J. Basic. Appl. Sci. Res., 1(11)1963-1969, 2011

ISSN 2090-4304
Journal of Basic and Applied
Scientific Research
www.textroad.com

Internal Optimal Control of a Heterogeneous One-Dimensional Linear Wave Equation System by Embedding Method

Ali Behzadi[1,*]

[1] Department of Mathematica, Behbahan Branch, Islamic Azad University, Behbahan, Iran

ABSTRACT

The paper purpose is heterogeneous one-dimensional linear wave equation physical system. By using an embedding method, the problem of finding the time optimal control is reduced to one consisting of minimizing a linear form over a set of positive Radon measures. Then we show that optimal size can be approximated by a finite combination of atomic sizes. Then, by approximation of this size, it is converted to a problem of finite linear programming and by its answer; a piece-wise control function is built approximately optimal. Numeral examples are also given in the paper.
KEY WORDS: Wave equation; approximation; radon measure; embedding method; optimal control; linear programming.

1- Introduction

Russel, Malanovski and Butkovskiy investigated control systems directed by wave equation. Farahi in 1996 [2]. Evaluated the boundary optimal control of linear wave equation, Observation, control, and stabilization of the control system is studied based on wave equation by LucMiller by escape function in 2003 [7]. In this paper we obtain internal optimal control of heterogeneous linear wave equation and by Fourier series and embedded method. Suppose a control system directed by linear wave equation and heterogeneous:

$$Ytt(x,t) - \alpha^2 Yxx(x,t) + u(x,t) \qquad (1)$$

In this equation (x,t) ➜ Y(x,t) function shows the system changes at condition x and t in (0,S)×(0,T). As S,T > 0 are fixed real numbers and α is the wave propagation velocity. The initial conditions are as the followings:

$$Y(x,0)=f(x) \qquad Yt(x,0)=h(x) \qquad (2)$$

We consider boundary conditions for $t \in [0,T]$as the followings:

$$Y(0,t)=0 \qquad Yt(S,t)=0 \qquad (3)$$

Measurable unknown function (x,t) ➜u(x,t) $\in R\ where\ (x,t) \in [0,S] \times [0,T]$ in(1) is considered as system control function.

Definition The control function u is called admissible if it is a Lebesgue function on $[0,S] \times [0,T]$and besides establishing equations (1),(2) and (3) satisfy in the following conditions:

a. For each $(x,t) \in [0,S] \times [0,T]$we have: $|u(x,t)| \leq 1$
b. The answer of equation(1) corresponded with initial conditions (2) and boundary conditions (3) satisfy in the following final conditions :

$Y(x,T)=g_1(X) \qquad Y_t(x,T)=g_2(x) \qquad (4)$

Where $g_1(x).g_2 \in L_2[0,S]$

We denote all admissible controls by u.

Note: condition (a) without reducing the generality of the problem show boundary of u function.

We aim to obtain admissible optimal control u$\in U$as for the given function $f_0 \in C(\Omega)$, where $\Omega = [0,S] \times [0,T] \times [-1,1]$, the following function is minimized:

$I(u)=\int_0^T \int_0^S f_0(t,x,u(t,x))dxdt \qquad (5)$

It is possible that the above problem doesn't have in u or in case an answer is existing, determining it is difficult[8]. To remove this problem and the similar problems, as we will see, problem is transformed into a problem of minimizing a linear function on set of Radon measures on Ω.

2- Converting the problem to a problem with integral constraint

***Corresponding Author:** Ali Behzadi, Department of Mathematica, Behbahan Branch, Islamic Azad University, Behbahan, Iran.
Email: ali.behzad@ymail.com

As it is shown in [?], the answer of wave equation (1) considering the initial and boundary conditions (2) and(3) are as the followings:

Y(x,t)=$\sum_{n=1}^{\infty}(\int_0^t C_n Q_n(\tau)\sin\big(K_n(t-\tau)\big)\,d\tau + A_n\cos(K_n t) + B_n\sin(K_n t))sin\frac{n\pi x}{S}$ (6)

For n=1,2,3,… we have:

$A_n=\frac{2}{S}\int_0^S f(x)\,sin\frac{n\pi x}{S}dx,\qquad K_n=\frac{nax}{S},\ C_n=\frac{1}{K_n}$

$B_n=\frac{2}{K_nS}\int_0^S \square(x)\,sin\frac{n\pi x}{S}dx,\qquad Q_n(t)=\frac{2}{S}\int_0^S u(x,t)\,sin\frac{n\pi x}{S}dx$

By analysis theorems, we have:

Y_t(x,t)=$\sum_{n=1}^{\infty}K_n(\int_0^t C_n Q_n(\tau)\cos\big(K_n(t-\tau)\big)\,d\tau - A_n\sin(K_n t) + B_n\cos(K_n t))sin\frac{n\pi x}{S}$ (7)

Because$g_1(x).\,g_2(x)\in L_2[0,S]$, so it is having half range Fourier sine series for each x$\in[0,S]$ as the followings [?]:

Y (x,T)=$g_{1(x)}\sum_{n=1}^{\infty}\alpha_n\,sin\frac{n\pi x}{S}$, Y_t(x,T)=$g_{2(x)}\sum_{n=1}^{\infty}b_n\,sin\frac{n\pi x}{S}$ (8)

For n=1,2,3,… we have

$\alpha_n=\frac{2}{S}\int_0^S g_1(x)\,sin\frac{n\pi x}{S}dx,\qquad b_n=\frac{2}{S}\int_0^S g_2(x)\,sin\frac{n\pi x}{S}dx$

According to equations (6), (7) and (8),the equality of coefficients for n=1,2,3,…is resulted as the followings:

$$\int_0^t C_n\sin\big(K_n(T-t)\big)\,Q_n(t)dt + A_n\cos\,(K_nT) + B_n\sin\,(K_nT)) = \alpha_n$$

$$\int_0^t C_n\cos\big(K_n(T-t)\big)\,Q_n(t)dt - A_n\sin\,(K_nT) + B_n\cos\,(K_nT)) = c_nb_n$$

By multiplying the two sides of equations respectively by Y=sin$K_nT + cosK_nT$,$X = sinK_nT - cosK_nT$and then, simplifying the expressions of the following equations we obtain:

$\int_0^t C_n\,(\sin K_nT + cosK_nT)Q_n(t)dt = A_n - B_n + \alpha_nX)) + \frac{b_n}{K_n}Y$ n=1,2,3,…

By defining $\gamma_n = A_{n-}B_n + \alpha_nX + \frac{b_n}{K_n}Y$ and replacing $Q_n(t)$ in the above expression for n=1,2,3,…, we have:

$$\int_0^t C_n\,(\sin K_nt + cosK_nt)(\int_0^S \frac{2}{S}u(x,t)\,sin\frac{n\pi x}{S}dx)\ dt = \gamma_n$$

Thus,

$\int_0^T\int_0^S \frac{2C_n}{S}(\sin K_nt + cosK_nt)sin\frac{n\pi x}{S}u(x,t)dxdt = \gamma_n$

As $\frac{2C_n}{S}=\frac{2}{n\pi x}$ we define:

$$\varphi_n\,(\mathbf{t.x.u(x,t)}) = \frac{2}{n\pi x}(\sin K_nt + cosK_nt)sin\frac{n\pi x}{S}u(x,t)$$

So establishing differential equation with partial derivatives (1) is correspond with the initial, boundary and final conditions (2), (3) and (4) by establishing the following equations:

$\int_0^T\int_0^S \varphi_n\,(\mathbf{t.x.u(x,t)})\mathbf{dxdt} = \gamma_n$ n=1,2,3,... (9)

3- Transforming the problem to measure space

It is possible that the above problem doesn't have any answer, although incase of having any answer, achieving it, is not possible. We try to find the optimal answer by expanding the space by transforming the problem to an equal problem in a bigger space and by being sure of the existence of the answer and its achievability. At first, any admissible control u(x,t) have the following mapping:

Λ_u(F):F $\int_0^T\int_0^S F(\mathbf{t.x.u(x,t)})\mathbf{dxdt}$, F∈C(Ω)

That is a positive linear function on C(Ω). By the Riesz's representation theorem, there is a unique positive regular Borel measure μ on Ω such that for each F∈C(Ω),

$\int_0^T\int_0^S F(\mathbf{t.x.u(x,t)})\mathbf{dxdt} = \int_\Omega \varphi_n\,(\mathbf{t.x.u(x,t)})\mathbf{d}\mu = \mu$ (F) (10)

So functional minimizing of (5) on u by minimizing

I(μ) $= \int_\Omega f_0\mathbf{d}\mu - \mu(f_0)\mathbf{d}\mu \in R$ (11)

On a set of μ measures (correspondent with admissible control of u) as these equations satisfy the following equations:

$\mu(\varphi n) = \gamma n$, n=1,2,3,… (12)

As transformation of u→ Λ_{u-} μ is one to one [?]. So, the image set of all admissible controls are equal to the set of all positive Radon measures satisfying in equation (12). So, there are yet all the previous problems of classic problem because they are transferred to a new space by one to one transform and there will be no change on its nature. Now to remove this problem, the image sets under the above transformation are expanded and we consider non-classic problem:

Among all positive Radon measures Q$\subset M^+(\Omega)$, we are searching for the measure of μ^+ such that equation (12) is established and minimizing of function $\mu \to \mu$ (f_0). It is worth to say that in this case minimizing is done on subsets of positive μmeasures that satisfy equations (12), and they are not obtained from Riesz representation theorem and these measures μ are required to have certain properties which are abstracted from the definition of admissible controls:

1) These measures should satisfy in equations (12), it means that :

$\mu(\varphi_n) = \gamma n$, $\forall n$: n=1,2,3,…

2) Suppose G∈ $C_1(\Omega)$ is arbitrary, $C_1(\Omega)$is the space of all functions that are not dependent on u) in this case the existing measures of Q should satisfy in the following equations:

$$\mu(G) = \int_\Omega G\mathbf{d}\mu = \int_0^T \int_0^S G(\mathbf{t.x.u(x,t)})\mathbf{dxdt} = \boldsymbol{\alpha_G}$$

Where u is an arbitrary number in [-1,1] and $\boldsymbol{\alpha_G}$is Lebesgue Integral of G(..,..u) on $[0,S] \times [0,T]$ is independent from u.

This property of Q is used in Ghoulia-Houri [3] and proving compactness of Q.

As it was said Q$\subset M^+(\Omega)$set is defined as the followings:

Q=$S_1 \cap S_2$

Where

$$S_1 = \{\mu \in M^+(\Omega): \quad \mu(\varphi_n) = \gamma_n, \quad n = 1,2,3,\ldots\}$$
$$S_2 = \{\mu \in M^+(\Omega): \quad \mu(G) = a_G, \quad G \in C_1(\Omega)\}$$

We topologies the space $M^+(\Omega)$by the weak $*$ $-$topology of the following proposition:

Proposition 1: if We topologies the space $M^+(\Omega)$by the weak $*$ $-$topology , then,

a. Q is compact.
b. $\mu \to \mu$ (f_0) function mapping Q set to R is continuous.

Proof: see [8]

The above proposition shows that $\mu \to \mu$ (f_0) is a linear and continuous function on compact set of Q. So, it choose its minimizing in one point of Q[1]. Thus, the following minimizing problem that is generalized from of minimizing problem (5) considering equation(9) in positive Radon space measure have the following minimizing such as μ^*on Q.

Minimize μ (f_0)

Subject to $\mu(\varphi_n) = \gamma_n$, n=1,2,3,… (13)

$$\mu(G) = a_G, \quad G \in C_1(\Omega)$$

Raising the issue of non-classic optimal control in measures space are having many benefits as due to linear nature of the measures that are the unknowns of the problem, we are facing with a linear programming problem (of infinite type, so we can use easily from known abilities of linearity properties. Also the answer of (3) is a general answer; because Q is a generalized set of u. So we have:

$$\inf_{\mu \in Q} I(\mu) \le \inf_{\mu \in u} I(u)$$

The above theorem assures optimal measure of μ^* but it should be said that obtaining this minimizing measures is not very easy. Because besides the fact that the number of constraints are infinite, the space of problem solution Q is also infinite, S, both these factors (constraints and solution space)are being approximated. As by it we can identify an answer approximate to the optimal answer. In the next section approximation process is expressed and we show how at first μ^* optimal measure is obtained by a suitable approximation. Then, by another admissible and suitable approximation find an approximate answer to the optimal answer by solving a finite linear programming.

4- Approximation of the optimal control by a piecewise- constant control

As it was said in the previous section, any admissible control and especially admissible piecewise- constant control u can be corresponding to one measure of $\mu_u$on $M^+(\Omega) \cap S_1 \cap S_2$. Let $Q1$ be the space of all measures as μ_u, then theorem of (1) of [3] indicates that if $M^+(\Omega)$is topologized by the weak*-topology then, $Q1$ is dense in $M^+(\Omega) \cap S_1 \cap S_2$. A basis of closed neighborhoods in the weak*-topology is given by sets of the form:

U ϵ {μ : |μ(Fn)| ≤ □, n = 1, 2, ..., k + 1},

where $k \in Z^+$, $\epsilon > 0$ and $Fn \in C(\Omega)$. It is therefore it is possible to find a measure μu, corresponding to a piecewise control u, in any weak*- neighborhood of μ∗. In particular, if we choose

$F1 = f_0, \quad F2 = \varphi_1 \quad ,F3 = \varphi_2, \ldots, Fk+1 = \varphi_K$

a piecewise constant control $uk(\ldots)$ can be found such that for each $\epsilon > 0$

$$\left|\int_0^T \int_0^S f_0\big(\mathbf{t.x.u_k(x,t)}\big)\mathbf{dxdt} - \mu^*(f_0)\right| \leq \epsilon$$

$\left|\int_0^T \int_0^S \varphi_n\big(\mathbf{t.x.u_k(x,t)}\big)\mathbf{dxdt} - \gamma_n\right| \leq \epsilon \quad$ n=1,2,3,…k

Therefore, by using the piecewise constant control $uk(\ldots)$, we can get within ϵ of the minimum value μ^* (f_0). Let x □[0,S], Y_k(x,T)and $\frac{\partial}{\partial t}$ Y_k(x,T)are respectively the final sate and its derivative attained by replacing *uk(...)* in equation (6). We can show that if ϵ is chosen small enough, and k large enough, then *the distance between* $g_1(0)$and $g_2(0)$ respectively with L_2[0,S], Y_k(x,T) and $\frac{\partial}{\partial t}$ Y_k(x,T) can be made as small as desired (final condition is estimated by using u_k with the desired approximation)

Proposition 2 Given δ > 0, we may choose $\epsilon > 0$ and k such that:

a. $\int_0^S (Y_k(x,T) - g1(x))^2\, dx \leq \delta,$

b. $\int_0^S (\frac{\partial}{\partial t} Y_k(x,T) - g2(x))^2\, dx \leq \delta$

Proof. See [19].

5- Approximation to the optimal measure

In this section, we obtain an approximation to the optimal measure $\mu*$ by the answer to the problem of finite linear programming. Consider the minimizing problem (13), this is an infinite-dimensional aspect; because in addition that solution space is having infinite dimension, the number of problem equations are infinite. To solve it we can use a linear programming problem with finite dimension and obtain the solution as approximated. At first we restrict the number of problem (10) constraints. To do this, we choose M_1 constraints from S1 set. In S_2 set as [8] we choose G functions as the followings:

$G_{ij}(t,x,u) = t^i x^j \qquad$ i,j=0,1,2,…

These functions in functional space dependent on x,t variables are dense on set of $[0,S] \times [0,T]$ and according to Weierstrass approximation theorem, each continuous function in this space can be approximated by finite combination of these functions. Thus, second type constraints are approximated as the followings:

$\mu G_{ij}) = aG_{ij} \qquad$ i,j=0,1,2,…

It is worth to mention that aG_{ij} is Lebesgue Integral $t^i x^j$ on$[0,S] \times [0,T]$. So the solution of linear programming problem (13) is approximated by the following solution:

Minimize μ (f_0)

Subject to $\mu(\varphi_n) = \gamma_n$, n=1,2,3,… ,M_1 (14)

$$\mu\big(G_{ij}\big) = a_{G_i} \quad , \quad i = 0,1,\ldots m_1 - 1$$

$$j = 0,1,\ldots m_2 - 1$$

$$m_1 m_2 = M_2$$

As we see at the end of this stage of approximation, the problem dimension is still infinite. As the second stage, solution of problem(14) is approximated with the solution of problem with finite dimension (linear programming). The rational number μ∗ in the above set at which the function μ → μ(f_0) attains its minimum has the following form [8]:

$\mu^* = \sum_{l=1}^{M_1+m_2} a_l^*\, \delta(z_l^*)$

With $z_l^* \in \Omega$ and $a_l^* \geq 0$, (l=1,2,⋯ ,M$_1$ + M$_2$). Here δ is atomic measure. Using this form of representation in expressing problem (14) based on atomic measure property cause that unknowns of the problem are changed from positive Radon measures to another two types of unknowns. They are including coefficients $\{a_l^*\}_{l=1}^{M_1+M_2}$ and $\{Z_L^*\}_{l=1}^{M_1+M_2}$ supports. Robio in [8] expresses that optimal measure can be approximated by the following measure:
$v^* = \sum_{l=1}^{M_1+M_2} a_l^* \, \delta(z^l)$

Where, $a_l^* \geq 0$ are unknown coefficients and z^l are values of count and dense such as W of Ω. We see that Z^j are triples with the form of (t^l, x^l, u^l) in which Z_l^* is approximated by them. So, linear programming problem (14) is changed into form:

Minimize $\sum_{l=1}^{N} a_l f_0(z_l)$
Subject to $\sum_{l=1}^{N} a_l\, n(z_l)=\gamma_n$, n=1,2,3,… ,M$_1$ (15)
$\sum_{l=1}^{N} a_l\, G_{ij}(z_l)= a_{G_{ij}}$, $i = 0,1, \ldots m_1$
$j = 0,1, \ldots m_2$
$m_1 m_2 = M_2$
$a_j \geq 0$ $j = 1,2, \ldots N$

It is observed that most of linear programming problems as the above form don't have possible answers and this is due to choosing G_{ij} functions as t^i, x^j. ([8] pages 53-54). This problem can be removed such as [8]and [2] by especial selections. Suppose intervals [0,S], [0,T] are divided equally into m$_1$ and m$_2$ such that
$M_2 = m_1 m_2$. We choose:
$I_{ij} = \left[\frac{T(i-1)}{m_1}, \frac{Ti}{m_1}\right) \times \left[\frac{S(j-1)}{m_2}, \frac{Sj}{m_2}\right) \quad i=1,2,\ldots m_1, j=1,2,\ldots m_2$
Assumpting z=(t,x,u)∈ W we define:
$$f_{ij}(z)=\begin{cases} 1, & z \in I_{ij} \\ 0, & z \notin I_{ij} \end{cases}$$
So the right side of second type constraints for this kind of selection from functions for each
$i = 1,2, \ldots m_1, j = 1,2, \ldots m_2$ are as the followings:
$$a_{ij} = a_{fij} = \int_0^T \int_0^S f_{ij}(\boldsymbol{z})\,\boldsymbol{dxdt} = \int_{\frac{T(i-1)}{m_1}}^{\frac{Ti}{m_1}} \int_{\frac{S(j-1)}{m_2}}^{\frac{Sj}{m_2}} \boldsymbol{dxdt} = \frac{TS}{m_1 m_2}$$
Consider that f_{ij} functions are not continuous but we consider the followings:

1) Each of f_{ij} functions , $(j = 1,2, \ldots | m_2,\ \ i = 1,2, \ldots m_1)$are limit following the positive and continuous functions of $\{f_{ijk}\}$, so, if μ is a positive Radon measure on $M^+(\Omega)$we have:
([9] $\mu(f_{ij})$=$\lim_{k\to\infty} \mu\, f_{ijk}$)
2) a set of these functions are considered for all positive and integer numbers m_1, m_2, the desired linear combinations of these functions can approximate any function as G_{ij} ([9]).

By choosing these functions of the second type of the constraints, linear programming problem (15) is as the following form:
Minimize $\sum_{l=1}^{N} a_l f_0(z_l)$
Subject to $\sum_{l=1}^{N} a_l\, \varphi n(z_l)=\gamma_n$, n=1,2,3,… ,M$_1$ (16)
$\sum_{l=1}^{N} a_l\, G_{ij}(z_l)= a_{ij}$, $i = 0,1, \ldots m_1$
$j = 0,1, \ldots m_2$
$m_1 m_2 = M_2$
$a_j \geq 0$ $j = 1,2, \ldots N$

Now by using answers of $\{a_1^*, a_2^*, \ldots, a_1^*\}$ obtained from solving problem (16), we can approximate optimal control by the given measure in(15) by piecewise constant control like the given algorithm in [8]. In the next section we investigate the efficiency of the above method by giving an example.

Example: Consider a thing wire with the length of π. This wire is constant in boundary points. Suppose at t=0 its condition is sinus and its velocity is zero. It is aimed to obtain the force being given from outside on wired as at timeπ, make the condition and its velocity zero and the least energy is also consumed.

Suppose at point x and time t forces u(x,t) are enforced on the wire, as we know, wires vibrations are as a wave. By adding the forces on the wire, the following equation is obtained [?]:
Y_{tt}(x,t)- Y_{xx}(x,t)=u(x,t)

The above problem is considered as an optimal control problem. Considering the assumptions of this problem, g_1(x)= g_2(x)=0 and $Y(0$,t)= $Y(\pi$,t)=0, h(x)=0, f(x)=sin(x) and the target function is $\int_0^{\pi}\int_0^{\pi} u^2(\boldsymbol{x},\boldsymbol{t})\boldsymbol{dxdt}$ $(f_0 = u^2)$. To solve at first intervals[–1,1],[0,S],[0,T] are respectively divided equally into M_1=9, M_2=9, M_3=11then, from each part of the interval we choose a point. By choosing $M_{2=}m_1m_2$=81, $M_{1=}$2, linear programming problem with 83 constraints and 891 variables are obtained as:

Minimize $\sum_{l=1}^{891} u_l^2\, a_l$

Subject to $\sum_{l=1}^{891}(\frac{2}{\pi}) sinxl(sint_l + cost_l)\, a_l \quad a_l = 1$

$\sum_{l=1}^{891}(\frac{1}{\pi}) sin2x_l(sin2t_l + cos2t_l)\, u_l \quad a_l = 0$

$$a_1 + a_2 + \cdots + a_{11} = \frac{\pi^2}{18}$$

$$a_{12} + a_{13} + \cdots + a_{22} = \frac{\pi^2}{18}$$

…

$$a_{881} + a_{882} + \cdots + a_{891} = \frac{\pi^2}{18}$$

$a_l \geq 0 \qquad\qquad l = 1,2, \ldots 891$

We solve the above linear programming by modified simplex method existing in Sub program of DLPRS of IMSL library software Fortran 90. The optimized target function 2/0902488× 10^{-13} is obtained. Now by obtained $a_l^*$by the above problem and the given divisions for intervals [0,T]and [0,S], control function is calculated and plotted (Fig.1), then we approximate wave condition at time π, Y(X,π) by piecewise constant control and 15 prime numbers of the answers of wave equation (6)(Fig.2).

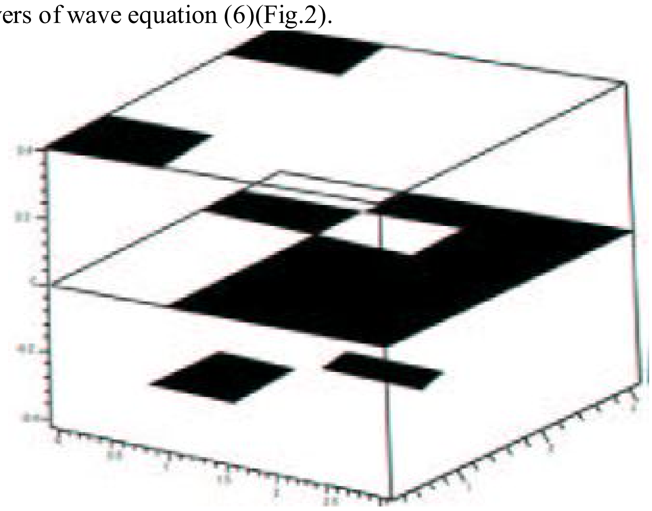

Figure (1): Optimal control

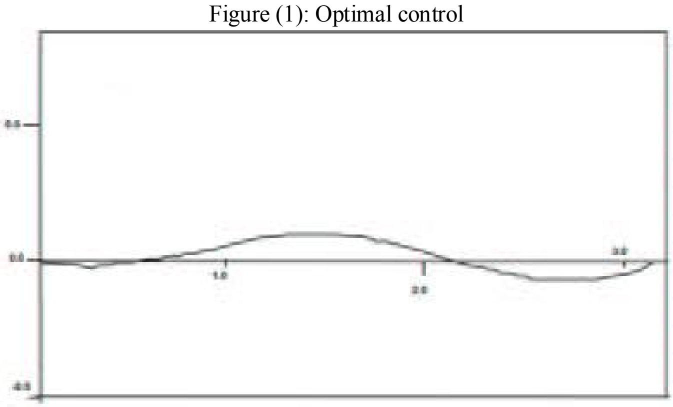

Figure (2): wave condition at the end of time interval

Acknowledgment

This research was done by financial support of Research & Technology Office, Behbahan Branch, Islamic Azad University.

REFERENCES

1. Bartel, R. G. & Sherbet, D. R., Introduction to Real Analysis, John Wiley and Sons, Second Edition, 1992.
2. Farahi, M. H., Rubio, J. E., Wilson, D., The Optimal control of the linear wave equation, INT. journal of Control, Vol. 63, 1996.
3. Ghouila-Houri, A., Surla generalization de la notation de commande d'un systeme guidable, Revue Francaise d'Automatique, Informatique, et Recherche operationalle, (4) 1967.
4. Gugat, M., Optimal Boundary control of networked hyperbolic systems, Evaluation of derivatives, advanced modeling and optimization 7 (2005) 9-37.
5. Gugat, M. L. Optimal boundary control of a string to rest in finite time, In: F. Jarre, C. Lamarechal, J. Zowe (Eds), Optimization and application, oberwolfach reports, 2006.
6. J. B. Rewlings and D. Q. Nayne, Model predictive control: theory and design, Nob Hill pulishing 2009.
7. Miller, L. Escape function conditions for the observation, control and stabilization of the wave equation, SIAM Journal, 2003.
8. Rubio, J. E., Control and Optimization: the linear treatment of nonlinear problems, Manchester University Press, Manchester, 1986.
9. Rudin, W., Real and Complex Analysis, McGraw-Hill series in higher mathematics (3rd Edition), 1987.