x)\leq s$ and $f_p(x)\geq t$. If $f_n(x)+s<-1$ and $f_p(x)+t>1$, then $\frac{x}{(s,t)}$ is said to be quasi-coincidence with f (denoted as $\frac{x}{(s,t)}qf$). If $\frac{x}{(s,t)}\in f$ or $\frac{x}{(s,t)}qf$, then we write $\frac{x}{(s,t)}\in\vee qf$. By $\frac{x}{(s,t)}\in\wedge qf$ we mean $\frac{x}{(s,t)}\in f$ and $\frac{x}{(s,t)}qf$. We write $\frac{x}{(s,t)}\overline{\alpha}f$ if the relation $\frac{x}{(s,t)}\alpha f$ does not hold.

Let $f_n(x)\geq-0.5$ and $f_p(x)\leq 0.5$ for all $x\in S$ $(s,t)\in[-1,0)\times(0,1]$ such that $\frac{x}{(s,t)}\in\wedge qf$. Then $f_n(x)\leq s$, $f_p(x)\geq t$, $f_n(x)+s<-1$ and $f_p(x)+t>1$. It follows that $-1>f_n(x)+s\geq f_n(x)+f_n(x)$ and $1<f_p(x)+t\leq f_p(x)+f_p(x)$ that is $f_n(x)<-0.5$ and $f_p(x)>0.5$. It follows that $\{\frac{x}{(s,t)}\mid\frac{x}{(s,t)}\in\wedge q\}=\phi$. From the above discussion we conclude that the case $\alpha=\in\wedge q$ will be omitted in this study.

Definition 3.1 A BFS f in S is called (α,β)-bipolar fuzzy left (right) ideal of S where $\alpha\neq\in\wedge q$ if the following conditions hold for all $(s,t)\in[-1,0)\times(0,1]$ and $x,y\in S$:

(1) If $x\leq y$, then $\frac{y}{(s,t)}\alpha f$ implies $\frac{x}{(s,t)}\beta f$,

(2) $\frac{y}{(s,t)}\alpha f\Rightarrow\frac{xy}{(s,t)}\beta f$ ($\frac{x}{(s,t)}\alpha f\Rightarrow\frac{xy}{(s,t)}\beta f$).

Definition 3.2 A BFS f in S is called (α,β)-BFBI the following conditions hold for all $(s,t)\in[-1,0)\times(0,1]$ and $x,y,z\in S$:

(1) If $x\leq y$, then $\frac{y}{(s,t)}\alpha f\Rightarrow\frac{x}{(s,t)}\beta f$,

(2) $\frac{x}{(s_1,t_1)}\alpha f$ and $\frac{y}{(s_2,t_2)}\alpha f\Rightarrow\frac{xy}{(\vee\{s_1,s_2\},\wedge\{t_1,t_2\})}\beta f$,

(3) $\frac{x}{(s_1,t_1)}\alpha f$ and $\frac{z}{(s_2,t_2)}\alpha f\Rightarrow\frac{xyz}{(\vee\{s_1,s_2\},\wedge\{t_1,t_2\})}\beta f$.

Theorem 3.3 Let f be a BFS in S. Then f is a BFBI if and only if for all $(s,t)\in[-1,0)\times(0,1]$ and $x,y,z\in S$ the following conditions hold simultaneously:

(1) If $x\leq y$, then $\frac{y}{(s,t)}\in f\Rightarrow\frac{x}{(s,t)}\in f$),

(2) $\frac{x}{(s_1,t_1)}\in f$ and $\frac{y}{(s_2,t_2)}\in f\Rightarrow\frac{xy}{(\vee\{s_1,s_2\},\wedge\{t_1,t_2\})}\in f$,

(3) $\frac{x}{(s,t)}\in f$ and $\frac{z}{(s,t)}\in f\Rightarrow\frac{xyz}{(s,t)}\in f$.

Proof. Suppose that f is a BFBI. If $x,y\in S$ such that $x\leq y$ and $\frac{y}{(s,t)}\in f$, then $f_n(y)\leq s$ and $f_p(y)\geq t$. By Definition 2.2 we have $f_n(x)\leq f_n(y)$ and $f_p(x)\geq f_p(y)$. This implies that $f_n(x)\leq s$ and $f_p(x)\geq t$ and so $\frac{x}{(s,t)}\in f$.

If $\frac{x}{(s_1,t_1)}\in f$ and $\frac{y}{(s_2,t_2)}\in f$, then $f_n(x)\leq s_1$, $f_p(x)\geq t_1$ and $f_n(y)\leq s_2$, $f_p(y)\geq t_2$. By Definition 2.2 we

have

$$f_n(xy) \le \vee\{f_n(x), f_n(y) \le \vee\{s_1, s_2\}$$

and

$$f_p(xy) \ge \wedge\{f_p(x), f_p(y)\} \ge \wedge\{t_1, t_2\},$$

it follows that $\frac{xy}{(\vee\{s_1,s_2\},\wedge\{t_1,t_2\})} \in f$.

If $x, y, z \in S$ such that $\frac{x}{(s_1,t_1)} \in f$ and $\frac{z}{(s_2,t_2)} \in f$, then $f_n(x) \le s_1$, $f_p(x) \ge t_1$ and $f_n(z) \le s_2$, $f_p(z) \ge t_2$. By Definition 2.2 (3) we have

$$f_n(xyz) \le \vee\{f_n(x), f_n(z) \le \vee\{s_1, s_2\}$$

and

$$f_p(xyz) \ge \wedge\{f_p(x), f_p(z)\} \ge \wedge\{t_1, t_2\},$$

this implies $\frac{xyz}{(\vee\{s_1,s_2\},\wedge\{t_1,t_2\})} \in f$.

Conversely let $x, y \in S$ such that $x \le y$. If $f_n(y) = s$ and $f_p(y) = t$, then $\frac{y}{(s,t)} \in f$ and hence by (1) we have $\frac{x}{(s,t)} \in f$. This implies $f_n(x) \le s = f_n(y)$ and $f_p(x) \ge t = f_p(y)$. Hence we have $f_n(x) \le f_n(y)$ $f_p(x) \ge f_p(y)$.

If $\frac{x}{(f_n(x),f_p(x))} \in f$ and $\frac{y}{(f_n(y),f_p(y))} \in f$, then by (2) we have $\frac{xy}{(\vee\{f_n(x),f_n(y)\},\wedge\{f_p(x),f_p(y)\})} \in f$. This implies that $f_n(xy) \le \vee\{f_n(x), f_n(y)$ and $f_p(xy) \ge \wedge\{f_p(x), f_p(y)\}$.

Let $x, y, z \in S$ such that $\frac{x}{(f_n(x),f_p(x))} \in f$ and $\frac{z}{(f_n(z),f_p(z))} \in f$, then by (3) we have $\frac{xyz}{(\vee\{f_n(x),f_n(z)\},\wedge\{f_p(x),f_p(z)\})} \in f$. It follows that

$$f_n(xyz) \le \vee\{f_n(x), f_n(z),\ f_p(xyz) \ge \wedge\{f_p(x), f_p(z)\}.$$

Proposition 3.4 (Ibrar et al., 2016) A BFS f in S is an $(\in, \in \vee q)$-BFGBI iff the following conditions hold simultaneously for all $x, y, z \in S$:

(1) $x \le y \Rightarrow f_n(x) \le \vee\{f_n(y), -0.5\}$ and $f_p(x) \ge \wedge\{f_p(y), 0.5\}$,

(2) $f_n(xyz) \le \vee\{f_n(x), f_n(z), -0.5\}$ and $f_p(xyz) \ge \wedge\{f_p(x), f_p(z), 0.5\}$.

Theorem 3.5 A BFS f in S is an $(\in, \in \vee q)$-BFBI iff the following conditions hold simultaneously for all $x, y, z \in S$:

(1) $x \le y \Rightarrow f_n(x) \le f_n(y) \vee -0.5$ and $f_p(x) \ge f_p(y) \wedge 0.5$,

(2) $f_n(xy) \le \bigvee\{f_n(x), f_n(y), -0.5\}$ and $f_p(xy) \ge \bigwedge\{f_p(x), f_p(y), 0.5\}$,

(3) $f_n(xyz) \le \bigvee\{f_n(x), f_n(z), -0.5\}$ and $f_p(xyz) \ge \bigwedge\{f_p(x), f_p(z), 0.5\}$.

Proof. The proof follows from Proposition 3.4.

Theorem 3.6 Every $(\in, \in)$-BFBI is $(\in, \in \vee q)$-BFBI.

Proof. It can be proved easily.

The below given example shows that every $(\in, \in \vee q)$-BFBI is not $(\in, \in)$-BFBI.

Example 3.7 Let $S = \{a, b, c, d, e\}$ be a set with the following multiplication table and order relation "$\le$"

Table 1

.	a	b	c	d	e
a	a	d	a	d	d
b	a	b	a	d	d
c	a	d	c	d	e
d	a	d	a	d	d
e	a	d	c	d	e

$\le := \{(a,a),(a,c),(a,d),(a,e),(b,b),(c,c),(c,e),(d,d),(e,e)\}$.

Then S is an OS. Let f be a BFS in S defined by Table 2 below:

Table 2

S	a	b	c	d	e
f_n	-0.75	-0.35	-0.72	-0.58	-0.65
f_p	0.8	0.3	0.7	0.5	0.6

Then f is $(\in,\in\vee q)$-BFBI but not an $(\in,\in)$-BFBI because $\frac{a}{(-0.7,0.7)}\in f$ and $\frac{e}{(-0.6,0.55)}\in f$ but $\frac{ae}{(\vee\{-0.7,-0.6\},\ \wedge\{0.7,0.55\})}=\frac{d}{(-0.6,0.55)}\notin f$.

Theorem 3.8 Every $(\in\vee q,\in\vee q)$-BFBI is an $(\in,\in\vee q)$-BFBI.

Proof. Let f be an $(\in\vee q,\in\vee q)$-BFBI of S. Let $x,y\in S$ such that $x\le y$ and $(s,t)\in[-1,0)\times(0,1]$. If $\frac{y}{(s,t)}\in f$ then $\frac{y}{(s,t)}\in\vee qf$ and hence by hypothesis we have $\frac{x}{(s,t)}\in\vee qf$. Next we consider $(s_1,t_1),(s_2,t_2)\in[-1,0)\times(0,1]$ and $\frac{x}{(s_1,t_1)}\in f$, $\frac{y}{(s_2,t_2)}\in f$. Then $\frac{x}{(s_1,t_1)}\in\vee qf$, $\frac{y}{(s,t)}\in\vee qf$ and hence by hypothesis this implies $\frac{xy}{(\vee\{s_1,s_2\},\wedge\{t_1,t_2\})}\in\vee qf$. Again we let $x,y,z\in S$ such that $\frac{x}{(s_1,t_1)}\in f$ and $\frac{z}{(s_2,t_2)}\in f$ implies $\frac{x}{(s_1,t_1)}\in\vee qf$ and $\frac{z}{(s_2,t_2)}\in\vee qf$. By hypothesis $\frac{xyz}{(\vee\{s_1,s_2\},\wedge\{t_1,t_2\})}\in\vee qf$. Hence $f=(S;f_n,f_p)$ is an $(\in,\in\vee q)$-BFBI.

Theorem 3.9 If f is a nonzero $(\in,\in\vee q)$-BFBI of S, then the set $S_\circ=\{x\in S\mid f_n(x)\ne 0\}\cap\{x\in S\mid f_p(x)\ne 0\}$ is a bi-ideal of S.

Proof. Let $x,y\in S$ such that $x\le y$ and $y\in S_\circ$. Then $f_n(y)\ne 0$ and $f_p(y)\ne 0$. Hence $f_n(y)<0$ and $f_p(y)>0$. Suppose that $f_n(x)=0$ or $f_p(x)=0$. Since $\frac{y}{(f_n(y),f_p(y))}\in f$ and $f_n(x)>f_n(y)$ or $f_p(x)<f_p(y)$. This implies that $\frac{x}{(f_n(y),f_p(y))}\overline{\in}f$, which is a contradiction. Also $f_n(x)+f_n(y)=f_n(y)\ge -1$ or $f_p(x)+f_p(y)=f_p(y)\le 1$. This implies that $\frac{x}{(f_n(y),f_p(y))}\overline{q}f$, which is a contradiction. Therefore, $f_n(x)\ne 0$ and $f_p(x)\ne 0$. Hence, $x\in S_\circ$.

If $x,y\in S_\circ$, then $f_n(x)\ne 0$, $f_n(y)\ne 0$, $f_p(x)\ne 0$ and $f_p(y)\ne 0$. So $f_n(x)<0$, $f_n(y)<0$, $f_p(x)>0$ and $f_p(y)>0$. Suppose that $xy\notin S_\circ$, then $f_n(xy)=0$ or $f_p(xy)=0$. Clearly $\frac{x}{(f_n(x),f_p(x))}\in f$ and $\frac{y}{(f_n(y),f_p(y))}\in f$. Since $f_n(xy)=0>f_n(x)\vee f_n(y)$ or $f_p(xy)=0<f_p(x)\wedge f_p(y)$. This implies $\frac{xy}{(f_n(x)\vee f_n(y),f_p(x)\wedge f_p(y))}\overline{\in}f$, a contradiction. Also $f_n(xy)+f_n(x)\vee f_n(y)=f_n(x)\vee f_n(y)\ge -1$ or $f_p(xy)+f_p(x)\wedge f_p(y)=f_p(x)\wedge f_p(y)\le 1$. This implies that $\dfrac{xy}{(f_n(x)\vee f_n(y),f_p(x)\wedge f_p(y))}\overline{q}\,f$, again a contradiction. Therefore, $f_n(xy)\ne 0$ and $f_p(xy)\ne 0$. Hence, $xy\in S_\circ$.

Let $x,y,z\in S$ such that $x,z\in S_\circ$. Then $f_n(x)\ne 0$, $f_n(z)\ne 0$, $f_p(x)\ne 0$ and $f_p(z)\ne 0$. So $f_n(x)<0$, $f_n(z)<0$, $f_p(x)>0$ and $f_p(z)>0$. If $xyz\notin S_\circ$, then $f_n(xyz)=0$ or $f_p(xyz)=0$. Clearly $\frac{x}{(f_n(x),f_p(x))}\in f$ and $\frac{z}{(f_n(z),f_p(z))}\in f$. Since $f_n(xyz)=0>f_n(x)\vee f_n(z)$ or $f_p(xyz)=0<f_p(x)\wedge f_p(z)$. This implies $\frac{xyz}{(f_n(x)\vee f_n(z),f_p(x)\wedge f_p(z))}\overline{\in}f$, a contradiction. Also $f_n(xyz)+f_n(x)\vee f_n(z)=f_n(x)\vee f_n(z)\ge -1$ or $f_p(xyz)+f_p(x)\wedge f_p(z)=f_p(x)\wedge f_p(z)\le 1$. This implies that $\dfrac{xyz}{(f_n(x)\vee f_n(z),f_p(x)\wedge f_p(z))}\overline{q}\,f$, a contradiction. Therefore $xyz\in S_\circ$.

Theorem 3.10 Let f be an $(\in,\in\vee q)$-BFBI of S and $\left(f_n(x),f_p(x)\right)\in(-0.5,0)\times(0,0.5)$ for all $x\in S$. Then f is an $(\in,\in)$-BFBI.

Proof. Suppose f be an $(\in,\in\vee q)$-BFBI of S.

Let $x,y\in S$ be such that $x\le y$ and $\frac{y}{(s,t)}\in f$ for $(s,t)\in[-1,0)\times(0,1]$, then $f_n(y)\le s$ and $f_p(y)\ge t$. By

Theorem 3.5 (1) we have $f_n(x) \le f_n(y) \vee -0.5 \le s$ and $f_p(x) \ge f_p(y) \wedge 0.5 \ge t$ and so $\frac{x}{(s,t)} \in f$.

Let $x, y \in S$ and $(s_1,t_1),(s_2,t_2) \in [-1,0) \times (0,1]$ be such that $\frac{x}{(s_1,t_1)} \in f$ and $\frac{y}{(s_2,t_2)} \in f$ then $f_n(x) \le s_1$, $f_n(y) \le s_2$ $f_p(x) \ge t_1$, and $f_p(y) \ge t_2$. By Theorem 3.5 (2) we have $f_n(xy) \le \vee\{f_n(x), f_n(y), -0.5\} \le \vee\{s_1, s_2\}$ and $f_p(xy) \ge \wedge\{f_p(x), f_p(y), 0.5\} \ge \wedge\{t_1, t_2\}$, which implies $\frac{xy}{(\vee\{s_1,s_2\}, \wedge\{t_1,t_2\})} \in f$.

Let $x, y, z \in S$ and $(s_1,t_1),(s_2,t_2) \in [-1,0) \times (0,1]$ such that $\frac{x}{(s_1,t_1)} \in f$ and $\frac{z}{(s_2,t_2)} \in f$ then $f_n(x) \le s_1$, $f_n(z) \le s_2$, $f_p(x) \ge t_1$ and $f_p(z) \ge t_2$. By Theorem 3.5 (3) we have $f_n(xyz) \le \vee\{f_n(x), f_n(z), -0.5\} \le \vee\{s_1, s_2\}$ and $f_p(xyz) \ge \wedge\{f_p(x), f_p(z), 0.5\} \ge \wedge\{t_1, t_2\}$. In which it follows that $\frac{xyz}{(\vee\{s_1,s_2\}, \wedge\{t_1,t_2\})} \in f$. Hence f is an $(\in,\in)$-BFBI.

Theorem 3.11 Let I be a bi-ideals of S and f a BFS in S such that

(i) $f_n(x) = 0 = f_p(x)$ for all $x \in S \setminus I$,

(ii) $(f_n(x), f_p(x)) \in [-1,-0.5] \times [0.5,1]$ for all $x \in I$.

Then f is $(q, \in \vee q)$-BFBI of S.

Proof. Let $x, y \in S$ such that $x \le y$ and $(s,t) \in [-1,0) \times (0,1]$. Let $\frac{y}{(s,t)} qf$ then $f_n(y) + s < -1$ and $f_p(y) + t > 1$. This implies that $y \in I$. Since I is bi-ideals of S, so $x \in I$. In order to check $\frac{x}{(s,t)} \in \vee qf$, we consider the following four cases:

1. $s \ge -0.5$ and $t \le 0.5$,
2. $s < -0.5$ and $t > 0.5$
3. $s < -0.5$ and $t \le 0.5$,
4. $s \ge -0.5$ and $t > 0.5$.

The first case induces $f_n(x) \le -0.5 \le s$ and $f_p(x) \ge 0.5 \ge t$ and hence $\frac{x}{(s,t)} \in f$. The second case implies that $f_n(x) + s < -1$ and $f_p(x) + t > 1$, it follows that $\frac{x}{(s,t)} qf$. Since $\frac{y}{(s,t)} qf$ so case (3) and (4) do not occur. Consequently, $\frac{x}{(s,t)} \in \vee qf$.

Let $x, y \in S$ such that $\frac{x}{(s_1,t_1)} qf$ and $\frac{y}{(s_2,t_2)} qf$ for $(s_1,t_1),(s_2,t_2) \in [-1,0) \times (0,1]$, then $f_n(x) + s_1 < -1$, $f_n(y) + s_2 < -1$, $f_p(x) + t_1 > 1$ and $f_p(y) + t_2 > 1$ and so $x, y \in I$. Since I is a bi-ideals of S, therefore $xy \in I$. In order to check $\frac{xy}{(\vee\{s_1,s_2\}, \wedge\{t_1,t_2\})} \in \vee qf$ we consider the following four cases:

1. $s_1 \vee s_2 \ge -0.5$ and $t_1 \wedge t_2 \le 0.5$,
2. $s_1 \vee s_2 < -0.5$ and $t_1 \wedge t_2 > 0.5$,
3. $s_1 \vee s_2 < -0.5$ and $t_1 \wedge t_2 \le 0.5$,
4. $s_1 \vee s_2 \ge -0.5$ and $t_1 \wedge t_2 > 0.5$.

The first case induces $f_n(xy) \le -0.5 \le s_1 \vee s_2$ and $f_p(xy) \ge 0.5 \ge t_1 \wedge t_2$ and so $\frac{xy}{(s_1 \vee s_2, t_1 \wedge t_2)} \in f$. The second case implies that $f_n(xy) + s_1 \vee s_2 < -1$ and $f_p(xy) + t_1 \wedge t_2 > 1$ that is $\frac{xy}{(s_1 \vee s_2, t_1 \wedge t_2)} qf$. Since $\frac{x}{(s_1,t_1)} qf$ and $\frac{y}{(s_2,t_2)} qf$ so cases (3) and (4) do not occur. Consequently, $\frac{xy}{(s_1 \vee s_2, t_1 \wedge t_2)} \in \vee qf$.

Let $x, y, z \in S$ such that $\frac{x}{(s_1,t_1)} qf$ and $\frac{z}{(s_2,t_2)} qf$ for $(s_1,t_1),(s_2,t_2) \in [-1,0) \times (0,1]$, then $f_n(x) + s_1 < -1$, $f_n(z) + s_2 < -1$, $f_p(x) + t_1 > 1$ and $f_p(z) + t_2 > 1$, in which it follows that $x, z \in I$. Since I is bi-ideals of S, therefore we have $xyz \in I$. Now to show that $\frac{xyz}{(s_1 \vee s_2, t_1 \wedge t_2)} \in \vee qf$ we consider the following four cases:

1. $s_1 \vee s_2 \ge -0.5$ and $t_1 \wedge t_2 \le 0.5$,
2. $s_1 \vee s_2 < -0.5$ and $t_1 \wedge t_2 > 0.5$,
3. $s_1 \vee s_2 < -0.5$ and $t_1 \wedge t_2 \le 0.5$,
4. $s_1 \vee s_2 \ge -0.5$ and $t_1 \wedge t_2 > 0.5$.

The first case induces $f_n(xyz) \le -0.5 \le s_1 \vee s_2$ and $f_p(xyz) \ge 0.5 \ge t_1 \wedge t_2$ and so $\frac{xyz}{(s_1 \vee s_2, t_1 \wedge t_2)} \in f$. The second case implies that $f_n(xyz) + s_1 \vee s_2 < -1$ and $f_p(xyz) + t_1 \wedge t_2 > 1$ and therefore $\frac{xy}{(s_1 \vee s_2, t_1 \wedge t_2)} qf$. Since $\frac{x}{(s_1, t_1)} qf$ and $\frac{z}{(s_2, t_2)} qf$ and therefore both Case (3) and Case (4) do not occur. Hence $\frac{xyz}{(\vee\{s_1, s_{2_1}, \wedge\{t_1, t_2\})} \in \vee qf$.

In general every $(\in, \in \vee q)$-BFBI may not be $(q, \in \vee q)$-BFBI as shown in the following example.

Example 3.12 Consider the OS of Example 3.7 and define BFS f in S as in Table 3.

Table 3

S	a	b	c	d	e
f_n	-0.9	-0.35	-0.72	-0.58	-0.65
f_p	0.8	0.3	0.7	0.5	0.6

Then f is an $(\in, \in \vee q)$-BFBI of S but not a $(q, \in \vee q)$-BFBI because $\frac{a}{(-0.15, 0.7)} qf$ and $\frac{e}{(-0.45, 0.52)} qf$ but $\frac{ae}{(-0.15 \vee -0.45, 0.7 \wedge 0.52)} = \frac{d}{(-0.15, 0.52)} \overline{\in \vee q} f$.

In the following theorem we provide a condition for an $(\in, \in \vee q)$-BFBI to be $(q, \in \vee q)$-BFBI.

Theorem 3.13 Every $(\in, \in \vee q)$-BFBI is an $(q, \in \vee q)$-BFBI of S for condition that $(s,t) \in [-0.5, 0) \times (0, 0.5]$.

Proof. Let f be an $(\in, \in \vee q)$-BFBI of S and $x, y \in S$ such that $x \le y$ and $\frac{y}{(s,t)} qf$ for $(s,t) \in [-0.5, 0) \times (0, 0.5]$. Then $f_n(y) + s < -1$ and $f_p(y) + t > 1$ that is $f_n(y) < -1 - s \le s$ and $f_p(y) > 1 - t \ge t$ and so $\frac{y}{(s,t)} \in f$. Since f is an $(\in, \in \vee q)$-BFBI of S, Therefore $\frac{x}{(s,t)} \in \vee qf$.

Let $x, y \in S$ such that $\frac{x}{(s_1, t_1)} qf$ and $\frac{y}{(s_2, t_2)} qf$ for some $(s_1, t_1), (s_2, t_2) \in [-0.5, 0) \times (0, 0.5]$, then $f_n(x) + s_1 < -1$, $f_n(y) + s_2 < -1$, $f_p(x) + t_1 > 1$ and $f_p(y) + t_2 > 1$. This implies that $f_n(x) < -1 - s_1 \le s_1$ $f_n(y) < -1 - s_2 \le s_2$, $f_p(x) > 1 - t_1 \ge t_1$ and $f_p(y) > 1 - t_2 \ge t_2$. Hence $\frac{x}{(s_1, t_1)} \in f$ and $\frac{y}{(s_2, t_2)} \in f$ and since f is an $(\in, \in \vee q)$-BFBI of S, therefore $\frac{xy}{(s_1 \vee s_2, t_1 \wedge t_2)} \in \vee qf$.

Let $x, y, z \in S$ such that $\frac{x}{(s_1, t_1)} qf$ and $\frac{z}{(s_2, t_2)} qf$ for $(s_1, t_1), (s_2, t_2) \in [-0.5, 0) \times (0, 0.5]$, then $f_n(x) + s_1 < -1$, $f_n(z) + s_2 < -1$, $f_p(x) + t_1 > 1$ and $f_p(z) + t_2 > 1$. This implies that $f_n(x) < -1 - s_1 \le s_1$, $f_n(z) < -1 - s_2 \le s_2$, $f_p(x) > 1 - t_1 \ge t_1$ and $f_p(z) > 1 - t_2 \ge t_2$ that is $\frac{x}{(s_1, t_1)} \in f$ and $\frac{z}{(s_2, t_2)} \in f$. Since f is an $(\in, \in \vee q)$-BFBI of S, therefore $\frac{xyz}{(s_1 \vee s_2, t_1 \wedge t_2)} \in \vee qf$ and hence f is $(q, \in \vee q)$-BFBI of S.

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