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Elasto-Plastic Axisymmetric Thermal Stress Analysis of Functionally Graded Cylindrical vessel

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ABSTRACT

Based on Tresca's yield criterion and small deformation theory, an axi-symmetric thermal elasto-plastic stress analysis in cylindrical vessels made of Functionally Graded Material (FGM) is presented. Primarily the separate distribution of elastic and plastic stress and deformation for an inhomogeneous cylinder is introduced. The elasto-plastic cylinder is assumed to be composed of several nested elastic and fully plastic cylinders. The deformation of each cylinder is correlated to its neighboring cylinders and consequently a system of equations is formed. Using this model, in some problems, the influence of temperature on the stress, strain and plastic zone patterns are studied. Elastic modulus and yield stress are assumed temperature-independent power functions of radius. Especially, the influence of temperature upon the evolution and growth of plastic zone is studied. It is shown that depending on the composition of thermo-mechanical properties of FG material, the interface line between elastic and plastic regions may take shape in different radii. The rate and direction of the growth of plastic zone are studied. For a wide range of material properties, the patterns of yielded zones are plotted.
KEYWORDS: FGM, Elastoplastic, Pressure vessel, interface line, thermal stress

1. INTRODUCTION

In harmony with the progress in science and production and the development of new materials, the study and analysis of newly developed materials looks to be more favorable. Similarly, the application of Functionally Graded Materials (FGM) is not apart from this category and in some fields such as the thermoplastic structural analysis there are few research work in this context. FGMs are heterogeneous materials that depended on the consumers' need their thermo-mechanical behavior in different points may be adjusted in advance. In practice, the continuous change in material properties can be obtained by gradual change in volume fraction of two constructional constituents usually consisted of ceramic and metallic parts. Therefore, the compound yet isotropic may be mechanically or thermally inhomogeneous. In parallel with the industrial importance of FGMs, the mathematical analyses of the FGM problems are also gaining importance. One of the assessable aspects of inhomogeneous structure behaviors is the effect of temperature. Therefore keeping all this facts in mind, in this work the influence of pressure and temperature on the plastic deformation of a heterogeneous typical cylinder as the representative of a thick walled pressure vessel is studied.

In the following, some of the nearly recent researches in this context are introduced. Some of the leading papers in the stress analysis of FG pressure vessels are those that study the effect of internal pressure on the distribution of stress in the vessel. Among these studies, one can point to the works by Tutuncu [1]. The steady state thermo-elastic distribution of stress in an axi-symmetric FGM media is studied by Zhang *et.al.* [2]. Nemat-Alla *et.al.* have studied the transient thermal stresses in a plate of functionally gradient materials [3]. By using of Fourier series in a non-symmetric problem, Jabbari *et.al.* have found the distribution of temperature and deformation in a cylindrical vessel [4]. The postbuckling analysis of pressure-loaded functionally graded cylindrical shells in thermal environments is studied by Shen [5]. Using thermoplastic stress analysis and optimization techniques Tanaka *et.al* .designed the properties of a FG material such that its thermal stresses gets minimized [6]. Awaji and Sivakumar [7] consider the transient distribution of thermal stresses in FGM cylinders. Bahtui and Eslami have considered the coupled thermal and deformation response of FGM cylinder [8]. In this analysis, energy equations and second order shear deformation theory of shells is employed. Ye *et. al.* studied the two dimensional axisymmetric thermoelastic problems in a FGM transversely isotropic vessel [9]. Araslan and Akis develop their research work in analytical solution of plain strain FGM pressure vessels in elastic and plastic regions [10]. Akis represents the elastoplastic analysis for spherical FGM pressure vessel [11]. Sadeghian and Ekhteraei Toussi represents the thermo-elastoplastic analysis for spherical FGM pressure vessel [12].In this work, a power law equation for the distribution of elastic modulus and yield stress is used.

Based on the knowledge of the authors, despite the development of different solution methods for the different mechanical properties yet a thermo plastic stress analysis for the pressure vessels is not performed. Therefore, in this paper for a cylindrical vessel made of FG material obeying an elastic perfectly plastic behavior and in a steady state thermal condition, the effect of thermo-mechanical loads upon the stress

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distribution and specially the position of elastic and plastic interface lines are studied. Therefore assuming a power law behavior for different material, mechanical and thermal properties across the radius, followed by obtaining the Navier equations of the problem and using a temperature distribution function, are the fundamentals of this analysis. Down to the dependency of the solution on the material properties and boundary conditions, the used magnitude of parameters are given in the tables. Some different graphs are represented in which the positions of plastic zone are shown and different patterns of yielding are recognizable. Using these graphs and monographs, the influence of different material parameters and boundary conditions upon the solutions can be studied.

2. Thermo-elasto-plastic deformation of FGM cylinder

At the first step and following the formulations provided for the elasto-plastic analysis of FGM cylinders in [10] hereafter the general solution of a more complicated problem of thermo-elasto-plastic deformation analysis of FGM cylinders is sought. To this ends, it is assumed that throughout a cylinder made of FG material the Poisson's ratio is constant, deformations are small and in the cylindrical coordinates of (r,θ,z) material properties distributed across the thickness based on a power law model. That is,

$$E(r) = \mathrm{E_o} r^{\mathrm{n}} \tag{1}$$

$$Y(r) = \mathrm{Y_o} r^{\mathrm{m}} \tag{2}$$

$$\alpha(r) = \alpha_{\mathrm{o}} r^{\mathrm{L}} \tag{3}$$

$$K(r) = \mathrm{K_o} r^{\mathrm{q}} \tag{4}$$

In which $\mathrm{E_o} = \bar{\mathrm{E}}/\mathrm{b^n}$, $\mathrm{Y_o} = \bar{\mathrm{Y}}/\mathrm{b^m}$, $\alpha_{\mathrm{o}} = \bar{\alpha}/\mathrm{b^L}$ and $\mathrm{K_o} = \bar{\mathrm{K}}/\mathrm{b^q}$.Moreover $\bar{\mathrm{E}}$, $\bar{\mathrm{Y}}$, $\bar{\alpha}$ and $\bar{\mathrm{K}}$ are some reference datum for elastic modulus, yielding stress, heat expansion coefficient and heat conduction constant. Here b is the outer radius of cylinder and $\mathrm{m}, \mathrm{n}, \mathrm{L}$ and q are the exponents of the property functions in *r* direction. Figure (1) shows a general outline of the problem.

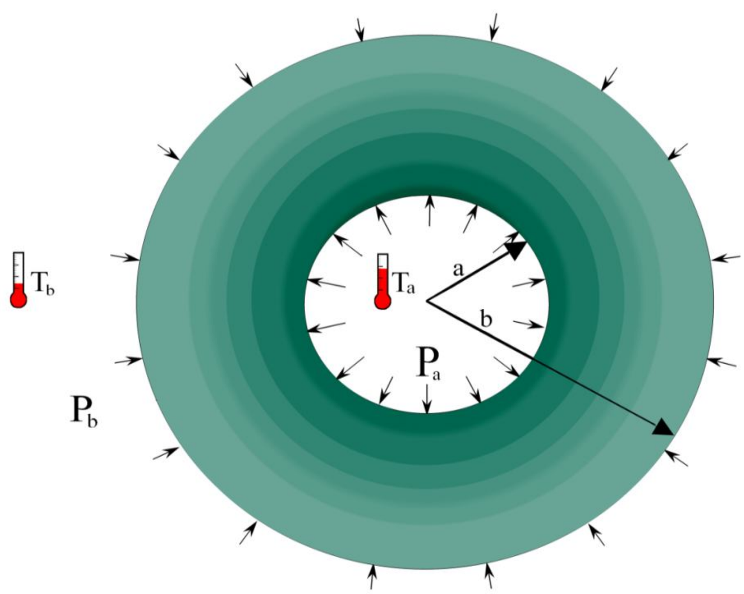

Fig. 1. Inhomogeneous cylindrical pressure vessel

Wherever in a cylindrical vessel cross sectional deformations are small, axial deformations are negligible and loading is axi-symmetric, the general stress strain relationship can be expressed as,

$$\varepsilon_r^T(r) = \frac{1-\nu^2}{E(r)}\left[\sigma_r(r) - \frac{\nu}{1-\nu}\sigma_\theta(r)\right] + \varepsilon_r^p(r) + (1+\nu).\alpha(r).T(r) \tag{5}$$

$$\varepsilon_\theta^T(r) = \frac{1-\nu^2}{E(r)}\left[\sigma_\theta(r) - \frac{\nu}{1-\nu}\sigma_r(r)\right] + \varepsilon_\theta^p(r) + (1+\nu)\alpha(r).T(r) \tag{6}$$

In which ε is strain, σ is stress and the superscripts T and p designates the 'Total value' and 'plastic parts' of the deformation, respectively. In an axi-symmetric problem one obtains,

$$\varepsilon_r^T\left(r\right)=\frac{\mathrm{d}u\left(r\right)}{\mathrm{d}r} \tag{7}$$

$$\varepsilon_\theta^T\left(r\right)=\frac{u\left(r\right)}{r} \tag{8}$$

2.1 Elastic solution:

In elastic deformation, using Eqs. (5) to (8) stress components are obtained to be,

$$\sigma_r\left(r\right)=\frac{E\left(r\right)}{\left(1+\nu\right)\left(1-2\nu\right)}\left[\left(1-\nu\right)\frac{\mathrm{d}u\left(r\right)}{\mathrm{d}r}+\nu\frac{u\left(r\right)}{r}-\left(1+\nu\right)\alpha\left(r\right)T\left(r\right)\right] \tag{9}$$

$$\sigma_\theta\left(r\right)=\frac{E\left(r\right)}{\left(1+\nu\right)\left(1-2\nu\right)}\left[\nu\frac{\mathrm{d}u\left(r\right)}{\mathrm{d}r}+\left(1-\nu\right)\frac{u\left(r\right)}{r}-\left(1+\nu\right)\alpha\left(r\right)T\left(r\right)\right] \tag{10}$$

In which $u\left(r\right)$ is the radial displacement function. The equilibrium equation for an axisymmetric problem is,

$$\frac{\mathrm{d}\sigma_r}{\mathrm{d}r}+\frac{\left(\sigma_r-\sigma_\theta\right)}{r}=0 \tag{11}$$

Substituting Eqs. (9) and (10) in (11) and using (1) and (3) the governing differential equation of displacement known as Navier equation is obtained.

$$\frac{\mathrm{d}^2u\left(r\right)}{\mathrm{d}r^2}+\left[\frac{\mathrm{n}+1}{r}\right]\frac{\mathrm{d}u\left(r\right)}{\mathrm{d}r}+\left[\frac{\nu\left(\mathrm{n}+1\right)-1}{r^2\left(1-\nu\right)}\right]u\left(r\right)=\frac{\left(1+\nu\right)\alpha_0r^L}{\left(1-\nu\right)}\left[\frac{\mathrm{n}+\mathrm{L}}{r}T\left(r\right)+\frac{\mathrm{d}T\left(r\right)}{\mathrm{d}r}\right] \tag{12}$$

The one dimensional cylindrical equation of steady state conduction heat transfer is [11],

$$\frac{\mathrm{d}^2T\left(r\right)}{\mathrm{d}r^2}+\left(\frac{K'\left(r\right)}{K\left(r\right)}+\frac{1}{r}\right)\frac{\mathrm{d}T\left(r\right)}{\mathrm{d}r}=0 \tag{13}$$

Thermal boundary conditions for a hollow FGM cylindrical vessel is,

$$\mathrm{C}_{11}T'\left(a\right)+\mathrm{C}_{12}T\left(a\right)=\mathrm{f}_1 \tag{13a}$$

$$\mathrm{C}_{21}T'\left(b\right)+\mathrm{C}_{22}T\left(b\right)=\mathrm{f}_2 \tag{13b}$$

In which C_{11} and C_{21} are thermal convection factors, C_{12} and C_{22} are heat conduction coefficient, a is inner radius, f_1 and f_2 are some definite constants for inner and outer radii. Substituting thermal conduction function from Eq. (4) into Eq. (13) provides,

$$\frac{\mathrm{d}^2T\left(r\right)}{\mathrm{d}r^2}+\left(\frac{\mathrm{q}+1}{r}\right)\frac{\mathrm{d}T\left(r\right)}{\mathrm{d}r}=\mathrm{o} \tag{14}$$

A general solution of Eq. (14) is,

$$T\left(r\right)=\mathrm{C}_1r^{-\mathrm{q}}+\mathrm{C}_2 \tag{15}$$

Using (15) and thermal boundary conditions (13a) and (13b), the parameters can be found to be,

$$\mathrm{C}_1=\frac{\mathrm{C}_{22}\mathrm{f}_1-\mathrm{C}_{12}\mathrm{f}_2}{\mathrm{C}_{12}\left(\mathrm{q}\,\mathrm{C}_{21}\mathrm{b}^{-(\mathrm{q}+1)}-\mathrm{C}_{22}\mathrm{b}^{-\mathrm{q}}\right)-\mathrm{C}_{22}\left(\mathrm{q}\,\mathrm{C}_{11}\mathrm{a}^{-(\mathrm{q}+1)}-\mathrm{C}_{12}\mathrm{a}^{-\mathrm{q}}\right)} \tag{16}$$

$$\mathrm{C}_2=\frac{\mathrm{f}_1\left(\mathrm{q}\mathrm{C}_{21}\mathrm{b}^{-(\mathrm{q}+1)}-\mathrm{C}_{22}\mathrm{b}^{-\mathrm{q}}\right)-\mathrm{f}_2\left(\mathrm{q}\mathrm{C}_{11}\mathrm{a}^{-(\mathrm{q}+1)}-\mathrm{C}_{12}\mathrm{a}^{-\mathrm{q}}\right)}{\mathrm{C}_{12}\left(\mathrm{q}\mathrm{C}_{21}\mathrm{b}^{-(\mathrm{q}+1)}-\mathrm{C}_{22}\mathrm{b}^{-\mathrm{q}}\right)-\mathrm{C}_{22}\left(\mathrm{q}\mathrm{C}_{11}\mathrm{a}^{-(\mathrm{q}+1)}-\mathrm{C}_{12}\mathrm{a}^{-\mathrm{q}}\right)} \tag{17}$$

Now, as the distribution of temperature is known, the general and particular solutions of Eq. (12) are found. A particular solution of the homogeneous part of Eq. (12) is,

$$u_g\left(r\right)=\mathrm{A}r^{\mathrm{s}} \tag{18}$$

According to Eq.(18) a general solution of Eq.(12) can be represented as,

$$u_g\left(r\right)=\mathrm{A}_1 r^{\mathrm{S}_1}+\mathrm{A}_2 r^{\mathrm{S}_2} \tag{19}$$

$$\mathrm{S}_{1,2}=\frac{-\mathrm{n}\pm\sqrt{\mathrm{n}^2-\dfrac{4\nu\left(\mathrm{n}+1\right)-4}{1-\nu}}}{2} \tag{20}$$

A_1 and A_2 are some constants related to the boundary conditions. On the other hand, the particular solution of equation (12) can be represented as,

$$u_p\left(r\right)=\mathrm{E}_1 r^{\mathrm{L}+1}+\mathrm{E}_2 r^{\mathrm{L-q}+1}. \tag{21}$$

By substituting Eq. (21) in Eq. (12), the constants E_1 and E_2 are found to be,

$$\mathrm{E}_1=\frac{\left(1+\nu\right)\alpha_{\mathrm{o}}\left(\mathrm{n}+\mathrm{L}\right)\mathrm{C}_2}{\left(1-\nu\right)\left[\mathrm{L}^2+\left(2+\mathrm{n}\right)\mathrm{L}\right]\ \ +\mathrm{n}} \tag{22}$$

$$\mathrm{E}_2=\frac{\left(1+\nu\right)\alpha_{\mathrm{o}}\left(\mathrm{n}+\mathrm{L}-\mathrm{q}\right)\mathrm{C}_1}{\left(1-\nu\right)\left[\left(\mathrm{L}-\mathrm{q}\right)^2+\left(2+\mathrm{n}\right)\left(\mathrm{L}-\mathrm{q}\right)\right]\ \ +\mathrm{n}} \tag{23}$$

Total solution of Eq. (12) which is a combination of the general solution of its relevant homogeneous form and its particular solution, is,

$$\begin{aligned}u_T\left(r\right)&=u_g\left(r\right)+u_p\left(r\right)\\&=\left(\mathrm{A}_1 r^{\mathrm{S}_1}+\mathrm{A}_2 r^{\mathrm{S}_2}\right)+\left(\mathrm{E}_1 r^{\mathrm{L}+1}+\mathrm{E}_2 r^{\mathrm{L-q}+1}\right)\end{aligned} \tag{24}$$

Now by using of the boundary conditions given for this problem the constants A_1 and A_2 are found. The pressure inside the cylinder is P_a and that of the outside is P_b. Substituting Eqs. (1), (3), (15) and (24) into Eq. (9), radial stress function in terms of radial variable, *r,* is found as,

$$\begin{aligned}\sigma_r\left(r\right)=\frac{\mathrm{E}_\mathrm{o}}{\left(1+\nu\right)\left(1-2\nu\right)}\Big[&\mathrm{A}_1 r^{\mathrm{n}+\mathrm{S}_1-1}\left(\left(1-\nu\right)\mathrm{S}_1+\nu\right)+\mathrm{A}_2 r^{\mathrm{n}+\mathrm{S}_2-1}\left(\left(1-\nu\right)\mathrm{S}_2+\nu\right)\\&+r^{\mathrm{L}+\mathrm{n}}\left\{\mathrm{E}_1\left(\mathrm{L}+1-\nu\mathrm{L}\right)-\mathrm{C}_2\alpha_\mathrm{o}\left(1+\nu\right)\right\}\\&+r^{\mathrm{L}+\mathrm{n}-\mathrm{q}}\left\{\mathrm{E}_2\left(\left(\mathrm{L}-\mathrm{q}\right)\left(1-\nu\right)+1\right)-\mathrm{C}_1\alpha_\mathrm{o}\left(1+\nu\right)\right\}\Big].\end{aligned} \tag{25}$$

Moreover, by using of presumed boundary conditions, the constants of the displacement function in (24) are found to be,

$$\mathrm{A}_1=\frac{\varphi\left(\mathrm{b},\mathrm{S}_2\right)\left[\mathrm{P}_\mathrm{a}-\zeta\left(\mathrm{a}\right)\right]-\varphi\left(\mathrm{a},\mathrm{S}_2\right)\left[\mathrm{P}_\mathrm{b}-\zeta\left(\mathrm{b}\right)\right]}{\varphi\left(\mathrm{a},\mathrm{S}_1\right)\varphi\left(\mathrm{b},\mathrm{S}_2\right)-\varphi\left(\mathrm{a},\mathrm{S}_2\right)\varphi\left(\mathrm{b},\mathrm{S}_1\right)} \tag{26}$$

$$\mathrm{A}_2=\frac{\varphi\left(\mathrm{a},\mathrm{S}_1\right)\left[\mathrm{P}_\mathrm{a}-\zeta\left(\mathrm{a}\right)\right]-\varphi\left(\mathrm{a},\mathrm{S}_2\right)\left[\mathrm{P}_\mathrm{b}-\zeta\left(\mathrm{b}\right)\right]}{\varphi\left(\mathrm{a},\mathrm{S}_1\right)\varphi\left(\mathrm{b},\mathrm{S}_2\right)-\varphi\left(\mathrm{a},\mathrm{S}_2\right)\varphi\left(\mathrm{b},\mathrm{S}_1\right)} \tag{27}$$

$$\varphi\left(r,s\right)=\frac{\mathrm{E}_\mathrm{o}}{\left(1+\nu\right)\left(2\nu-1\right)}\left[\left(\nu-1\right)\mathrm{S}-2\nu\right]r^{\mathrm{n}+\mathrm{s}-1} \tag{28}$$

$$\begin{aligned}\zeta\left(r\right)=\frac{\mathrm{E}_\mathrm{o}}{\left(1+\nu\right)\left(2\nu-1\right)}\Big[&\left(\mathrm{C}_2\alpha_\mathrm{o}\left(1+\nu\right)+\mathrm{E}_1\left(\nu\mathrm{L}-\mathrm{L}-1\right)\right)r^{\mathrm{q}}\\&+\left(\mathrm{C}_1\alpha_\mathrm{o}\left(1+\nu\right)+\mathrm{E}_2\left(\left(\mathrm{L}-\mathrm{q}\right)\left(\nu-1\right)-1\right)\right)\Big]r^{\mathrm{n}+\mathrm{L}-\mathrm{q}}\end{aligned} \tag{29}$$

Consequently, in elastic deformation, the tangential stresses as well as the tangential and normal strains are found to be,

$$\sigma_\theta(r) = \frac{\mathrm{E_o}}{(1+\nu)(1-2\nu)}\Big[\mathrm{A_1} r^{\mathrm{n+S_1-1}}\big((\mathrm{S_1}-1)\nu+1\big) + \mathrm{A_2} r^{\mathrm{n+S_2-1}}\big((\mathrm{S_2}-1)\nu+1\big)$$
$$+ r^{\mathrm{L+n}}\left\{\mathrm{E_1}(1+\nu\mathrm{L}) - \mathrm{C_2}\alpha_\mathrm{o}(1+\nu)\right\}$$
$$+ r^{\mathrm{L+n-q}}\left\{\mathrm{E_2}\big((\mathrm{L-q})\nu+1\big) - \mathrm{C_1}\alpha_\mathrm{o}(1+\nu)\right\}\Big] \quad (30)$$

$$\varepsilon_r(r) = \mathrm{A_1S_1}r^{\mathrm{S_1-1}} + \mathrm{A_2S_2}r^{\mathrm{S_2-1}} + \mathrm{E_1}(\mathrm{L}+1)r^{\mathrm{L}} + \mathrm{E_2}(\mathrm{L-q}+1)r^{\mathrm{L-q}} \quad (31)$$

$$\varepsilon_\theta(r) = \mathrm{A_1}r^{\mathrm{S_1-1}} + \mathrm{A_2}r^{\mathrm{S_2-1}} + \mathrm{E_1}r^{\mathrm{L}} + \mathrm{E_2}r^{\mathrm{L-q}} \quad (32)$$

2.2 Plastic solution:

Now for the fully plastic conditions by using of the Tresca's criterion the stresses and strains are found. In situations, where tangential stress is greater than radial stress the Tresca's criterion is simplified as,

$$\sigma_\theta(r) - \sigma_r(r) = Y(r) \quad (33)$$

Using Eq.(2), equilibrium equation in Eq. (11) and Tresca's criterion in Eq. (33) as in [9] one can deduce that,

$$\sigma_r(r) = \frac{r^{\mathrm{m}}\mathrm{Y_o}}{\mathrm{m}} + \mathrm{Q_1} \quad (34)$$

$$\sigma_\theta(r) = \frac{(1+\mathrm{m})r^{\mathrm{m}}\mathrm{Y_o}}{\mathrm{m}} + \mathrm{Q_1} \quad (35)$$

In which $\mathrm{Q_1}$ is an optional constant depended on the boundary conditions. In general, plastic deformations are assumed incompressible. That is in this problem we may write, $\varepsilon_r^p(r) + \varepsilon_\theta^p(r) + \varepsilon_j^p(r) = 0$. Now, using Eqs.(7 and 8) the amount of $\varepsilon_r^T(r) + \varepsilon_\theta^T(r) + \varepsilon_z^T(r)$ is calculated to be,

$$\varepsilon_r^T(r) + \varepsilon_\theta^T(r) + \varepsilon_z^T(r) = \frac{(1+\nu)(1-2\nu)}{\mathrm{E}}\big(\sigma_r(r) + \sigma_\theta(r)\big) + 2(1+\nu)\alpha(r).T(r) \quad (36)$$

Then using Eqs. (7) and (8) one obtains,

$$\frac{\mathrm{d}u(r)}{\mathrm{d}r} + \frac{u(r)}{r} = \frac{(1+\nu)(1-2\nu)}{\mathrm{E}}\big(\sigma_r(r) + \sigma_\theta(r)\big) + 2(1+\nu)\alpha(r).T(r) \quad (37)$$

Substituting stress components from Eq. (34) and (35) into the Eq. (37) one obtains,

$$\frac{\mathrm{d}u(r)}{\mathrm{d}r} + \frac{u(r)}{r} = \left[\frac{(1+\nu)(1-2\nu)}{\mathrm{E_o}}.\frac{(2+\mathrm{m})\mathrm{Y_o}}{\mathrm{m}}\right]r^{\mathrm{m-n}} + \frac{2\mathrm{Q_1}(1+\nu)(1-2\nu)}{\mathrm{E_o}}r^{\mathrm{-n}} + 2(1+\nu)\alpha_\mathrm{o}T(r)r^{\mathrm{L}} \quad (38)$$

Substituting heat transfer equation (15) into Eq. (38) the general form of plastic displacement function is obtained as,

$$\frac{\mathrm{d}u(r)}{\mathrm{d}r} + \frac{u(r)}{r} = \left[\frac{(1+\nu)(1-2\nu)}{\mathrm{E_o}}.\frac{(2+\mathrm{m})\mathrm{Y_o}}{\mathrm{m}}\right]r^{\mathrm{m-n}} + \frac{2\mathrm{Q_1}(1+\nu)(1-2\nu)}{\mathrm{E_o}}r^{\mathrm{-n}}$$
$$+ 2(1+\nu)\alpha_\mathrm{o}\mathrm{C_1}r^{\mathrm{L-q}} + 2(1+\nu)\alpha_\mathrm{o}\mathrm{C_2}r^{\mathrm{L}} \quad (39)$$

Similarly, to solve Eq. (39) the well-known method of combining the solution of particular and general solution is used. Here the related homogeneous differential equation happens to be,

$$\frac{\mathrm{d}u(r)}{\mathrm{d}r} + \frac{u(r)}{r} = 0 \quad (40)$$

or

$$\frac{\mathrm{d}u(r)}{u(r)} = -\frac{\mathrm{d}r}{r} \quad (41)$$

This has a solution as,

$$u_p(r) = \frac{\mathrm{C_3}}{r} \quad (42)$$

In which C_3 is a constant. The particular solution of Eq. (39) can be shown to be,

$$u_p(r) = P_1 r^{L-q+1} + P_2 r^{L+1} + P_3 r^{m-n+1} + P_4 r^{1-n} \tag{43}$$

In which P_1, P_2, P_3, P_4 can be obtained by substituting Eq. (43) in Eq. (39). That is,

$$\frac{du(r)}{dr} + \frac{u(r)}{r} = P_1(L - q + 2)r^{L-q} + P_2(L+2)r^{L} + P_3(m - n + 2)r^{m-n} + P_4(2 - n)r^{-n} \tag{44}$$

$$P_1 = \frac{2(1+\nu)\alpha_o C_1}{L - q + 2} \tag{45}$$

$$P_2 = \frac{2(1+\nu)\alpha_o C_2}{L + 2} \tag{46}$$

$$P_3 = \frac{(1+\nu)(1-2\nu)(2+m)Y_o}{mE_o(m - n + 2)} \tag{47}$$

$$P_4 = \frac{3Q_1(1+\nu)(1-2\nu)}{E_o(2-n)} \tag{48}$$

Therefore the complete solution of the plastic deformation in cylinders is,

$$u(r) = u_g(r) + u_p(r) = \frac{C_3}{r} + P_1 r^{L-q+1} + P_2 r^{L+1} + P_3 r^{m-n+1} + P_4 r^{1-n} \tag{49}$$

Having the deformation function, plastic strains can be obtained as well.

$$\varepsilon_\theta^p(r) = \frac{u(r)}{r} - (1+\nu)\alpha(r).T(r) - \left(\frac{1-\nu^2}{E}\right).\left[\sigma_\theta(r) - \frac{\nu}{1-\nu}\sigma_r(r)\right] \tag{50}$$

$$\varepsilon_r^p(r) = \frac{du(r)}{dr} - (1+\nu)\alpha(r).T(r) - \left(\frac{1-\nu^2}{E}\right).\left[\sigma_r(r) - \frac{\nu}{1-\nu}\sigma_\theta(r)\right] \tag{51}$$

In Eq. (47), it is obvious that if $m = 0$ the radial and tangential stresses are indefinite. Therefore, in this special case the problem is solved separately. Substituting the Tresca's criterion in equilibrium equation provides,

$$\frac{d\sigma_r(r)}{dr} = \frac{Y(r)}{r} \tag{52}$$

Therefore, the stress components are found to be,

$$\sigma_r(r) = Q_1 + Y_o \ln r \tag{53}$$

$$\sigma_\theta(r) = Q_1 + [1 + \ln r]Y_o \tag{54}$$

2. RESULTS AND DISCUSSIONS

In this section the method of thermo-elasto-plastic deformation analysis of FGM cylindrical vessels developed in the previous section is used to explain some aspects of the mechanical behavior of FGM cylindrical vessels. In all case studies, it is assumed that a steady state distribution of temperature prevails. Moreover, the influence of heat upon the material properties is negligible. Poisson's ratio is a constant throughout the thickness while the elastic modulus, yield stress, heat conduction and heat expansion coefficient are some power function of radius. In each case following the definition and solution of a problem, the resulted distribution of stress is used to find the position of the plastic zone boundaries.

For an inhomogeneous or especially a FGM cylindrical vessel, the boundary of plasticized zone is not easily detectable and it mainly depends on the composition of the FG material. Based on Tresca's criterion and elastic solution of the problem, the yielding start point is a point that during the loading process, the following equation forms earlier.

$$\sigma_\theta(r_p) - \sigma_r(r_p) = Y(r_p) \tag{55}$$

In contrast to the homogeneous cylinders in which plastic zone initiates from inside, the first yielding point of an inhomogeneous cylinder may be anywhere along the thickness of the vessel. In addition, the growth rate of plastic zone down to the internal pressure or temperature is an important factor. For example in the case of a middle plastic zone, the inward or outward growth rates may be different. To guarantee that the process of transient heat transfer does not seriously affect this analysis, only cases wherein the direction of plastic zone growth is in harmony with the growth of temperature are considered. For example, this analysis does not consider the situation in which an increase of temperature results in a decrease of plastic zone.

It needs to mention that in all cases, the thermal boundary condition is of the Dirichlet type with outer temperature $T(b)=0^{oc}$ and steady state heat transfer prevails. In all examples, the physical parameters given in Table (1) are the same.

Table 1. Physical Characteristics

$E_o(Pa)$	$Y_o(Pa)$	$P_b(Pa)$	$b(m)$	ν	f_2
2×10^{11}	4.3×10^{8}	0	1	0.3	0
$\alpha_o(^\circ C)^{-1}$	$C_{11}(^\circ C)^{-1}$	$C_{12}(^\circ C)^{-1}$	$C_{21}(^\circ C)^{-1}$	$C_{22}(^\circ C)^{-1}$	
1.2×10^{-6}	0	1	0	1	

The following equations are used to reduce the number of independent variables.

$$\bar{r}=\frac{r}{b} \quad , \quad \bar{\sigma}_\theta=\frac{\sigma_\theta}{Y_o a^m} \quad , \quad \bar{\sigma}_r=\frac{\sigma_r}{Y_o a^m} \quad , \quad \bar{\varepsilon}_r=\frac{\varepsilon_r.E_o.b^n}{Y_o a^m} \quad , \quad \bar{\varepsilon}_\theta=\frac{\varepsilon_\theta.E_o.b^n}{Y_o a^m} \tag{56}$$

Initially the situation where plastic zone commences from outside is studied. To do this, a vessel with the properties given in table (2) is considered.

Table 2. internal pressure and physical properties when plasticity begins from outside

n	m	L	q	$a(m)$	$P_a(Pa)$
2.9	-3.4	-2.9	-2.2	0.4	8×10^{8}

To find the distribution of strain, stress and plastic zone curves primarily a system of equations consisting of Eqs.(24, 25, 30, 34, 35 and 49) should be solved simultaneously. There are five unknowns A_1 , A_2 , Q_1 , C_3 and r_p in these equations. By using, the prescribed boundary conditions as below the unknown parameters can be obtained.

$$\sigma_r^e(a)=P_a \ ; \ \sigma_r^p(b)=P_b \ ; \ u^e(r^p)=u^p(r^p) \ ; \ \sigma_r^e(r^p)=\sigma_r^p(r^p) \ ; \ \sigma_\theta^e(r^p)=\sigma_\theta^p(r^p)$$

According to the conditions shown in Table (2) and the steady state temperature, the plots of different stress, plastic strain components and the size of yielded zone are provided in Figs. (2-4). It can be seen in the figures that the increasing of temperature causes the thickness of the yielded zone to be increased accordingly.

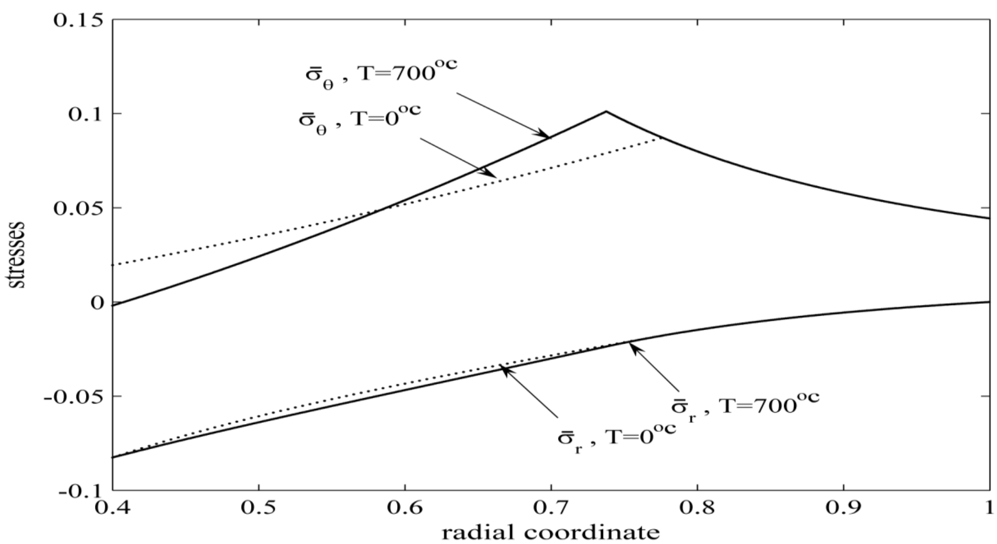

Fig. 2 The curve of stress vs. radial distance in different temperatures where plastic zone initiates from outside

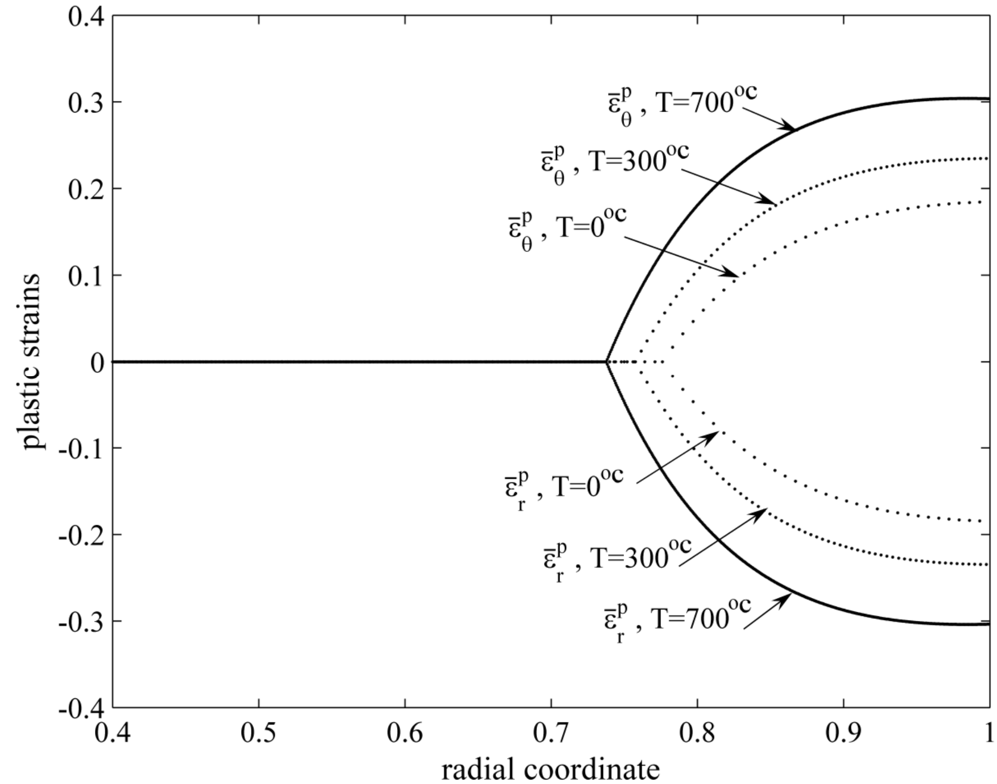

Fig. 3 A plot of strain components in different radii and some typical temperatures for an inhomogeneous cylinder with the material parameters given in Table 2, where plastic zone commences from outside.

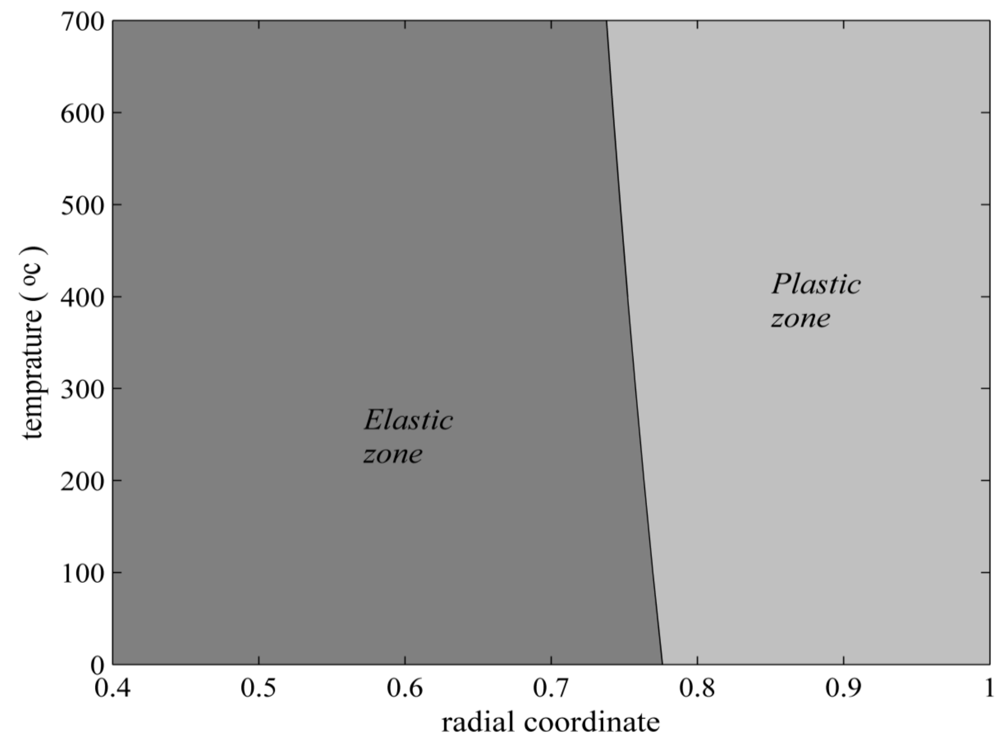

Fig. 4 The elastic-plastic interface line position in different temperatures where plastic zone starts from outside.

Once more, it can be seen that by increasing the temperature, the thickness of the yielded zone is increased.
The next situation is where plastic zone starts from an intermediate radius between inside and outside. For this case study the selected parameters are as those given in table 3.

Table 3. Internal pressure and material properties for the case where plastic zone starts from an intermediate radius

n	m	L	q	$a(m)$	$P_a(Pa)$
-4.2	-5.87	2.5	2.5	0.4	1.54×10^{10}

In this case, the solution is relatively more complex. In this case there are eight unknowns such that for the first elastic zone ($a < r < r_1^p$) the unknowns are r_1^p, A_1 and A_2, in the middle plastic zone where ($r_1^p \langle r \langle r_2^p$) the unknowns are Q_1 and C_3 and in the secondary elastic zone where ($r_2^p \langle r \langle b$) the unknowns are r_2^p, A_1 and A_2. These quantities can be found by using the eight following boundary conditions.

$$\sigma_r^e(a) = P_a\,;\ \sigma_r^e(b) = P_b\,;\ \sigma_r^e(r_1^p) = \sigma_r^p(r_1^p)\,;\ \sigma_\theta^e(r_1^p) = \sigma_\theta^p(r_1^p)$$

$$\sigma_r^e(r_2^p) = \sigma_r^p(r_2^p)\,;\ \sigma_\theta^e(r_2^p) = \sigma_\theta^p(r_2^p)\,;\ u^e(r_1^p) = u^p(r_1^p)\,;\ u^e(r_2^p) = u^p(r_2^p)$$

In the aforementioned equations the r_1^p is the first elastic plastic interface line from inside and r_2^p designates the second interface line. Meanwhile e and p indices shows the elastic and plastic deformations. For the prescribed magnitude of the parameters, the plots of stress and strains are given in Figure 5 and 6 respectively. Additionally in Figure 7, it can be seen that by increasing of temperature the interface is changed.

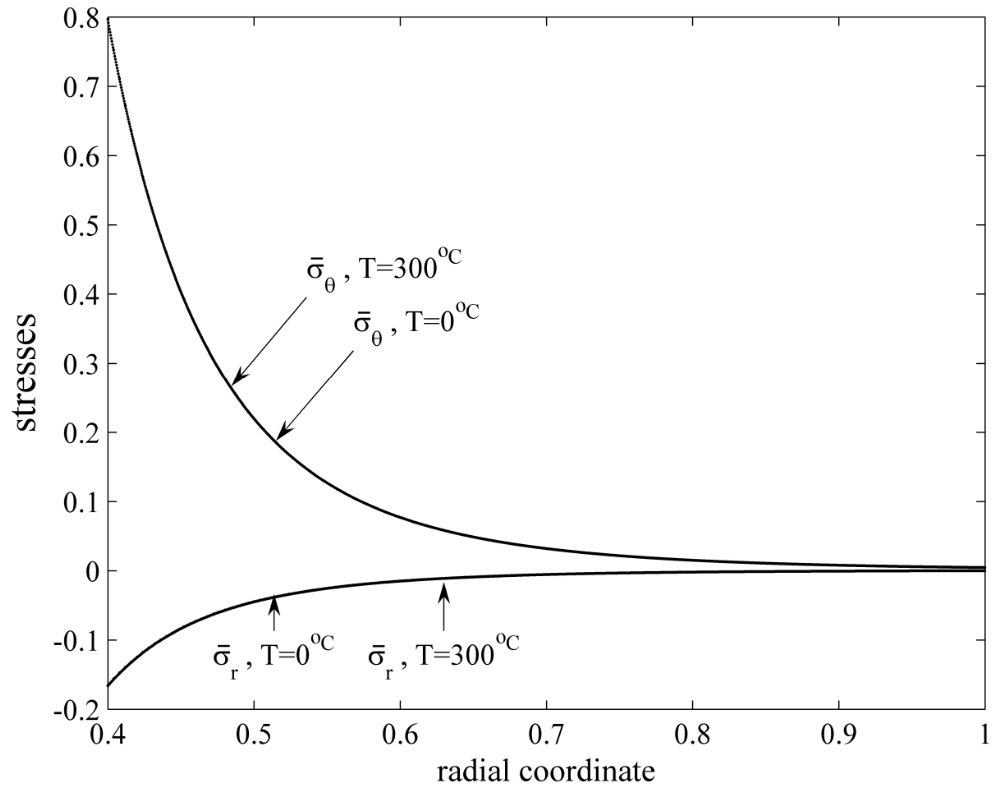

Fig. 5 The plot of stress vs. radius of the cylindrical vessel in different temperature where plastic zone starts in an intermediate radius between inside and outside.

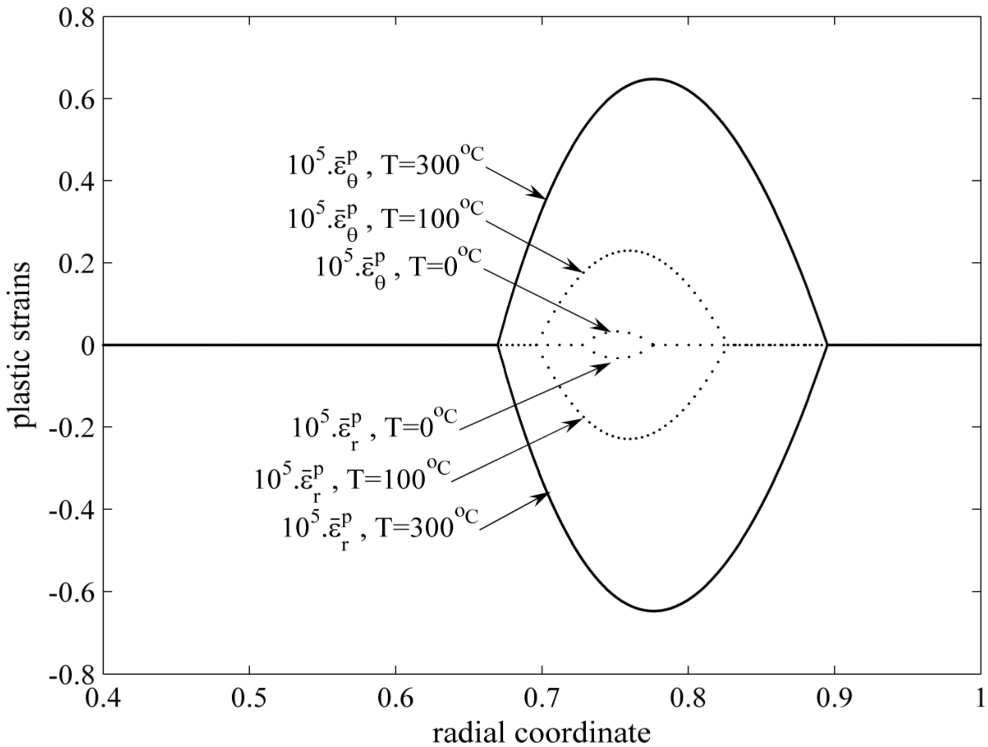

Fig. 6 Strain vs. radius of the cylindrical vessel in different temperature where plastic zone starts in an intermediate radius between inside and outside.

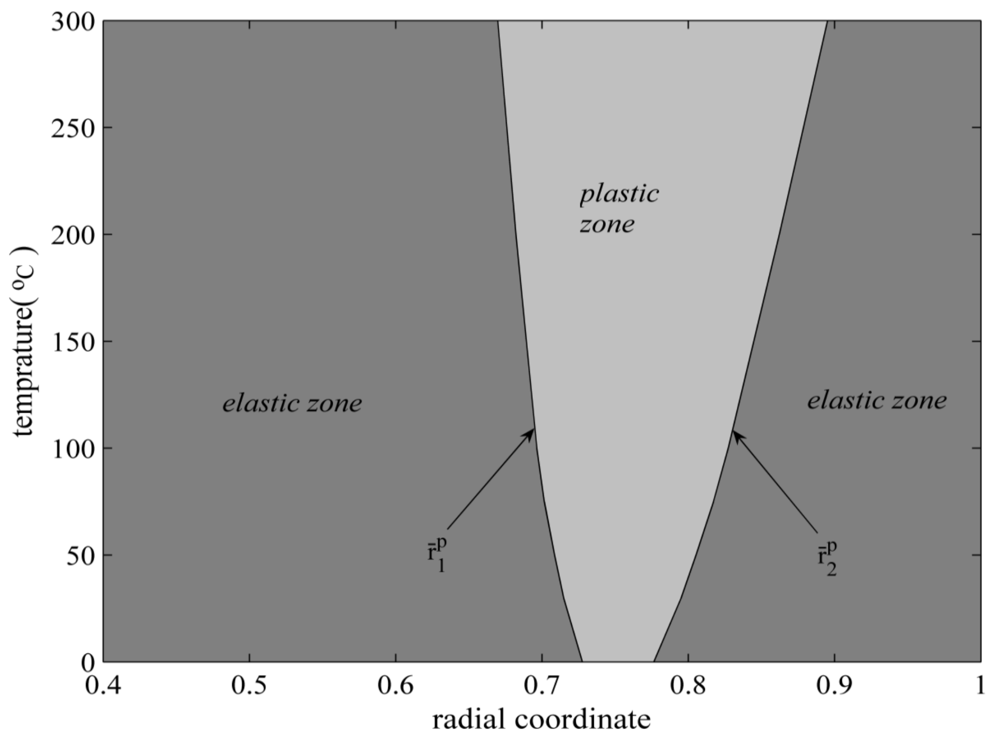

Fig. 7 The position of interface line vs. temperature where plastic zone starts in an intermediate radius between inside and outside.

4. Yield pattern monograph

As it is explained in the previous section, there may be different patterns for the commencement of plastic zone. That is the plastic zone may initiates from inside, outside or an intermediate radius. Fig.(8) shows a special monograph in which according to the material composition indices, introduced in Eqs. (1 and 2), the probable pattern of yielding is illustrated. In this figure the signs (i), (ii) and (iii) show the pattern of yielding from inside, outside and an intermediate radius of the vessel. It needs to be emphasized that the information provided in Fig.(8) is restricted to the first yielding incidence. On the other hand, the increase of internal pressure or temperature may cause additional plastic zones to be appeared.

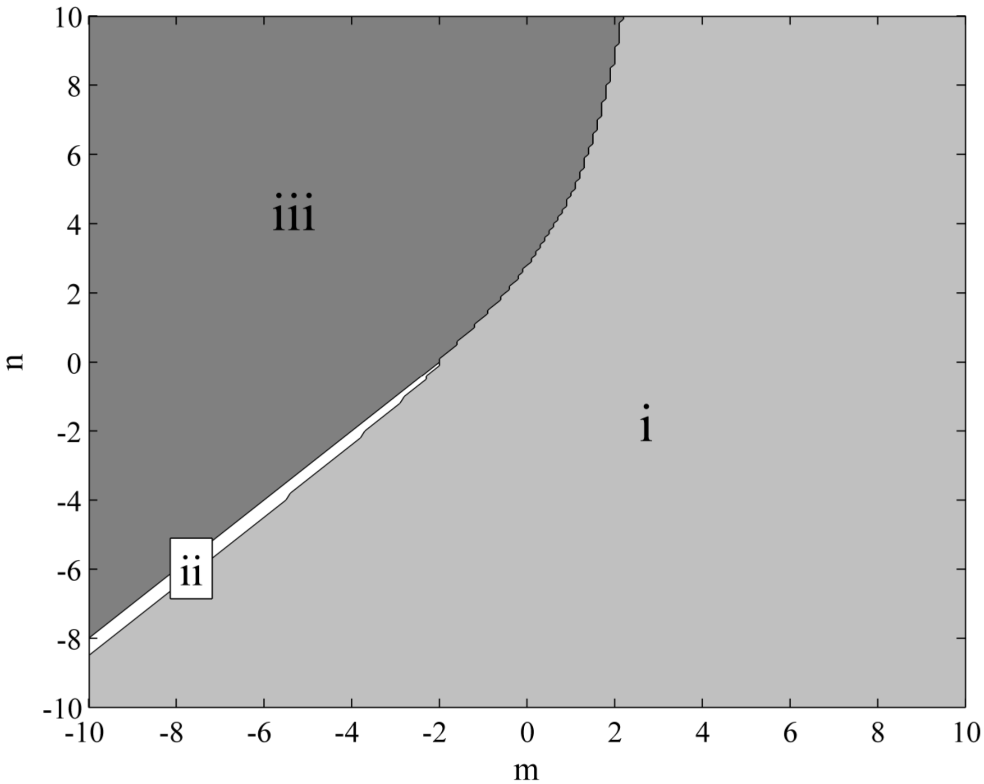

Fig. 8 The monograph of yielding pattern in different m and n material exponents

5. Conclusion

In this paper, different effect of temperature and pressure upon the distribution of stress and plastic deformation on a cylindrical vessel made of inhomogeneous material is studied. The Tresca's failure criterion and small deformation theory is used for the elasto-plastic analysis. Two main features of the material properties, i.e. elastic modulus and yield strength are taken to be some power functions of radius and temperature invariant. Poisson's ratio is taken to be a constant irrelevant to temperature. Based on the analysis represented in this paper the temperature as well as the pressure can influence the position and growth of the plastic zone interface line. In contrary to the homogeneous materials this time the expansion of the plastic zone is not necessarily from inside and towards the outside. The analysis shows that different modes of yield patterns are feasible. In the case of inhomogeneous material, plastic zone may starts from inside, from outside or from an intermediate radius between inside and outside. In this case depended on the composition of the material parameters yet the plastic zone may grow with different rates towards inside or outside wall of the vessel.

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