J. Appl. Environ. Biol. Sci., 7(10)195-206, 2017

ISSN: 2090-4274
Journal of Applied Environmental and Biological Sciences
www.textroad.com

A New Hybrid Technique for the Solution of a Maxwell Fluid with Fractional Derivatives in a Circular Pipe

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Received: June 3, 2017
Accepted: August 13, 2017

ABSTRACT

In this article, fractional Maxwell fluid model is considered. The constant shear stress and velocity function of a Maxwell model with fractional derivatives are studied. The fluid is placed within an infinitely long circular pipe with radius R . The motion in long circular pipe is produced because of constant shear stress action. The semi analytical results are found with the help of Laplace transformation and modified Bessel equation. The hybrid technique we are using has less computational effort, time cost and easy to implement in engineering applications as compared to others methods commonly used. The solutions are obtained in transformed domain which are expressed in modified Bessel functions $I_0(\cdot)$ and $I_1(\cdot)$ of first kind of order zero and one respectively. In this paper, inverse Laplace transformation has been calculated numerically by using MATLAB. The semi analytical solutions for a Maxwell fluid model with fractional derivatives are reduced to the similar solutions of ordinary Maxwell model and Newtonian model as limiting cases. The graphical illustrations represent the behavior of material parameters on our solutions.
KEYWORDS: Maxwell fluid; Velocity field; Laplace transformation; Modified Bessel function; Shear stress.

Nomenclature

g – Velocity of the fluid, [ms^{-1}]
ζ – Shear stress of the fluid, [$Pascal$]
t – Time, [Sec]
λ – Relaxation time, [Sec]
E – Extra shear stress, $[Pascal]$
μ – Dynamic viscosity, [Nms^{-2}]
ν – Kinematic viscosity, [m^2s^{-1}]
R – Radius of circular cylinder, $[meter]$
α – Fractional parameter, $[---]$
ρ – Fluid density, $[kgms^{-3}]$
ν – Fluid kinematics viscosity, $[m^2s^{-1}]$
f – Constant, $[---]$

1. INTRODUCTION

Nowadays, problems of fluid mechanics related to numerical results are widely used because of accessibility of computer softwares. The numerical results are inconsiderable if these results are not compared with experimental or analytical results. Navier-Stokes theories, which are based on non-linear differential equations are consider as fundamental equations in fluid mechanics. The best way to examine the rheological properties of non-Newtonian models is through Navier-Stokes equations. Many non-Newtonian fluids have been proposed because of their implementations in engineering and industry.

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For the motion of incompressible viscoelastic fluids, the manipulation of fractional derivatives has been firstly presented by Germant [1]. The books of Chanderashker [2], Drazin and Reid [3] are considered standard for the exact solutions of non-Newtonian models in cylindrical areas. For Oldroyd-B models, exact results related to flow of non-Newtonian fluid models in cylindrical areas discussed by Waters and King [4], for second grade fluid by Ting [5] and for Maxwell models by Srivastava [6]. The applications of fractional approach in the study of viscoelastic fluids is examined by Bagley and Torvik [7]. Friedrich [8] showed that the molecular theory of non-Newtonian fluid models has been well described by converting ordinary derivatives into fractional derivatives. The movement of a Maxwell model with fractional derivatives within two parallel plates has been presented by Makris *et al* [9]. The investigation of rheological properties of differential type fluids by Rajagopal [10] and rate type fluid models by Dunn and Rajagopal [11] have acquired acceptance for experimentalists and theoreticians. It is worthy to mention here that Palade *et al* [12] constructed constitutive equation for an unsteady incompressible fluid which reduces to generalized Maxwell fluid model by deformations hypothesis.

Furthermore, mathematical model of Maxwell fluid with fractional derivatives has been found quite handy in polymer industry to develop transition in amorphous materials and glass states, like quartz and polystyrene. The movement of non-Newtonian models with fractional derivatives along a plane surface is given by Tan and Xu [13]. Motion of Burger's fractional model in Rayleigh Stokes equation is discussed by Kashif *et al* [14]. Vieru *et al* [15] examined Maxwell model motion with fractional derivatives corresponding to two parallel side walls. Jamil *et al* [16] discussed the exact solution of fractional Maxwell fluid related to Stokes' first problem. The motion of Maxwell fluid lying inside two rotating cylinders is presented by Fetecau *et al* [17]. Influence of magnetic field for Maxwell fractional fluid is examined by Abro *et al* [18].

Some important recent attempts to acquire exact analytic solutions of non-Newtonian fluid models with fractional derivatives are listed here [19-23]. Exact results of a fractional Maxwell fluid for a longitudinal motion within two coaxial cylinders is discussed by Awan *et al* [24], whereas motion of a fractional Maxwell fluid model in a cylinder having a nonlinear velocity is discussed Athar *et al* [25]. Numerical solutions of metallic wire coating using Non-Newtonian model motion is discussed by Khan *et al* [26].The analytical results of a Maxwell fluid over an oscillating plane is studied by Abro *et al* [27]. Dufour and Soret influence on unsteady motion of an Oldroyd-B fluid using boundary conditions discussed by Ashraf *et al* [28]. Kaladhar *et al* procured analytical solution of natural convection flow in porous channel with Soret, Hall and Joule effects [29]. The effect of radiations on nanofluid with boundary layer equations is presented by Mat *et al* [30]. Magnetohydrodynamic motion of Non-Newtonian through a moving surface boundary is studied by Tariq *et al* [31].

Maxwell fluid is considered as viscoelastic fluid. The most of basic biological models such as slurries, gastro-intestinal suspensions and blood are considered viscoelastic in applications. These fluids are used in bioengineering, chemical engineering and polymer industry. An important role played by viscoelastic fluid of fractional models with viscoelastic properties in rectifying simulations. In computational viscoelasticity, a familiar trend is the viscoelastic fluids with fractional derivatives in recent years. The Maxwell fluid with fractional derivatives is considered and found fractional Maxwell fluid by substituting the ordinary time derivatives of strain-stress by fractional order derivative in general.

The goal of this article is to examine the semi analytical results of unsteady longitudinal movement of Maxwell fluid model with fractional derivatives in a circular pipe. Initially the fluid in the circular pipe of radius R is at rest and at time $t = 0^+$, because of the shear stress, it starts to flow longitudinally. The semi analytical solutions are procured with the tool of Laplace transformation and modified Bessel equation. The inverse Laplace transformation has been achieved through MATLAB. These solutions are examined in transformed form with modified Bessel functions $I_0(\cdot)$ and $I_1(\cdot)$ meet all boundary and initial conditions. We obtained the known solutions relating to a Newtonian fluid for $\lambda \to 0,\ \alpha \to 1$ and ordinary Maxwell fluid for $\alpha \to 1$ having the same motion are the limiting cases of our solutions. The obtained general results for constant shear stress and velocity field for the Maxwell fluid motion with fractional derivatives show more accuracy by using this hybrid technique as compared to other techniques presented in other articles. For validation of our obtained results, comparison with other two numerical algorithms, namely the Stehfest's algorithm [33] and Tzou's algorithm [34] is presented in Table 1. Finally resulted are depicted by graphical demonstrations, which shows the impact of parameters on shear stress and velocity field.

The paper presentation is arranged as; the mathematical portion of the problem is presented in Sect. 2. Semi analytical results in transformed domain using the Laplace transform are examined in Sect. 3and some limiting cases in Sect. 4. Numerical solutions and graphical influences are studied in Sect. 5for important physical parameters. Some valuable results in the end in Sect. 6. At the end references are enlisted.

2. Basic governing equations

Assuming that an unsteady Maxwell fluid moving longitudinally along the axis of the circular pipe. The property of incompressibility is consequently fulfilled for the velocity field. Assume that Maxwell fluid model with fractional derivatives which is initially at rest in an infinitely long circular pipe with radius R. In the presence of shear stress, at $t = 0^+$, it starts to move due to applied constant shear stress on the surface of the circular pipe.

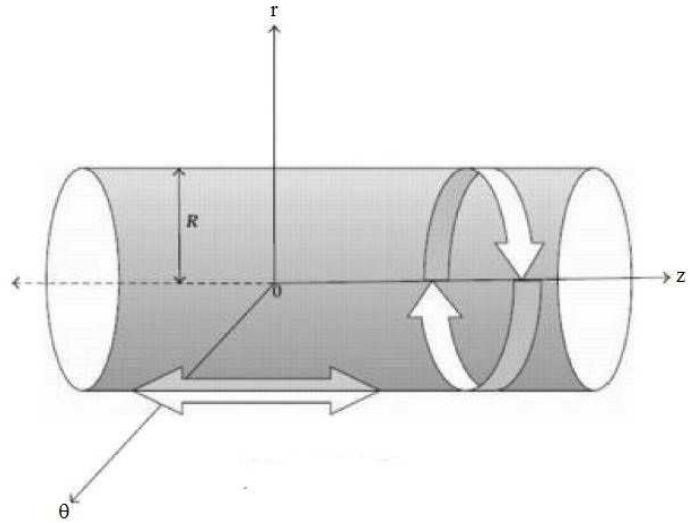


Fig 1. Geometry of the problem

The extra-stress T and velocity field G for the movement of fluid [17] are considered as

$$G = G(r,t) = g(r,t)e_z, \; T = T(r,t), \tag{1}$$

where e_z is the unit vector of the cylindrical coordinate system (r,θ,z) along the $z-$direction.

Moreover, when the fluid starts to move, we have

$$G(r,0) = 0, \qquad T(r,0) = 0, \tag{2}$$

The governing equations for Maxwell fluids [17] are

$$\left(1+\lambda\frac{\partial}{\partial t}\right)\zeta(r,t) = \mu\frac{\partial g(r,t)}{\partial r}, \tag{3}$$

$$\left(1+\lambda\frac{\partial}{\partial t}\right)\frac{\partial g(r,t)}{\partial t} = \nu\left(\frac{\partial}{\partial r}+\frac{1}{r}\right)\frac{\partial g(r,t)}{\partial r}, \tag{4}$$

where, the only non-trivial shear stress is $\zeta(r,t) = T_{rz}(r,t)$, λ is relaxation time, ν is the kinematics viscosity, μ is the dynamics viscosity. The Caputo fractional differential operator [14] is defined as

$$D_t^{\alpha}u(y,t) = \begin{cases} \dfrac{1}{\Gamma(1-\alpha)}\displaystyle\int_0^t \dfrac{1}{(t-\tau)^{\alpha}}\dfrac{\partial u(y,\tau)}{\partial \tau}d\tau; & 0<\alpha<1 \\ \dfrac{\partial u(y,t)}{\partial t}; & \alpha = 1 \end{cases}, \tag{5}$$

Where $\Gamma(\cdot)$ denotes the Gamma function.

By using the Caputo fractional differential operator D_t^{α} instead of inner time derivatives, the equations related to the Maxwell fluid having fractional derivatives can be obtained as

$$\left(1+\lambda^{\alpha}D_t^{\alpha}\right)\zeta\left(r,t\right)=\mu\frac{\partial g\left(r,t\right)}{\partial r}, \tag{6}$$

$$\left(1+\lambda^{\alpha}D_t^{\alpha}\right)\frac{\partial g\left(r,t\right)}{\partial t}=\nu\left(\frac{\partial}{\partial r}+\frac{1}{r}\right)\frac{\partial g\left(r,t\right)}{\partial r}, \tag{7}$$

The initial and boundary conditions for above mentioned governing Eqs. (7) and (8) are

$$g\left(r,0\right)=\left.\frac{\partial g(r,t)}{\partial t}\right|_{t=0}=0,\quad \zeta\left(r,0\right)=0;\ r\in\left[0,R\right], \tag{8}$$

$$\left(1+\lambda^{\alpha}D_t^{\alpha}\right)\zeta\left(R,t\right)=\mu\left.\frac{\partial g\left(r,t\right)}{\partial r}\right|_{r=R}=f,\quad \text{where f is a constant.} \tag{9}$$

Eqs. (6) and (7) involve fractional derivatives, these equations are solved by using the tool of Laplace transformation and modified Bessel equation.

3. Solution of the problem

3.1 Calculation of the Velocity field

Taking the Laplace transformation of Eqs. (7) and (9), we have

$$\frac{\partial^2\overline{g}(r,s)}{\partial r^2}+\frac{1}{r}\frac{\partial\overline{g}(r,s)}{\partial r}-\frac{\left(s+\lambda^{\alpha}s^{\alpha+1}\right)}{\nu}\overline{g}(r,s)=0, \tag{10}$$

$$\left.\frac{\partial\overline{g}(r,s)}{\partial r}\right|_{r=R}=\frac{f}{\mu s}, \tag{11}$$

where 's'is the Laplace transform parameter. Eqs. (10) and (11) can be written as

$$\frac{\partial^2\overline{g}(r,s)}{\partial r^2}+\frac{1}{r}\frac{\partial\overline{g}(r,s)}{\partial r}-a(s)\overline{g}(r,s)=0, \tag{12}$$

$$\left.\frac{\partial\overline{g}(r,s)}{\partial r}\right|_{r=R}=b(s), \tag{13}$$

where $a(s)=\dfrac{\left(s+\lambda^{\alpha}s^{\alpha+1}\right)}{\nu}$ and $b(s)=\dfrac{f}{\mu s}$.

The Eq. (12) represents the modified Bessel equation. The solution of the Eq. (12) by using condition (13) is written as

$$\overline{g}(r,s)=\frac{b(s)I_0\left(r\sqrt{a(s)}\right)}{\sqrt{a(s)}I_1\left(R\sqrt{a(s)}\right)}\ . \tag{14}$$

The expression in Eq. (14) is in a complicated form of modified Bessel functions of first kind of order zero and one. For the solution of Eq. (14), it is very difficult to found the inverse Laplace transform analytically. To solve this problem, we used some numerical package to obtain the inverse Laplace transform. Here, we have found the inverse Laplace transform numerically through MATLAB. Since the analytical and numerical methods are used in this paper therefore, our technique is referred as semi analytical.

3.2 Calculation of the Shear Stress

Taking Laplace transform of Eq. (6), we have

$$\overline{\zeta}\left(r,s\right)=\frac{\mu}{\left(1+\lambda^{\alpha}s^{\alpha}\right)}\frac{\partial\overline{g}\left(r,s\right)}{\partial r}, \tag{15}$$

Differentiating Eq. (14) with respect to 'r', we get

$$\frac{\partial \overline{g}(r,s)}{\partial r} = b(s)\frac{I_1\left(r\sqrt{a(s)}\right)}{I_1\left(R\sqrt{a(s)}\right)}. \tag{16}$$

Substituting this in Eq. (15), we obtain

$$\overline{\zeta}\left(r,s\right) = b(s)\frac{\mu}{\left(1+\lambda^{\alpha}s^{\alpha}\right)}\frac{I_1\left(r\sqrt{a(s)}\right)}{I_1\left(R\sqrt{a(s)}\right)}. \tag{17}$$

After taking inverse Laplace transform of Eq. (17), we obtain the solution of shear stress. The expression of Eq. (17) is of the same type of Eq. (14). Here, again we have found the inverse Laplace transform numerically through MATLAB.

4. Special Cases

We can take the following limiting cases of our general solutions.

4.1 Ordinary Maxwell Fluid

Using $\alpha \to 1$ in Eqs. (14) and (17), we acquire the solutions of velocity field $g(r,t)$ and time dependent shear stress $\varsigma(r,t)$ respectively equivalent to ordinary Maxwell fluid.

4.2 Newtonian Fluid

When we take $\lambda \to 0,\ \alpha \to 1$ in Eq. (14) and (17), the solutions of velocity field $g(r,t)$ and time dependent shear stress $\varsigma(r,t)$ equivalent to Newtonian fluid are procured respectively.

5. NUMERICAL RESULTS AND DISCUSSION

The semi analytical solutions of velocity field and tangential stress of a fractional Maxwell fluid in an infinitely long circular pipe are fixed. In order to provide the validation of results, we presented in Table 1, numerical results of fluid velocity, obtained with MATLAB program and with other two numerical algorithms, namely the Stehfest's algorithm [33] and Tzou's algorithm [34]. According with Stehfest's algorithm, the inverse Laplace transform is given by:

$$u(y,t) = \frac{\ln(2)}{t}\sum_{j=1}^{2m} d_j\,\overline{u}\left(y, j\frac{\ln(2)}{t}\right), \text{ where } m \text{ is a positive integer.} \tag{18}$$

$$\text{where } d_j = (-1)^{j+m}\sum_{i=\left[\frac{j+1}{2}\right]}^{\min(j,m)} \frac{i^m(2n)!}{(m-i)!\,i!\,(i-1)!\,(j-i)!\,(2i-j)!}, \tag{19}$$

and $[r]$ denotes the integer value function or bracket function. Whereas, the Tzou's algorithm is based on the Riemann-sum approximation. In this method, the inverse Laplace is given by

$$u(r,t) = \frac{e^{4.7}}{t}\left[\frac{1}{2}\overline{u}\left(r,\frac{4.7}{t}\right) + \mathrm{Re}\left\{\sum_{k=1}^{N_1}(-1)^k\,\overline{u}\left(r,\frac{4.7+k\pi i}{t}\right)\right\}\right], \tag{20}$$

here $\mathrm{Re}(\cdot)$ is the real part, i is the imaginary unit and $N_1 \geq 1$ is a natural number. The values of inverse Laplace transform obtained with MATLAB, Eqs. (22) and (23) are given in the following table.

Table 1: Comparison between different numerical algorithms

r	$g(r,t)$ **(Talbot)[32]**	$g(r,t)$ **(Stehfest's)[33]**	$g(r,t)$ **(Tzou's)[34]**
0	0	0	0
0.05	0.085864	0.084678	0.085014
0.1	0.12379	0.122081	0.122565
0.15	0.197265	0.194541	0.195312
0.2	0.321678	0.317237	0.318493
0.25	0.518062	0.510909	0.512932
0.3	0.811661	0.800455	0.803625
0.35	1.229326	1.212353	1.217154
0.4	1.795877	1.771082	1.778096
0.45	2.530082	2.49515	2.505032
0.5	3.441265	3.393752	3.407193

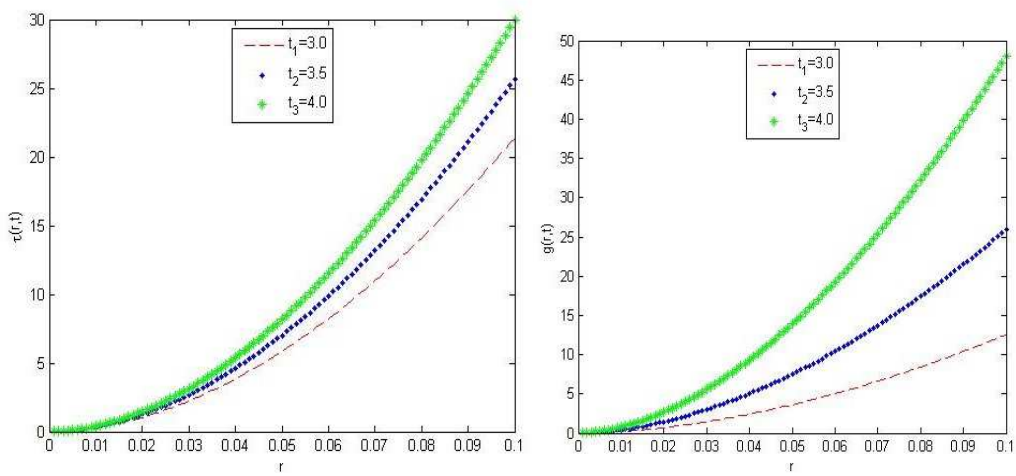
Table 2: Comparison of solutions with existing results

r	$g(r,t)$ **Present paper**	$g(r,t)$ **(I. Siddique) [35]**
0	0	0
0.05	0.003145	0.003233
0.1	0.005236	0.005390
0.15	0.009656	0.009712
0.2	0.017920	0.018065
0.25	0.032280	0.032371
0.3	0.055679	0.055473
0.35	0.091502	0.091149
0.4	0.143096	0.143061
0.45	0.213153	0.213287
0.5	0.303114	0.303499

It is observed from the Table 1 that the results obtained with different numerical algorithms has a good agreement between them. Table 2 show the comparison of obtained results of present scheme with previous results in literature [35].

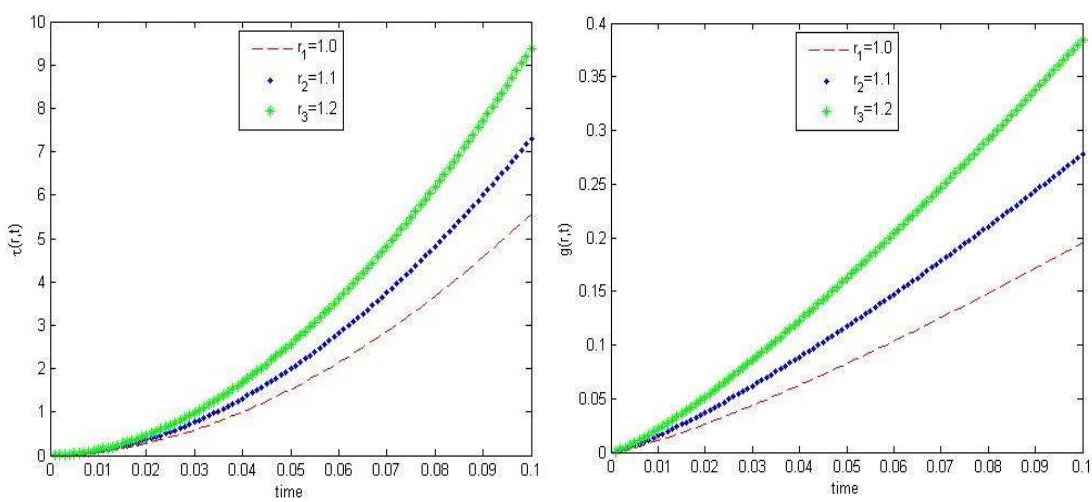
It is worthy pointing out that the integral transform methods previously used for finding the exact solutions of such kind of models have some flaws. For example, the graphical solutions obtained by Hankel transformation in [36] does not satisfy its initial and boundary conditions. Recently, Abdullah *et al* [37] used the numerical Laplace method in an efficient way to solve fractional Maxwell model. Sheng *et al* [38] found that the numerical inverse Laplace algorithms are reliable for fractional differential equations. Tahir *et al* [39], Raza *et al* [40], Tong *et al* [41] and Jiang *et al* [42] used the numerical inverse Laplace algorithm successfully to solve fluid models.

Finally, we have plotted some graphs for velocity function and tangential stress of the fluid by using Eqs. (14) and (17) respectively, to see the effect of various material parameters on our results. Figs. 2(a) and 2(b) depicted for different values of time, the shear stress $\varsigma(r,t)$ and velocity field $g(r,t)$ both are directly proportional to r. Other graphs have been plotted against the values of t. Figs. 3(a) and 3(b) shows that the shear stress and the velocity field are increasing function to t. From Figs. 4(a) and 4(b), we conclude that the shear stress and the velocity field are increasing function to λ. The influence of the kinematic viscosity is shown in Figs. 5(a) and 5(b), which show that the kinematics viscosity is increasing function to both shear stress and velocity field. Figs. 6(a) and 6(b) represent shear stress and velocity field are decreasing function to R. Figs. 7(a) and 7(b) shows that the shear stress and velocity field are increasing function to α. The impact of the dynamics viscosity ν is shown in Fig. 8, which shows that dynamics viscosity is also decreasing function to velocity field. Fig. 9depicts the comparison of velocity field $g(r,t)$ between Maxwell fluid with fractional derivatives, Newtonian and ordinary Maxwell fluid. It is seen from the Fig. 9 that Newtonian fluid is swiftest and Maxwell fluid with fractional derivatives is the slowest. Fig 10 is drawn for the graphical comparison of our obtained numerical results of new hybrid scheme with previous results of literature. It is to be noted that there is a good accordance between both results. For the sake of simplicity, we have used same physical parameters in numerical simulations. These are the parameters we used in this paper $R = 0.5,\ \nu = 0.0357541,\ \mu = 15,\ f = 90,\ \lambda = 5.$



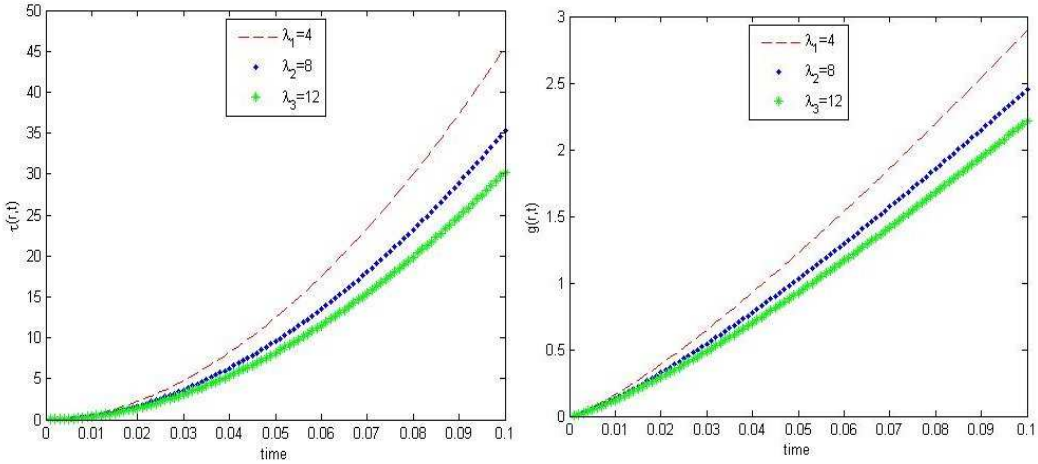
(a) **(b)**

Fig 2: Shear stress and velocity profiles with different values of time versus '*r*'.



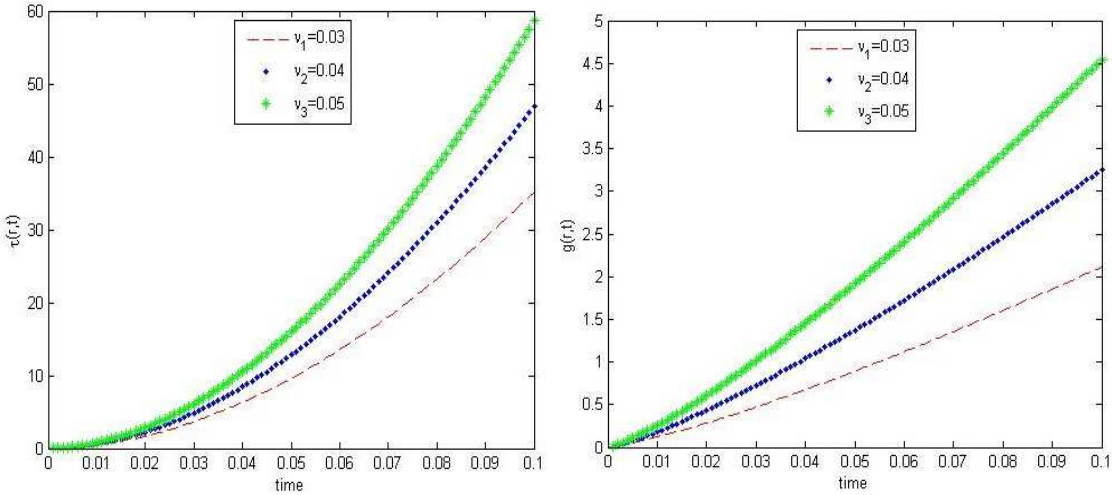
(a) (b)

Fig 3: Shear stress and velocity profiles with different values of '*r*' versus time.



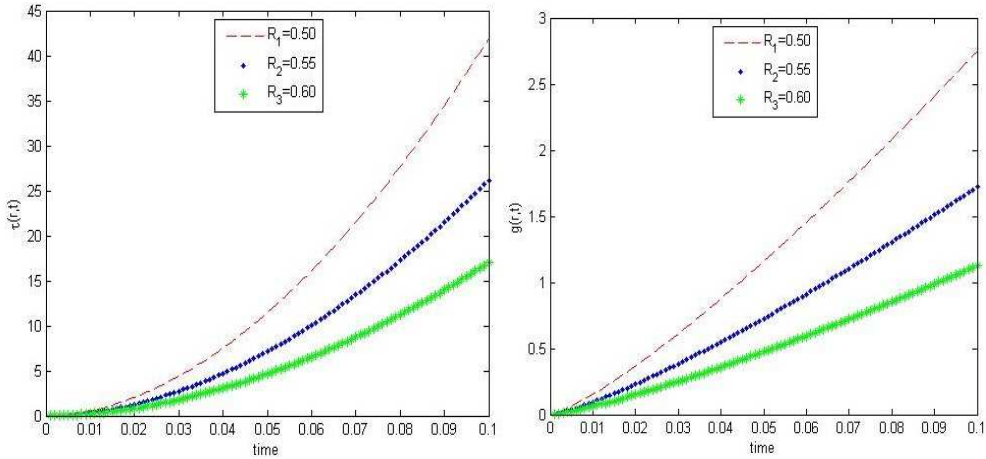
(a) (b)

Fig 4: Shear stress and velocity profiles with different values of λ versus time.



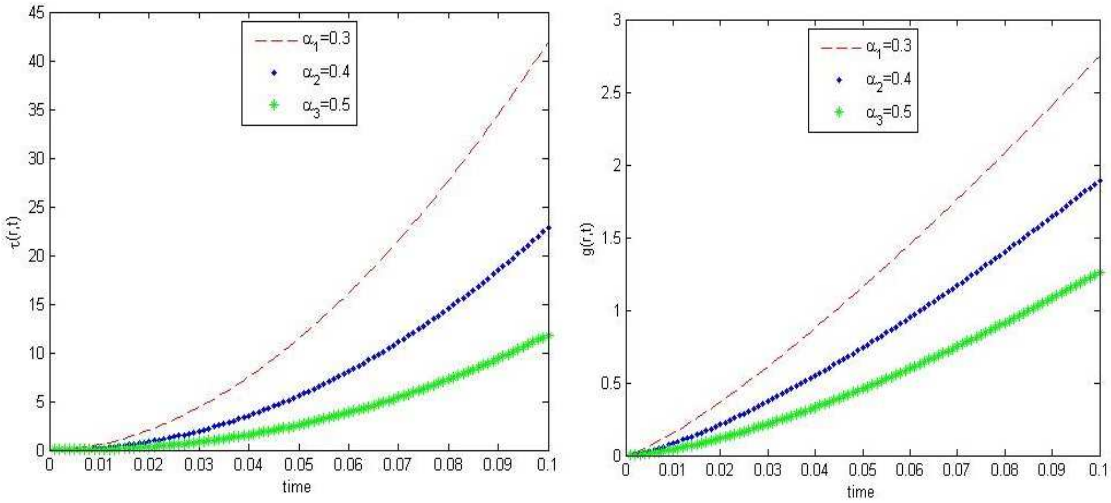
(a) (b)

Fig 5: Shear stress and velocity profiles with different values of ν versus time.



(a) (b)

Fig 6: Shear stress and velocity profiles with different values of R versus time.



(a) (b)

Fig 7: Shear stress and velocity profiles with different values of α versus time.

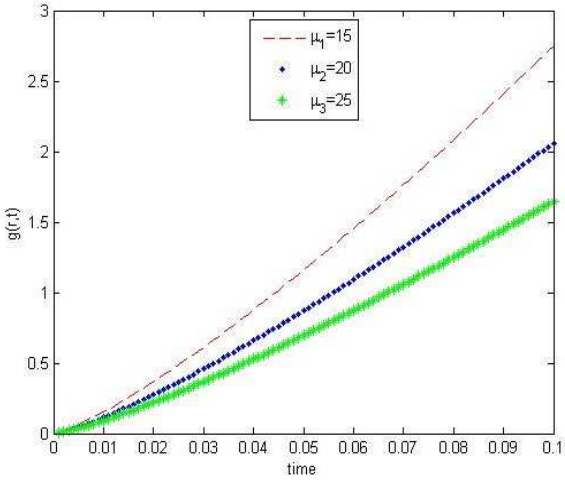


Fig 8: Velocity profile with different values of μ versus time.

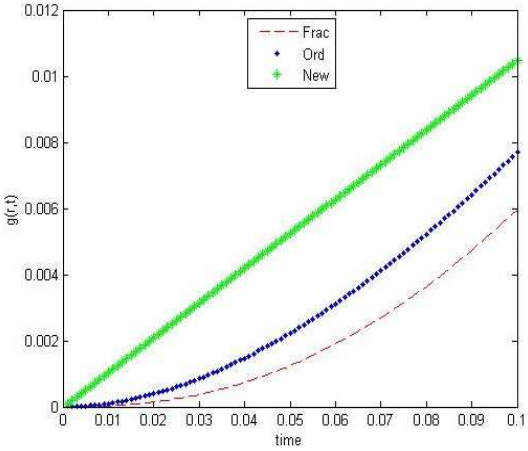


Fig 9: Comparison graph of Maxwell fluid with fractional derivatives, Maxwell fluid with ordinary derivatives and Newtonian fluid.

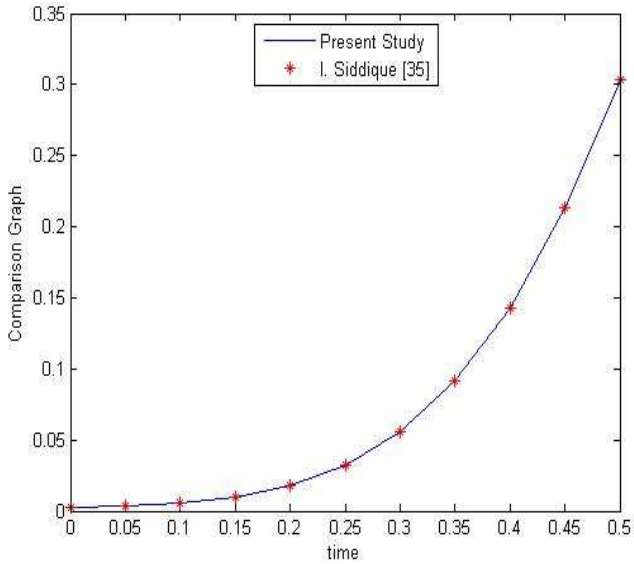


Fig 10: Comparison graph of velocity field of present scheme with solution of Siddique [35].

6. Conclusion

In this paper, the motion of Maxwell fluid governed by the fractional differential equations with Caputo derivative is studied. The flow domain is the inner of a circular pipe and flow is generated by the longitudinal stress-force given on the pipe surface. By applying the Laplace transform with respect to the time variable *t*, the semi analytical solutions for the fluid velocity and longitudinal shear stress are obtained in terms of the modified Bessel functions of first kind $I_0(\cdot)$ and $I_1(\cdot)$. Since the Laplace transforms of the velocity and of the shear stress are enough complicated, we have obtained the inverse Laplace transforms by means of the numerical procedures. Firstly, we used a MATLAB numerical code, then to provide validation of results, we have used other two numerical algorithms, namely the Stehfest's [33] algorithm and Tzou's [34] algorithm.

Here are some main findings of the study. From Table 1, we found that there is a good agreement between the results obtained for inverse Laplace transform with three numerical methods. Table 2 show that the results obtained with present scheme has good accordance with existing results. It is important to observe that the fluid layers situated close pipe surface have a significant motion, while the fluid situated in the central area of the pipe has a very slow motion. The fluids modeled with fractional derivatives flow faster than the ordinary fluid. The shear stress has the behavior similar with velocity therefore, it is increasing when the fractional parameter decreases. The physical parameters t, r and ν are directly proportional to both shear stress and velocity field. The physical parameters R, α, μ and λ are inversely proportional to both shear stress and velocity field.

7. REFERENCES

[1] A. Germant, On fractional differentials, Philosophical Magazine, 25 (1938) 540-549.

[2] S. Chandrasekhar, Hydrodynamic and Hydromagnetic Stability, Oxford University Press, Oxford (1961).

[3] P.G. Drazin, W.H. Reid, Hydromagnetic Stability, Cambridge University Press, Cambridge (1981).

[4] N.D. Waters, M.J. King, The unsteady flow of an elastic-viscous liquid in a straight pipe of circular cross-section, J. Phys. D. Appl. Phys., 4 (1971) 207-211.

[5] T.W. Ting, Certain non-steady flows of second-order fluids, Arch. Ration. Mech. Anal., 14 (1963) 1-23.

[6] P.N. Srivastava, Non-steady helical flow of a visco-elastic liquid, Arch. Mech., 18 (1966) 145-150.

[7] R.L. Bagley, P.J. Torvik, A theoretical basis for the applications of fractional calculus to viscoelasticity, J. Rheol., 27 (1983) 201-210.

[8] C. Friedrich, Relaxation and retardation functions of a Maxwell model with fractional derivatives, Rheol. Acta, 30 (1991) 151-158.

[9] N. Makris, D.F. Dargush and M.C. Constantinou, Dynamical analysis of generalized viscoelastic fluids, J. Eng. Mech., 119 (1993) 1663-1679.

[10] K.R. Rajagopal, Mechanics of non-Newtonian fluids, recent developments in theoretical fluids mechanics, Pirman Research Notes in Mathematics, Longman, New York, 291 (1993) 129-162.

[11] J.E. Dunn, K.R. Rajagopal, Fluids of differential type: Critical review and thermodynamic analysis, Int. J. Eng. Sci., 33 (1995) 689-729.

[12] L.I. Palade, R.R. Attane, P. Huilgol, B. Mena, Anomalous stability behavior of a properly invariant constitutive equation which generalize fractional derivative models, Int. J. Eng. Sci., 37 (1999) 315-329.

[13] W. Tan, M. Xu, Plane surface suddenly set in motion in a viscoelastic fluid with the fractional Maxwell model, Acta Mech. Sin. 18 (2002) 342-349.

[14] K. A. Abro, M. Hussain, M.M. Baig, K.U.R. Chana, Flow of generalized Burger's fluid in Rayleigh Stokes problem, Appl. Math. Comput. J. Appl. Environ. Biol. Sci., 7 (2017) 55-63.

[15] D. Vieru, Corina Fetecau, C. Fetecau, Flow of a viscoelastic fluid with the fractional Maxwell model between two side walls perpendicular to a plate, Appl. Math. Comput. 200 (2008) 459-464.

[16] M. Jamil, A. Rauf, A.A. Zafar, N.A. Khan, New exact analytical solutions for Stokes first problem of Maxwell fluid with fractional derivative approach, Comput. Math. Appl., 62 (2011)1013-1023.

[17] C. Fetecau, Corina Fetecau, M. Jamil, A. Mahmood, Flow of fractional Maxwell fluid between coaxial cylinders, Arch. Appl. Mech., 81 (2011) 1153-1163.

[18] K.A. Abro, M. Hussain, M.M. Baig, Impacts of magnetic field on fractionalized viscoelastic fluid, J. Appl. Environ. Biol. Sci., 6 (2016) 84-93.

[19] M. Jamil, A.U. Awan, D. Vieru, Unsteady helical flows of Maxwell fluids via prescribed shear stresses, Bulletin of the Polytechnic institute of Iasi, Tome LVII (LXI), Fasc., 1 (2011) 137-148.

[20] A. Mahmood, N.A. Khan, I. Siddique, A. Zada, A.U. Awan, A note on the unsteady torsional sinusoidal flow of fractional viscoelastic fluid in an annular cylinder, J. King Saud Uni. Sci., 23 (2011) 341-347

[21] W. Akhtar, I. Siddique, A. Sohail, Exact solutions for the rotational flow of a generalized Maxwell fluid between two circular cylinders, Commun. Nonlinear Sci. Numer. Simulat., 16 (2011) 2788-2795.

[22] C. Fetecau, Corina Fetecau, M. Jamil, A. Mahmood, Flow of fractional Maxwell fluid between coaxial cylinders, Arch Appl. Mech., 81 (2011) 1153-1163.

[23] W. Akhtar, C. Fetecau and A.U. Awan, Exact solutions for the poiseuille flow of a generalized Maxwell fluid induced by time dependent shear stress, ANZIAM J., 51 (2010) 416-429.

[24] A.U. Awan, M. Imran, M. Athar, M. Kamran, Exact analytical solutions for a longitudinal flow of a fractional Maxwell fluid between two coaxial cylinders, Punjab Uni. J.Math, 45 (2013) 9-23.

[25] M. Athar, A.U. Awan, Corina Fetecau, Mehwish Rana, Unsteady flow of a Maxwell fluid with fractional derivatives in a circular cylinder moving with a nonlinear velocity, Quaes. Math., 37 (2014) 139-156.

[26] Z. Khan, H.U. Rasheed, Analytical solution for metallic wire coating using Sisko fluid flow, J. Appl. Environ. Biol. Sci., 7 (2017) 87-101.

[27] K.A. Abro, A.A. Shaikh, Exact analytical solutions for Maxwell fluid over an oscillating plane, Sci. Int., 27 (2015) 923-929.

[28] M.B. Ashraf, T. Hayat, A. Alsaedi, S.A. Shehzadb, Soret and Dufour effects on the mixed convection flow of an Oldroyd-B fluid with convective boundary conditions, Results in Phys., 6 (2016) 917-924.

[29] K. Kaladhar, Ch. RamReddy, D. Srinivasacharya, T. Pradeepa, Analytical study for Soret, Hall and Joule heating effects on natural convection flow saturated porous medium in a vertical channel, Math. Sci., 10 (2016) 139-148.

[30] K. Kaladhar, Ch. RamReddy, D. Srinivasacharya, T. Pradeepa, Radiation effect on Marangoni convection boundary layer flow of a nanofluid, Math. Sci., 21 (2012) 6 pages.

[31] N. Tariq, S. Hussain, N. Yasmin, Magnetohydrodynamic flow of Casson fluids over a moving boundary surface, J. Appl. Environ. Biol. Sci., 7 (2017) 192-200.

[32] A. Talbot, The accurate numerical inversion of Laplace transforms. IMA J. Appl. Math. 23 (1979) 97-120.

[33] H. Stehfest, Algorithm 368: Numerical inversion of Laplace transforms, Commun. ACM, 13 (1970) 47-49.

[34] D.Y. Tzou, Macro to Microscale Heat Transfer: The Lagging Behavior, Taylor and Francis, Washington, (1997).

[35] I. Siddique, Exact solutions for the longitudinal flow of a generalized Maxwell fluid in a circular cylinder, Arch. Mech, 62 (2010) 305-317.

[36] A.U. Awan, C. Fetecau, Q. Rubab, Axial Couette flow of a generalized Oldroyd-B fluid dueto a longitudinal time-dependent shear stress, Quaes. Math, 33 (2010) 429-441.

[37] M. Abdullah, N. Raza, A. R. Butt, E. U. Haque, Semi-analytical technique for the solution of fractional Maxwell fluid, Can. J. Phys., 95 (2017) 472-478.

[38] H. Sheng, Y. Li, Y.Q. Chen, Application of numerical inverse Laplace transform algorithmsin fractional calculus, J. Franklin Inst., 348 (2011) 317-330.

[39] M. Tahir, M.A. Imran, N. Raza, M. Abdullah, M. Aleem, Wall slip and non-integer order derivative effects on the heat transfer flow of Maxwell fluid over an oscillating vertical plate with new definition of fractional Caputo-Fabrizio derivatives, Results in Phy., 7 (2017) 1887-1898.

[40] N. Raza, M. Abdullah, A.R. Butt, A.U. Awan, E.U. Haque, Flow of a second grade fluid with fractional derivatives due to a quadratic time dependent shear stress, Alex. Eng. J.,

http://dx.doi.org/10.1016/j.aej.2017.04.004.

[41] D.K. Tong, X.M. Zhang, X.H. Zhang, Unsteady helical flows of a generalized Oldroyd-B fluid. J. Non-Newtonian Fluid Mech., 156 (2009) 75-83.

[42] Y. Jiang, H. Qi, H. Xu and X. Jiang, Transient electro osmotic slip flow of fractional Oldroyd-B fluids, Micro-fluid Nanofluid, 21 (2017) 1-10.