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The Evaluation of Integer-Valued Polynomial Ring Elasticity

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ABSTRACT

The paper aims to obtain elasticity of Int (D). Here the factorization of R ring elements as the product of irreducible elements (if existing) in especial case of integer-valued polynomial ring is evaluated. Domain *D* is atomic if each non-unit of *D* is a product of irreducible element. It is well known that Noetherian domains, integer domain with ascending chain condition on principal ideals are atomic domains. Some examples of non-unique factorization domain and non-atomic domain are given. By divergence of $\sum \frac{1}{p}$series we show that Int (D) is an atomic domain and if its elements are divided by factorization length, different numbers are obtained, in other words, elasticity of this atomic domain is infinite.

KEY WORDS: Atomic domain; integer-valued polynomial ring; factorization length; elasticity.

1- INTRODUCTION AND STATEMENT OF THE PROBLEM

If the factorization of non-unit elements in D ring is as unique irreducible elements, D is called UFD (unique factorization domain). But it is possible to have two factorization in one atomic domain in which irreducible factors are not similar in its two factorization. We define the elasticity of D as:

P(D)=sup$\left\{\frac{m}{n}\middle| x_1x_{2\ldots}=y_{m\ \ y_2\ldots}y_n \text{where each } x_iy_{j\text{ are inducible elements of D.}}\right\}$

This concept was introduced by Valenza [11] for the ring of integers in an algebraic number field.

Then, it was studied by Steffan on Dedekind domain with finite quotient group and also by Anderson and others [1-3]. The number of irreducible factors of nonzero non-unit element of R is factorization length of that element. If for each non-zero non-unit element a ∈ R the numbers of irreducible factors are equal in two factorization, but R is not UFD, R is called half factorization domain (HFD). Factorization length quotient in UFD and HFD is one (p(R)=1). In atomic domain of R, any non-zero non-unit element is factorized to irreducible factors. Thus, quotient factorization length of the elements is bigger or equal to one. In other words we can say that:

$$1 \leq p(R) \leq \infty$$

Suppose that elasticity of R is rational number of m/n, p(R) =$\frac{m}{n}$, we can say that elasticity of R is realized, if irreducible elements $\beta_{n,\ldots,}\beta_2, \beta_1$and $\alpha_m, \ldots \alpha_2, \alpha_1$ are on R where

$\alpha_1, \alpha_2 \ldots \alpha_m$=$\beta_1, \beta_2 \ldots \beta_{n,,}$

2- Integer-valued polynomial

Definition 2-1 Suppose D is an infinite integral domain with quotient field k

Int (D){f|f ∈ K[X], ∀α ∈ Df(a) ∈ D}

Theorem 2.1 Int (D) is subring of K[X]

Proof: See [3]

Example 2.1 in integer domain of Z with quotient field Q we have:

1) $f(x) = \frac{(x-1)(x-2)}{2}$ belongs to Int (Z). Because for each x∈ D we have:

X=2k-1or x=2k k∈ Z

In both cases f(x) ∈ Z,thus, $f(x) = \frac{(x-1)(x-2)}{2}$belongs to Int (Z). Because according to Theorem for each x∈ Zwe have $x^p \equiv x(p)$ thus $x^p - x$ is product of p, it means that $\frac{(x^p-x)}{p}$ is an integer.

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2) For each n∈ N

$$\frac{(x-1)(x-2)\dots(x-n+1)}{n!}$$

In Int (Z).

Definition 2-2: Int (D) is called integer –valued polynomials on D.

Lemma 2.1 Let I is ideal of Int (D) and a∈ D, thus

I(a)=$\{f(a)|f \in I\}$

is an ideal of D.

Proof: see [4]

Definition 2.3: I(a) ideal is the ideal of I values on D.

Lemma 2.2 If f(x), f(y) are polynomial with small or equal to n, on Q[x] and

F(0)=g(0), g(n) = f(n) ,… g(1) =f(1)

Then,

f(x) =g(x)

Proof: If polynomial f(x) -g(x) is not zero, the maximum degree of equation f(x) -g(x) =0, thus, it has the maximum n root. We suppose that f(x) -g(x) =0 has the minimum n+1 root and this is contradictory. Thus f(x) =g(x) for each x.

Lemma 2.3: If $b_{n,\dots}b_2, b_1$is integer, polynomial

g(x)=$c_0\binom{x}{0} + c_1\binom{x}{1} + \dots + c_n\binom{x}{n}$

Where, for each I, c_I belongs to Z, as

g(0)=b_0, g(1)= b_1,…,g(n)= b_n

Proof: See [1,1]

Theorem 2.2: (Polia) subunit Z is the base for Int (Z)

$$B = \left\{\binom{x}{n} \middle| n = 0,1,2,\dots\dots\dots.\right\}$$

Proof: At first we show that elements of B are independent on Z. Suppose that

f(x)=$a_0 + a_1\binom{x}{1} + \dots + a_n\binom{x}{n}$

is a linear combination of B elements. If f(x)=0, then, coefficient $\binom{x}{n}$or $a_n$should be zero. Because $\binom{x}{n}$is the only one including (x-n+1). So, a_n 1 should be zero. By repeating this reason we can say that coefficients are zero.

Now we show that B elements produce Int (Z). Suppose that f(x) is a polynomial in Int(Z)with the degree of n. If we put

f(0)=b_0, $f(1) = b_1$, $\dots, f(n-1) = b_n$ 1

According to the previous Theorem polynomial

g(x)=$c_0 + c_1\binom{x}{1} + \dots + c_n\binom{x}{n}$

is existing where F(0)=g(0), g(n) = f(n) ,… g(1) =f(1), so for each x we have f(x)=g(x) it means that B elements produce Int(Z).

3- Irreducible polynomials

Theorem 3-1 supposes f is a polynomial in Int (D) and is irreducible in K[x]. So the followings are equal:

1) f is irreducible in Int (D).
2) for each a∈ D,$\frac{f}{a} \in \mathrm{Int}(D)$if and if only a is inverse in D.

Proof: Let f is irreducible in Int(D)and g=$\left(\frac{f}{a}\right) \in \mathrm{Int}(D)$. g is not constant, so it is not inverse, thus, a is inverse. Inversely, if f is irreducible in K[X] and f=gh in Int(D). As g,h∈ K[X],thus, it is necessary that one of them is inverse in K, for example, h=a∈ Dand

g=$\left(\frac{f}{a}\right) \in \mathrm{Int}(D)$

so the second condition is established.

Result 1.3 Suppose f is a polynomial in Int (D) where,

1) f is irreducible in K[X].
2) For each ideal maximal M of D, a∈ D where, f(a)∉M

So f is irreducible in Int (D)
Proof: Suppose a∈ D, $(\frac{f}{a}) \in \mathrm{Int}(D)$, for each maximal ideal M of D is a as f(∈, thus, $\alpha \neq$ M we put
F(a)=ma∈ M, m=$\frac{f(a)}{a} \in D$
(that is contradictory). So a is inverse in D, and according to the previous Theorem f is an irreducible element of Int(D).
Example3.1 For each a∈ D, (X-a) is irreducible in Int(D)
It is clear that (X-a) is irreducible in K[X]. If a=0, for each maximal ideal M, we have a ∉M, Thus, f(a)∉M and if a is inverse, we put a=0 and if a is not inversed, two cases are considered:
1) If a ∈M, a=1
2) If a ∉M, a=0

So for each maximal ideal M of D, a ∈D is existing where $(a - a)$ ∉M, thus, (X-a) is an irreducible factor of Int (D).
Note: About the inverse form of the above results, condition 2 is not necessary. Consider the following example.
Example 3.2: Let D is Dedekind domain with non-prime ideal M (u, v). If $\frac{D}{M}$ is finite, Int (D) is not clear. In other words, Int(D)≠D[x], we suppose that f=uX+v. It is clear that f is irreducible in K[X]. If
$\frac{f}{a} \in \mathrm{Int}(D)$, a∈D

Thus $\frac{f(1)}{a} = \frac{u+v}{a}$, $\frac{f(0)}{a} = \frac{v}{a}$

Both are on D, thus, a counts u,v elements and the prime ideal D_a is including M(u,v), thus, a is inverse. So, f is irreducible in Int(D). But for each a∈D, we have f(a)=au+v, where f(a) ∈M, it means that condition 2 from the above result is not necessary.
Result 3.2: Suppose f is a polynomial in Int(D)where,
1) f is irreducible in K[X].
2) a∈D such that f(a) is inversed in D.

Then, f is irreducible in Int(D)
Proof: According to the previous Theorem, we achieve it.
The next Theorem is obtained from the previous results. This Theorem is including many examples with the assumption of discrete valuation of v on K. It is obvious that this result is due to the element of b of D where $v(b) \neq 0$. (v is not empty on K element).
Theorem 3.2: Suppose that discrete valuation of v is on K. Then, for each a∈D and b∈D where $v(b) \neq 0$ and each integer and positive number of m that prime in comparison to v(b), polynomial $f(x)=(X-a)^m+b$ is irreducible in Int(D).
Proof: See [4].

4- Integer domain without irreducible element

Below is an example of integer domain without irreducible element. So it is not atomic domain.
Example 4.1: Suppose $a_1 = 2^{\frac{1}{2}}$ and $a_n = 2^{\frac{1}{2n}}$, it means that $a_n$is 2^n th of number 2.

$\alpha_1 = \sqrt{2}$ $\alpha_1^2 = 2$
$\alpha_2 = \sqrt{\alpha_1}$ $\alpha_2^2 = \alpha_1$
$\alpha_{n+1} = \sqrt{\alpha_n^2}$ $\alpha_{n+1}^2 = \alpha_n$

Define that T=R$[\alpha_1, \alpha_{2,\ldots\ldots}]$in which R is the ring of real numbers. When X∈T is non-zero and non-unit, then, x is written as in which $\alpha_i \in R$
X=$\alpha_0 + \alpha_1\alpha_1 + \cdots + \alpha_n\alpha_{1n}$
We have $\alpha_{n-1=}\alpha_n^2$, $\alpha_{n-2=}\alpha_{n-1}^2$, so instead of $\alpha_i$where $1\leq i\leq n-1$we substitute power of α_n, thus, x=$\alpha_n(\frac{\alpha_0\alpha_n^{2n-1}}{2}+\alpha_1\alpha_n^{2n-3} + \cdots \alpha_n)$ we show that each T is decomposed, so T hasn't any irreducible.

5- Domain that is not unified factorization and $\sum\frac{1}{p}$ divergence

Example: 5.1 R=Z+2Z[X] is not UFD.
Proof: We put
F(x)=6+16x+10x^2
This element is irreducible in R. Because factorization of this element in Z[X]is as the followings:
F(x)=6+16x+10x^2=2(3+5x)(1+x)
And if is the product of gh (g,h∈ R), then, for example g(x)=2(3+5x) and in this case H is not inside R. (2+2X) is irreducible in R. Now in the following equation
2(6+16x+10x^2)=(2+2x)(6+10x)
We have two factorization for one element (decomposed to irreducible factors), so R cannot be UFD. Now by the following Lemma we show that $\sum p\frac{1}{p}$ in which p is the set of prime numbers and it is divergent. This divergent series are applied in proving infinity of elasticity of Int(D) ring at the end of the article.
Lemma 5.1: $\prod_{p\leq x}(1-\frac{1}{p})$ in which p is the set of all prime numbers and it is divergent.
Proof: Suppose 2≤ x and p ≤ x is a prime number ≤ X.
$P_x=\prod_{p\leq x}(1-\frac{1}{p})^{-1}$
We have
$\log Px=-\sum_{p\leq x}\log(1-\frac{1}{p})$
we put
$s_1=\sum_{p\leq x}\ \frac{1}{p}\quad s_2=\sum_{p\leq x}\ \sum_{h=2}^{\infty}\frac{1}{hp^h}$

$\sum_{h=2}^{\infty}\frac{1}{hp^h} = \sum_{h=1}^{\infty}\frac{1}{hp^h} - \frac{1}{p}$
Thus
$S_1+s_2=\sum_{p\leq x}(\frac{1}{p}+\sum_{h=2}^{\infty}\frac{1}{hp^h}) = \sum_{p\leq x}-\log(1-\frac{1}{p})) = \log Px$
Also for p prime number we have:
$0\leq \sum_{h=2}^{\infty}\frac{1}{hp^h} \leq \sum_{h=2}^{\infty}\frac{1}{p^h}=\frac{1}{p(p-1)}$
$0\leq s_2 \leq \sum_{p\leq x}\frac{1}{p(p-1)} \leq \sum_{h=2}^{\infty}\frac{1}{n(m-1)}$
Thus, $0 \leq s_2 \leq 1$ and s_2 divergent.
If s_1 is convergent, as s_2 is convergent, thus log PX is convergent, that convergence of $\prod_p(1-\frac{1}{p})^{-1}$ is resulted and this is contradicting with the previous lemma. So s_1 is not convergent.

6- Elasticity of integer-valued polynomial ring
Elasticity in an atomic domain is finding the smallest upper bound for the two factorization length of irreducible factors of non-zero, non-unit elements. We show that elasticity of integer-valued polynomial ring is infinite.
Suppose D is an integer domain and K is quotient field. D is atomic if and only if every non-zero and non-unit element is factorized as the product of irreducible elements.
Here we discuss about Int(D) inverse elements are inverse elements of D.
Proof: Let $0\neq f \in Int(D)$be inverse, thus, f degree is 1, if f=a, for each a∈ D the result is obtained.
Lemma6.2: D is irreducible on Int(D) if and if only it is irreducible in D.

Proof: Let a$\in D$, in this case if a is decomposed in Int(D), it is decomposed in Int(D). Because D is subring of Int(D), So if a is irreducible in D, it is the same in Int(D).
Theorem 6.1: If Int(D) is atomic, D is atomic
Proof: Let D is subring of Int(D), if a$\in D$ and a is written as product of irreducible elements of D in Int(D), so D is atomic.
Theorem 6.2: Int(D) ring satisfy ACCP, if and if only D satisfy ACCP.
Proof: See [15]

Thus, if D satisfy ACCP (and at especial case D is Noetherian) then, Int(D) is atomic. By assuming atomic nature of Int(D), we discuss about elasticity property of Int(D). We obtain upper bound for the following value.

P(Int(D))=sup$\left\{\frac{m}{n}\middle| f_1 f_2 \ldots\ldots = f_m = g_1 g_2 \ldots\ldots gn \text{ where each } f_i g_j \text{ are inducible elements of D.}\right\}$

Note: If a$\in D$ and a is decomposed in Int(D), then the factors of decompose are in D thus, each irreducible element of D is irreducible in Int(D)
Theorem 6.3: Elasticity of Int(D)is larger or equal to elasticity of D.
P(D)$\leq p(Int(D))$
Proof: As irreducible elements of D are irreducible in Int(D), the result is obtained.
Theorem 6.4: Suppose D is BFD, then for each $f \in Int(D) and f \in Int(D)$we have:
$l_{Int(D)}(f(x)) \leq lK[X]\big(f(X)\big) + l_{D(f(a))} \leq deg\big(f(x)\big) + l_D(f(a))$
In this case we have $l_D(0)=0$ and if α is inverse, $l_D(\alpha)=\infty$
Proof: See [15]
Theorem 6.5: Elasticity of Int(D) is infinite
Proof: We have the following polynomial formula.
$\binom{X}{n}=\frac{1}{n!}\prod_{i=0}^{n-1}(x-1)$
As Int (D)has one base with integer coefficients.
$n!\binom{X}{n}=\frac{1}{n!}\prod_{i=0}^{n-1}(x-1)$

They are integer-valued polynomial where (X-1) is irreducible. If n is irreducible factor in the right, while in the left n! is irreducible factors of Z that are irreducible in Int(Z). To finish the proof we claim that irreducible factors in the left for each m are bigger than mn. Suppose n=p! Where p is prime number. Consider 1, 2,…, n elements of n! factors. $\frac{n}{2}$ of them count 2 and $\frac{n}{3}$ of them count 3and $\frac{n}{5}$ of them count 5 and thus $\frac{n}{p}$ of them count p. Now of the existing n! in the left, there are n$(\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{p})$ irreducible factors and $\sum_{i=1}^{\infty}\frac{1}{pi}$in which pi are prime numbers and divergent. So it is true. Because in the left there are n $(\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{p})$ factors and in the right there are n irreducible factors and the division of two inverse series are prime numbers and this amount is infinite. So elasticity of Int (Z) is infinite.

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