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A COMMON FIXED POINT THEOREM ON ORDERED PARTIAL METRIC SPACES

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ABSTRACT

A common fixed point result for weakly increasing mappings satisfying generalized contractive type of Zhang in ordered partial metric spaces are derived. Also as an application we prove the existence and uniqueness of common solutions for a couple of integral equations.
KEY WORDS: Fixed point, partial metric.

1. INTRODUCTION

In the last years, the extension of the theory of fixed point to generalized structures as cone metric, partial metric and quasi-metric spaces has received a lot of attention. One of the most interesting is partial metric space. Partial metric space is a generalized metric space in which each object does not necessarily have to have a zero distance from itself [9]. A motivation behind introducing the concept of a partial metric was to obtain appropriate mathematical models in the theory of computation and, in particular, give a modified version of the Banach contraction principle, more suitable in this context [9]. Subsequently, Valero [13], Oltra and Valero [11] and Altun et al [2] gave some generalizations of the result of Matthews. Romaguera [12] proved the Caristi type fixed point theorem on this space. The purpose of this paper is to present a general fixed point theorem for two pairs of mappings on two partial metric spaces satisfying implicit relations. Our result generalizes the main result from [7] and [10]. Also as an application of this theorem we prove the existence and uniqueness of common solutions for a couple of integral equations.

First, we recall some definitions and results which is needed in the sequel. The reader interested in fixed point theory in partial metric spaces is referred to [1, 8, 9, 11, 12, 13] and references therein.

A partial metric on a nonempty set X is a mapping $p: X \times X \to \mathbb{R}^+$ such that for all $x, y, z \in X$:
(P1) $x = y$ if and only if $p(x,x) = p(x,y) = p(y,y)$,
(P2) $p(x,x) \leq p(x,y)$,
(P3) $p(x,y) = p(y,x)$,
(P4) $p(x,y) \leq p(x,z) + p(z,y) - p(z,z)$.

A partial metric space is a pair (X,p) such that X is a nonempty set and p is a partial metric on X. It is clear that, if $p(x,y) = 0$, then from (P1) and (P2) $x = y$. But if $x = y$, $p(x,y)$ may not be 0. A basic example of a partial metric space is the pair $(\mathbb{R}^+, p)$, where
$p(x,y) = \max\{x,y\}$ for all $x, y \in \mathbb{R}^+$.
Let (X,d) and (X,p) denote a metric space and partial metric space, respectively.

Example 1. *Mappings* $\rho_i: X \times X \to \mathbb{R}^+$ $(i \in \{1,2,3\})$ *defined by*

$$\rho_1(x,y) = d(x,y) + p(x,y)$$
$$\rho_2(x,y) = d(x,y) + max\{\omega(x), \omega(y)\}$$
$$\rho_3(x,y) = d(x,y) + a$$

define partial metrics on X, where $\omega: X \to \mathbb{R}^+$ *is an arbitrary function and* $a \geq 0$.

Other examples of the partial metric spaces which are interesting from a computational point of view may be found in [5] and [9].

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Each partial metric p on X generates a T_0 topology τ_p on X which has as a base the family of open p-balls

$$\{B_p(x,\varepsilon): x \in X, \varepsilon > 0\},$$

where $B_p(x,\varepsilon) = \{y \in X : p(x,y) < p(x,x) + \epsilon\}\ for\ all\ \ x \in X$ and $\varepsilon > 0$.

A sequence $\{x_n\}$ in a partial metric space (X,p) is said: (i) converge to a point $x \in X$ if and only if $p(x,x) = \lim_{n\to\infty} p(x,x_n)$ (ii) Cauchy sequence if $\lim_{n,m\to\infty} p(x_n,x_m)$ exists.

A partial metric space (X,p) is said to be complete if every Cauchy sequence $\{x_n\}$ in X converges, with respect to τ_p , to a point $x \in X$ and moreover $p(x,x) = \lim_{n,m\to\infty} p(x_n,x_m)$.

Suppose that $\{x_n\}$ is a sequence in partial metric space (X,p), then we define $L(x_n) = \{x | x_n \to x\}$.

The following example shows that a convergent sequence $\{x_n\}$ in a partial metric space X may not be Cauchy. In particular, it shows that the limit of a convergent sequence is not unique.

Example 2. *Let* $X = [0,\infty)$ *and* $p(x,y) = \max\{x,y\}$. *Let*

$$x_n = \begin{cases} 0, & n = 2k, \\ 1, & n = 2k+1. \end{cases}$$

Then clearly it is a convergent sequence and for every $x \geq 1$ *we have* $\lim_{n\to\infty} p(x_n,x) = p(x,x)$, *therefore* $L(x_n) = [1,\infty)$. *But* $\lim_{n,m\to\infty} p(x_n,x_m)$ *does not exist.*

The following Lemma shows that under certain conditions the limit is unique.

Lemma 1. *Let* $\{x_n\}$ *be a convergent sequence in the partial metric space X such that* $x_n \to x$ *and* $x_n \to y$. *If*

$$\lim_{n\to\infty} p(x_n,x_n) = p(x,x) = p(y,y),$$

then $x = y$.

Proof. As

$$p(x,y) \leq p(x,x_n) + p(x_n,y) - p(x_n,x_n),$$

therefore

$$p(x_n,x_n) \leq p(x,x_n) + p(x_n,y) - p(x,y).$$

By given assumptions, we have $\lim_{n\to\infty} p(x_n,x) = p(x,x)$, $\lim_{n\to\infty} p(x_n,y) = p(y,y)$, and $\lim_{n\to\infty} p(x_n,x_n) = p(x,x)$. Therefore

$$p(x,x) \leq p(x,x) + p(y,y) - p(x,y)$$

which shows that $p(y,y) \leq p(x,y) \leq p(y,y)$. Also,

$$p(x,y) \leq p(y,x_n) + p(x_n,x) - p(x_n,x_n)$$

implies that

$$p(x_n,x_n) \leq p(y,x_n) + p(x_n,x) - p(x,y)$$

which on taking limit as $n \to \infty$ gives

$$p(y,y) \leq p(y,y) + p(x,x) - p(x,y)$$

and

$$p(x,x) \leq p(x,y) \leq p(x,x).$$

Thus $p(x,x) = p(x,y) = p(y,y)$, therefore $x = y$.

Lemma 2. *Let* $\{x_n\}$ *and* $\{y_n\}$ *be two sequences in the partial metric space X such that*

$$\lim_{n\to\infty} p(x_n,x) = \lim_{n\to\infty} p(x_n,x_n) = p(x,x),$$

and

$$\lim_{n\to\infty} p(y_n,y) = \lim_{n\to\infty} p(y_n,y_n) = p(y,y),$$

then $\lim_{n\to\infty} p(x_n,y_n) = p(x,y)$. *In particular* , $\lim_{n\to\infty} p(x_n,z) = p(x,z)$ *for every* $z \in X$.

Proof. As $\{x_n\}$ and $\{y_n\}$ coverage to a $x \in X$ and $y \in X$ respectively, therefore for each $\epsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that $p(x,x_n) < p(x,x) + \frac{\epsilon}{2}$, $p(y,y_n) < p(y,y) + \frac{\epsilon}{2}$, $p(x,x_n) < p(x_n,x_n) + \frac{\epsilon}{2}$

and

$$p(y,y_n) < p(y_n,y_n) + \frac{\epsilon}{2}$$

for $n \geq n_0$. Now

$$p(x_n,y_n) \leq p(x_n,x) + p(x,y_n) - p(x,x)$$

$$\leq p(x_n,x)+p(x,y)+p(y,y_n)-p(y,y)-p(x,x)$$
$$< p(x,y)+\frac{\epsilon}{2}+\frac{\epsilon}{2}=p(x,y)+\epsilon,$$

and so we have

$$p(x_n,y_n)-p(x,y)< \epsilon.$$

Also,

$$p(x,y)\leq p(x,x_n)+p(x_n,y)-p(x_n,x_n)$$
$$\leq p(x,x_n)+p(x_n,y_n)+p(y_n,y)-p(y_n,y_n)-p(x_n,x_n)$$
$$<\frac{\epsilon}{2}+\frac{\epsilon}{2}+p(x_n,y_n)=p(x_n,y_n)+\epsilon$$

implies that

$$p(x,y)-p(x_n,y_n)<\epsilon.$$

Hence for all $n\geq n_0$, we have $|p(x_n,y_n)-p(x,y)|<\epsilon$. Hence the result follows.

Lemma 3. *If p is a partial metric on X, then mappings $p^s,p^m: X\times X\to\mathbb{R}^+$ given by*

$$p^s(x,y)=2p(x,y)-p(x,x)-p(y,y)$$

and

$$p^m(x,y)=\max\{p(x,y)-p(x,x),p(x,y)-p(y,y)\}$$

define equivalent metrics on X.

Proof. It is easy to see that p^s and p^m are metrics on X. Obviously,

$$p^m(x,y)\leq p^s(x,y)$$

for every $x,y\in X$. As for every positive real numbers a and b, we have $a+b\leq 2\max\{a,b\}$, therefore

$$p^s(x,y)=2p(x,y)-p(x,x)-p(y,y)$$
$$\leq 2\max\{p(x,y)-p(x,x),p(x,y)-p(y,y)\}=2p^m(x,y).$$

Hence

$$\frac{1}{2}p^s(x,y)\leq p^m(x,y)\leq p^s(x,y).$$

So p^s and p^m are equivalent.

Lemma 4 ([9], [11]). *Let (X,p) be a partial metric space.*

(a) $\{x_n\}$ is a Cauchy sequence in (X,p) if and only if it is a Cauchy sequence in the metric space (X,p^s).

(b) A partial metric space (X,p) is complete if and only if the metric space (X,p^s) is complete. Furthermore, $\lim_{n\to\infty}p^s(x_n,x)=0$ if and only if

$$p(x,x)=\lim_{n\to\infty}p(x_n,x)=\lim_{n,m\to\infty}p(x_n,x_m).$$

Lemma 5. *If $\{x_n\}$ is a convergent sequence in (X,p^s), then it is a convergent sequence in the partial metric space (X,p).*

Proof. As, $\lim_{n\to\infty}p^s(x_n,x)=0$, and $p(x_n,x_n)\leq p(x_n,x)$ for every n and $x\in X$, therefore

$$p(x_n,x)-p(x,x)\leq p^s(x_n,x)$$

implies that

$$\limsup_{n\to\infty}p(x_n,x)-p(x,x)\leq\lim_{n\to\infty}p^s(x_n,x)$$

and consequently, $\lim_{n\to\infty}p(x_n,x)=p(x,x)$.

1. Main Result

We begin this section giving the concept of weakly increasing mappings (see [4]).

Definition 1. Let $(X,\preccurlyeq)$ be a partially ordered set. Two mappings $S,T: X\to X$ are said to be weakly increasing if $Sx\preccurlyeq TSx$ and $Tx\preccurlyeq STx$ for all $x\in X$.

Note that, two weakly increasing mappings need not be nondecreasing. There exist some examples to illustrate this fact in [3].

In the sequel, we use the following notations :

(i) $\mathcal{F}$ denote the set of all functions $F: [0,\infty) \to [0,\infty)$ such that F is nondecreasing, continuous, and $F(0) = 0 < F(t)$ for every $t > 0$;

(ii) Ψ denote the set of all functions $\psi: [0,\infty) \to [0,\infty)$ such that ψ is nondecreasing, right continuous, and $\psi(t) < t$ for every $t > 0$.

Our main result is as follows:

Theorem 1. *Let $(X, \preccurlyeq)$ be a partially ordered set and suppose that there exists a partial metric p in X such that (X, p) is a complete partial metric space. Let $S, T: X \to X$ are two weakly increasing mappings such that*

$$F\big(p(Tx, Sy)\big) \le \psi(F\big(\phi(x,y)\big)) \tag{2.1}$$

for all comparable $x, y \in X$, where $F \in \mathcal{F}$, $\psi \in \Psi$ and $\phi(x,y) = \max\left\{p(x,y), p(x,Tx)p(y,Sy), \frac{p(x,Sy)+p(y,Tx)}{2}\right\}$. *(2.2)*

Then T and S have a common fixed point.

Proof. Let x_0 be an arbitrary point of X. If $x_0 = Sx_0$ the proof is finished. We can define a sequence in X as follows:

$$x_{2n+1} = Sx_{2n} \text{ and } x_{2n+2} = Tx_{2n+1} \quad \text{for } n \in \{0,1,\cdots\}.$$

Without lost of generality we can suppose that the successive term of $\{x_n\}$ are different. Otherwise we are again finished. Note that, since S and T are weakly increasing, we have

$$x_1 = Sx_0 \preccurlyeq TSx_0 = Tx_1 = x_2 = Tx_1 \preccurlyeq STx_1 = Sx_2 = x_3$$

and continuing this process we have

$$x_1 \preccurlyeq x_2 \cdots \preccurlyeq x_n \preccurlyeq x_{n+1} \preccurlyeq \cdots .$$

Now we claim that

$$F\big(p(x_{n+1}, x_n)\big) < F\big(p(x_n, x_{n-1})\big). \tag{2.3}$$

Setting $x = x_{2n}$ and $y = x_{2n+1}$ in (2.2), we have

$$\phi(x_{2n+1}, x_{2n}) = \max\left\{\begin{matrix} p(x_{2n+1}, x_{2n}), p(Tx_{2n+1}, x_{2n+1}), \\ p(Sx_{2n}, x_{2n}), \dfrac{p(x_{2n+1}, Sx_{2n}) + p(Tx_{2n+1}, x_{2n})}{2} \end{matrix}\right\}$$
$$= \max\{p(x_{2n+1}, x_{2n}), p(x_{2n+2}, x_{2n+1})\}.$$

Therefore, from (2.1)

$$F\big(p(x_{2n+2}, x_{2n+1})\big) = F\big(p(Tx_{2n+1}, Sx_{2n})\big) < \psi\Big(F\big(\phi(x_{2n+1}, x_{2n})\big)\Big)$$
$$= \psi\Big(F\big(p(x_{2n+1}, x_{2n})\big)\Big) < F\big(p(x_{2n}, x_{2n+1})\big). \tag{2.4}$$

Similarly, we have

$$F\big(p(x_{2n+1}, x_{2n})\big) < F\big(p(x_n, x_{n-1})\big). \tag{2.5}$$

Thus from (2.4) and (2.5), we get

$$F\big(p(x_{n+1}, x_n)\big) < F\big(p(x_n, x_{n-1})\big)$$

for all $n \in \mathbb{N}$. Now, since

$$F\big(p(x_{n+1}, x_n)\big) < \psi\Big(F\big(p(x_n, x_{n-1})\big)\Big) < \cdots < \psi^n\Big(F\big(p(x_1, x_0)\big)\Big)$$

we obtain

$$\lim_{n\to\infty} F(p(x_{n+1}, x_n)) = 0,$$

which implies

$$\lim_{n\to\infty} p(x_{n+1}, x_n) = 0. \tag{2.6}$$

Therefore

$$\lim_{n\to\infty} p^s(x_n, x_{n+1}) = 2\lim_{n\to\infty} p(x_n, x_{n+1}) - \lim_{n\to\infty} p(x_n, x_n) - \lim_{n\to\infty} p(x_{n+1}, x_{n+1}),$$

which shows that $\lim_{n\to\infty} p^s(x_n, x_{n+1}) = 0$.

Next, we claim that $\{x_n\}$ is a Cauchy sequence in (X, p). We proceed by negation and suppose that $\{x_n\}$ is not a Cauchy sequence. That is, there exists $\epsilon > 0$ such that $p^s(x_n, x_m) \ge \epsilon$ for infinite values of m and n with $m < n$. This assures that there exist two sequences $\{m(k)\}, \{n(k)\}$ of natural numbers, with $m(k) < n(k)$, such that for each $k \in \mathbb{N}$

$$p^s\big(x_{2m(k)}, x_{2n(k)+1}\big) > \epsilon. \tag{2.7}$$

It is not restrictive to suppose that $n(k)$ is the least positive integer exceeding $m(k)$ and satisfying (2.7). We have

$$\begin{aligned}\epsilon &< p^s\left(x_{2m(k)}, x_{2n(k)+1}\right)\\ &\le p^s\left(x_{2m(k)}, x_{2n(k)-1}\right) + p^s\left(x_{2n(k)-1}, x_{2n(k)}\right) + p^s\left(x_{2n(k)}, x_{2n(k)+1}\right)\\ &\le \epsilon + p^s\left(x_{2n(k)-1}, x_{2n(k)}\right) + p^s\left(x_{2n(k)}, x_{2n(k)+1}\right)\end{aligned}$$

and letting $k \to \infty$, we have $\lim_{n\to\infty} p^s(x_{2m(k)}, x_{2n(k)+1}) = \epsilon$. Hence $x_{2m(k)}, x_{2n(k)+1}) = \frac{\epsilon}{2}$. We note,

$$\begin{aligned}&p^s\left(x_{2m(k)}, x_{2n(k)+1}\right) - p^s\left(x_{2m(k)}, x_{2m(k)+1}\right) - p^s\left(x_{2n(k)+2}, x_{2n(k)+1}\right)\\ &\le p^s\left(x_{2m(k)+1}, x_{2n(k)+2}\right)\\ &\le p^s\left(x_{2m(k)}, x_{2n(k)+1}\right) + p^s\left(x_{2m(k)}, x_{2m(k)+1}\right) + p^s\left(x_{2n(k)+2}, x_{2n(k)+1}\right).\end{aligned}$$

And thus $p^s(x_{2m(k)+1}, x_{2n(k)+2}) \to \epsilon$ as $k \to \infty$. Hence $p(x_{2m(k)+1}, x_{2n(k)+2}) \to \frac{\epsilon}{2}$ as $k \to \infty$. We have

$$\begin{aligned}&\phi\left(x_{2n(k)+1}, x_{2m(k)}\right)\\ &= \max\left\{\begin{matrix} p\left(x_{2n(k)+1}, x_{2m(k)}\right), p\left(x_{2n(k)+1}, x_{2n(k)+2}\right),\\ p\left(x_{2m(k)}, x_{2m(k)+1}\right),\\ \dfrac{p\left(x_{2n(k)+1}, x_{2m(k)+1}\right) + p\left(x_{2m(k)}, x_{2n(k)+2}\right)}{2}\end{matrix}\right\},\\ &\le \max\left\{\begin{matrix} p\left(x_{2n(k)+1}, x_{2m(k)}\right), p\left(x_{2n(k)+1}, x_{2n(k)+2}\right),\\ p\left(x_{2m(k)}, x_{2m(k)+1}\right),\\ \dfrac{p\left(x_{2n(k)+1}, x_{2m(k)+1}\right)}{2} + \dfrac{p\left(x_{2m(k)}, x_{2m(k)+1}\right) + p\left(x_{2m(k)+1}, x_{2n(k)+2}\right)}{2}\\ - \dfrac{p\left(x_{2m(k)+1}, x_{2m(k)+1}\right)}{2}\end{matrix}\right\},\end{aligned}$$

and so letting as $k \to \infty$ we have $\lim_{k\to\infty} \phi(x_{2n(k)+1}, x_{2m(k)}) \le \frac{\epsilon}{2}$.

Therefore we have,

$$\begin{aligned}F\left(p\left(x_{2m(k)+1}, x_{2n(k)+2}\right)\right) &= F\left(p\left(Sx_{2m(k)}, Tx_{2n(k)+1}\right)\right)\\ &\le \psi\left(F\left(\phi\left(x_{2n(k)+1}, x_{2m(k)}\right)\right)\right)\end{aligned}$$

and letting as $k \to \infty$ in the above equation, by continuity of F and right continuity of ψ, we get:

$$F\left(\frac{\epsilon}{2}\right) \le \psi\left(F\left(\frac{\epsilon}{2}\right)\right) < F\left(\frac{\epsilon}{2}\right),$$

a contradiction. Therefore $\{x_n\}$ is a Cauchy sequence in the partial metric space (X, p). Since (X, p) is complete then from Lemma 4, the sequence $\{x_n\}$ converges in the metric space (X, p^s), say $\lim_{n\to\infty} p^s(x_n, z) = 0$. Again from Lemma 4, we have

$$p(z, z) = \lim_{n\to\infty} p(x_n, z) = \lim_{n,m\to\infty} p(x_n, x_m).$$

Now, we show that $z = Sz = Tz$. Setting $x = x_{2n+2}$ and $y = z$ in (2.1), we have

$$F\left(p(x_{2n+2}, Sz)\right) = F\left(p(Tx_{2n+1}, Sz)\right) \le \psi(F\left(\phi(x_{2n+1}, z)\right),$$

Where

$$\begin{aligned}\phi(x_{2n+1}, z) &= \max\left\{\begin{matrix} p(x_{2n+1}, z), p(Tx_{2n+1}, x_{2n+1}),\\ p(Sz, z), \dfrac{p(x_{2n+1}, Sz) + p(Tx_{2n+1}, z)}{2}\end{matrix}\right\}\\ &= \max\left\{\begin{matrix} p(x_{2n+1}, z), p(x_{2n+2}, x_{2n+1}),\\ p(Sz, z), \dfrac{p(x_{2n+1}, Sz) + p(Tx_{2n+2}, z)}{2}\end{matrix}\right\}\end{aligned}$$

and so letting as $n \to \infty$ we have $\lim_{n\to\infty} \phi(x_{2n+1}, z) = p(Sz, z)$.

Therefore we have

$$\lim_{n\to\infty} F(p(x_{2n+2}, Sz)) = F(p(z, Sz)) \le \psi(F\left(p(Sz, z)\right)) < F(p(Sz, z)),$$

a contradiction. Therefore, $p(Sz, z) = 0$, hence $z = Sz = Tz$.

Now we show that, if S or T has a fixed point, then it is a common fixed point of S and T. Indeed, let z be a fixed point of S. Now assume $p(z, Tz) > 0$. If we use the inequality (2.1), for $x = y = z$, we have

$$F\big(p(Tz,z)\big) = F\big(p(Tz,Sz)\big) \le \psi\Big(F\big(\phi(z,z)\big)\Big) = \psi(F\big(p(Tz,z)\big) < F\big(p(Tz,z)\big)$$

which is a contradiction. Thus $p(z,Tz) = 0$ and so z is a common fixed point of S and T. Similarly, if z is a fixed point of T, then it is also fixed point of S.

Corollary 1. *Let $(X,\preccurlyeq)$ be a partially ordered set and suppose that there exists a partial metric p in X such that (X,p) is a complete partial metric space. Let $S,T: X \to X$ are two weakly increasing mappings such that*

$$p(Tx,Sy) \le K\,\phi(x,y)$$

for all comparable $x,y \in X$, where $0 < K < 1$ and

$$\phi(x,y) = \max\left\{p(x,y), p(x,Tx), p(y,Sy), \frac{p(x,Sy)+p(y,Tx)}{2}\right\}$$

Then T and S have a common fixed point.

Theorem 2. *Consider the integral equations*

$$x(t) = \int_a^b g\big(s,x(t)\big)ds, \qquad t \in [a,b],$$
$$x(t) = \int_a^b h\big(s,x(t)\big)ds, \qquad t \in [a,b].$$

Let $\ll$ be a partial order relation on $\mathbb{R}^n$, and

(i) $g,h: [a,b] \times \mathbb{R}^n \to \mathbb{R}^n$ *are continuous,*

(ii) $x: [a,b] \to \mathbb{R}^n$ *is continuous,*

(iii) *for each* $s \in [a,b]$,

$$g(s,x(t)) \ll h(s, \int_a^b g\big(s,x(t)\big)ds),$$

$$h(s,x(t)) \ll g(s, \int_a^b h\big(s,x(t)\big)ds),$$

(iv) *there exist a continuous function* $p: [a,b] \to \mathbb{R}^+$ *such that*

$$|h(s,u) - g(s,v)| \le p(s)|u-v|$$

for each $s \in [a,b]$ *and comparable* $u,v \in \mathbb{R}^n$,

(v) $\int_a^b p(s)ds \le K,\ for\ 0 < K < 1.$

Then, the integral equations have a unique common solution $x^* \in C([a,b],\mathbb{R}^n)$.

Proof. Let $X = C([a,b],\mathbb{R}^n)$ with the usual supremum norm; that is, $\| x \| = \max_{t\in[a,b]}|x(t)|$, for $x \in C([a,b],\mathbb{R}^n)$. Define an order on X by

$$x \preccurlyeq y\ iff\ x(t) \ll y(t)\ for\ any\ t \in [a,b]\ and\ x,y \in X.$$

Then, $(X,\preccurlyeq)$ is a partially ordered set. Also, $(X,\|.\|)$ is a complete metric space.
Define $T,S: X \to X$ by

$$Tx(t) = \int_a^b g\big(s,x(t)\big)ds,$$
$$Sx(t) = \int_a^b h\big(s,x(t)\big)ds.$$

Now, from (iii), we have for all $t \in [a,b]$,

$$\begin{aligned} Tx(t) &= \int_a^b g\big(s,x(t)\big)ds \\ &\ll \int_a^b h\left(s, \int_a^b g\big(s,x(t)\big)ds\right)ds \\ &= \int_a^b h\big(s,Tx(t)\big)ds \\ &= STx(t), \end{aligned}$$

and

$$\begin{aligned} Sx(t) &= \int_a^b h(s,x(t))ds \\ &\ll \int_a^b g\left(s, \int_a^b h(s,x(t))ds\right)ds \\ &= \int_a^b g(s,Sx(t))ds \\ &= TSx(t). \end{aligned}$$

Thus, we have $Tx \preccurlyeq STx$ and $Sx \preccurlyeq TSx$ for all $x \in X$. This shows that T and S are weakly increasing. Also, for each comparable $x, y \in X$, we have

$$\begin{aligned} |Tx(t) - Sy(t)| &= |\int_a^b g(s,x(t))ds - \int_a^b h(s,y(t))ds| \\ &\leq \int_a^b |g(s,x(t)) - h(s,y(t))|ds \\ &\leq \int_a^b p(s)|x(t) - y(t)|ds = |x(t) - y(t)| \int_a^b p(s)ds \\ &\leq K|x(t) - y(t)|. \end{aligned}$$

Therefore, we get

$$\begin{aligned} \sup_{t\in[a,b]} |Tx(t) - Sy(t)| &= \| Tx - Sy \| \\ &\leq K \sup_{t\in[a,b]} |x(t) - y(t)| = K \| x - y \| \\ &\leq K max\left\{ \begin{matrix} \| x - y \|, \| x - Tx \|, \\ \| y - Ty \|, \frac{\|x-Sy\|+\|y-Tx\|}{2} \end{matrix} \right\} \end{aligned}$$

for each comparable $x, y \in X$. Therefore, all conditions of Corollary(1)(with $p(x,y) = \| x - y \|$) are satisfied. Thus, T and S have a common fixed point in $X = C([a,b], \mathbb{R}^n)$. That is, there exists $x^* \in X$ such that $Tx^* = Sx^* = x^*$. Thus, the conclusion follows.

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