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Function Spaces of Topological Inverse Semigroups

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ABSTRACT

Let S be topological inverse semigroup and G_S be the topological maximal subgroup of S. Following Munn [12], we have $\frac{S}{\rho} \simeq G_S$. In this paper we characterize the universal $\mathcal{P}$-compactification $S^{\mathcal{P}}$ of S relative to the universal $\mathcal{P}$-compactification $G_S^{\mathcal{P}}$ of G_S . As a consequence, we give some interesting results as $G_S^{sap} \simeq \frac{S^{sap}}{\hat{\rho}}$.

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1. INTRODUCTION

The notion of semigroup compactification as a generalization of almost periodic compactification was initiated by Weil [16, 17]. Semigroup compactifications are extremely useful tools in characterizing function spaces on topological semigroups and in particular those spaces associated with semigroup compactifications; See [1, 4, 5, 8, 14, 15] for instance. A large class of semigroups which has been studied extensively from various points of view is inverse semigroups. The symmetries of a local nature, and applications of inverse semigroups crop up almost everywhere in mathematics-evidence of this is the number of related textbooks in analysis, geometry, topology, algebra, category theory, etc, [2, 6, 9, 10, 13]. These facts led to the motivation to study functions paces of inverse semigroups.

This paper is organized as follows: In Section 2, we introduce our notations. In Sections 3, we investigate to the compactification spaces of $\frac{S}{\rho}$ where S is a topological inverse semigroup and then we characterize the spaces of functions of S relative to its maximal subgroup.

2. Preleminaries

In this paper all topological spaces will be assumed to be Hausdorff. We shall follow the terminology of [1, 3, 7]. For a semigroup S, the right translation ρ_s and the left translation λ_s on S are defined by $\rho_t(\mathrm{s}) = \mathrm{st} = \lambda_s(\mathrm{t})$, $(s, t \in S)$. A semigroup S, equipped with a topology, is said to be right topological if all of the right translations are continuous, semitopological if all of the left and right translations are continuous. Suppose S is a semitopological semigroup and (ψ, X) is a semigroup compactification of S that is, is a compact Hausdorff right topological semigroup and $\psi : S \to X$ is a continuous homomorphism such that $\overline{\psi(S)} = X$ a$\psi(S) \subseteq \Lambda(X)$ where $\Lambda(X) = \{t \in X : s \to ts : X \to X, \text{ is continuous}\}$ is the topological center of X.

We say that (ψ, X) has the left [right] joint continuity property if the mapping $(s, x) \to \psi(s)x$ $[(x, s) \to x\psi(s)]$ is continuous. The space of all bounded continuous complex valued functions on S is denoted by $C(S)$\$. For $f \in C(S)$ and $s \in S$ the right (respectively, left) translation of f by s is the function $R_s f = f o \rho_s$ (respectively, $L_s f = f o \lambda_s$). A left translation invariant unital C^*-subalgebra $\mathcal{F}$ of $C(S)$ (i.e., $L_s f \in \mathcal{F}$ for all $s \in S$ and $f \in \mathcal{F}$) is called m-admissible if the function $s \to (T_\mu)(s) = \mu(L_s(f))$ belongs to $\mathcal{F}$ for all $f \in \mathcal{F}$ and $\mu \in S^{\mathcal{F}}$ (=the spectrum of $\mathcal{F}$). If $\mathcal{F}$ is m-admissible then $S^{\mathcal{F}}$ under the multiplication $\mu\nu = \mu o T_\nu$ $(\mu, \nu \in S^{\mathcal{F}})$, furnished with the Gelfand topology is a compact Hausdorff right topological semigroup and it makes $S^{\mathcal{F}}$ a compactification (called the $\mathcal{F}-$compactification) of S.

Let S' and S'' be compactifications of S. Then S' is a factor of S'' if the identity map on S has an extension $\varphi : S'' \to S'$. A compactification with a given property $\mathcal{P}$ is called a $\mathcal{P}-$ compactification. A universal $\mathcal{P}-$ compactification of S is a $\mathcal{P}$-compactification of which, every $\mathcal{P}$-compactification of S is a factor. Universaln $\mathcal{P}$-compactifications, if they exist, are unique (up to isomorphism). We denote the universal $\mathcal{P}$-compactification of S

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by $S^{\mathcal{P}}$. If S is a semigroup, then by $E(S)$ we denote the subset of idempotents of S. A semigroup S is called inverse if for any $x \in S$ there exists a unique $x^* \in S$ such that $xx^*x = x$ and $x^*xx^* = x^*$. An element x^* of S is called inverse to x and is denoted by x^{-1}. If S is an inverse semigroup, then the map which takes $x \in S$ to the inverse element of x is called the inversion. A topological (semitopological) inverse semigroup is a topological (semitopological) semigroup S that is algebraically an inverse semigroup with continuous inversion. Obviously, any topological (inverse) semigroup is a semitopological (inverse) semigroup.

3 compactification of topological inverse semigroup

Let S be a inverse semigroup, for every s_1, s_2 in S, define the relation ρ on $S \times S$ by $s_1 \rho s_2$ if and only if $s_1 e = s_2 e$ for some $e \in E(S)$. Following Munn [12], ρ is a congruence on $S \times S$ and $\frac{S}{\rho} \simeq G_S$ where G_S is a maximal subgroup of S . In this section by using this method we consider the structure of the compactification spaces of topological inverse semigroup. We fix these notations for the rest of this section.

Lemma 3.1. *Let S be a topological (compact) inverse semigroup with compact $E(S)$, then $G_S \simeq \frac{S}{\rho}$ is a topological (compact) group.*

Proof. First, we show that ρ is closed, let $\{x_\alpha\}$ and $\{y_\alpha\}$ be nets in S such that $x_\alpha \to x$, $y_\alpha \to y$ and $x_\alpha \rho y_\alpha$. Then there exists $\{e_\alpha\}$ in $E(S)$ such that $x_\alpha e_\alpha = y_\alpha e_\alpha$. Compactness of $E(S)$ allows us to assume that $e_\alpha \to e$ for some $e \in E(S)$ and by joint continuity of product of S, $x_\alpha e_\alpha \to xe,\ y_\alpha e_\alpha \to ye$. Therefore $xe = ye$, that is $x\rho y$. Now by proposition 1.3.8 [5], G_S is a topological (compact) group.

The first author in [16] have shown that if Ω is an extension of G by S, i.e. $\frac{\Omega}{G} \simeq S$ where G is a topological group and $S = M^0(G,P)$ is completely 0-simple semigroup, then $\frac{\Omega}{G} = \frac{\Omega}{\rho} \simeq S$ where ρ is suitable congruence on Ω . Then it is shown that $s^{ap} \simeq \frac{\Omega^{ap}}{\hat{G}}$, $s^{sap} \simeq \frac{\Omega^{sap}}{\hat{G}}$ and $s^p \simeq \frac{\Omega^p}{\hat{G}}$ where p is a property of compactification. On the other hand $G_S \simeq \frac{S}{\rho}$ where S is a topological inverse semigroup. In this setting there is a natural question' whether there is the similar results for topological inverse semigroup?.

In the following theorems we will show that the analogues results are hold for topological inverse semigroups.

Theorem 3.2. *Suppose S is a topological inverse semigroup with compact $E(S)$. Suppose G_S is the maximal subgroup o f S and $\tau = \{(x_1,x_2) \in X \times X \mid \exists e \in E(S),\ x_1\psi(e) = x_2\psi(e)\}$. Then $\frac{X}{\tau}$ is a topological group compactification of G_S where (ψ, X) is a topological group compactification of S.*

Proof. It is not far to see that τ is a congruence. Suppose $\{(x_\alpha, y_\alpha)\}$is a net in τsuch that $(x_\alpha, y_\alpha) \to (x,y)$. Then for each α, there exist $e_\alpha \in E(S)$ such that $x_\alpha\psi(e_\alpha) = x_\alpha\psi(e_\alpha)$. Since $E(S)$ is compact so there exists $e \in E(S)$ such that $e_\alpha \to e$. Now $x_\alpha\psi(e_\alpha) \to x\psi(e), y_\alpha\psi(e_\alpha) \to y\psi(e)$. This implies that $x\tau y$. Thus τ is closed congruence. Now Proposition 1.3.8 [5], shows that $\frac{X}{\tau}$ is a compact Hausdorff topological group. It is clear that if $s_1\rho s_2$ $(s_1, s_2 \in S)$, then $\psi(s_1)\tau\psi(s_2)$. Thus ψ preserves congruence so there exists a continuous homomorphism $\hat{\psi} : \frac{S}{\rho} \to \frac{X}{\tau}$ such that $\hat{\pi} o \psi(S) = \hat{\psi} o \pi(S)$ where $\pi : S \to \frac{S}{\rho}$, $\hat{\pi} : X \to \frac{X}{\tau}$ are the natural quotient maps. Thus

$$\overline{\hat{\psi}(\tfrac{S}{\rho})} = \overline{\hat{\psi} o \pi(S)} = \overline{\hat{\pi} o \psi(S)} \supseteq \hat{\pi}(\overline{\psi(S)}) = \hat{\pi}(X) = \tfrac{X}{\tau}$$

and

$$\hat{\psi}(\tfrac{S}{\rho}) = \hat{\psi} o \pi(S) = \hat{\pi} o \psi(S) \subseteq \hat{\pi}(\Lambda(X)) = \Lambda(\hat{\pi}(X)) = \Lambda(\tfrac{X}{\tau})$$

This implies that $\frac{X}{\tau}$ is a compactification of $\frac{S}{\rho} \simeq G_S$.

Theorem3.3. *Suppose S is a topological inverse semigroup with compact $E(S)$ and G_S is the maximal subgroup of S. Let $(\varepsilon_{G_S}, G_S^{sap})$ and (ε_S, S^{sap}) be the strongly almost periodic compactifications of G_S and S respectively . Then $G_S^{sap} \simeq \frac{S^{sap}}{\tau}$.*

Proof. In Theorem 3.2 put $X = S^{sap}$, then $(\hat{\varepsilon_S}, \frac{S^{sap}}{\tau})$ is a topological group compactification of $\frac{S}{\rho} \simeq G_S$, where $\tau = \{(x_1,x_2) \in S^{sap} \times S^{sap} \mid \exists e \in E(S),\ x_1\varepsilon_S(e) = x_2\varepsilon_S(e)\}$ .Now $\hat{\varepsilon_S} : S \to \frac{S^{sap}}{\tau}$ and

$\varepsilon_{G_S} : S \to \frac{S}{\rho}^{sap} \simeq G_S^{sap}$ so the universal property of the sap-compactification $(\varepsilon_{G_S}, G_S^{sap})$, [Theorem 4.3.7, 1] shows that there exists a continuous homomorphism $\theta : G_S^{sap} \longrightarrow \frac{S^{sap}}{\tau}$ such that $\theta \circ \varepsilon_{G_S}(s) = \hat{\varepsilon_S}(s)\,(s \in S)$. Since $\varepsilon_{G_S} \circ \pi : S \to G_S^{sap}$ is continuous homomorphism so the universal property of S^{sap} implies that there exists a continuous homomorphism $\mu : S^{sap} \to G_S^{sap}$ such that $\mu \circ \varepsilon_S(s) = \varepsilon_{G_S} \circ \pi(s)\,(s \in S)$. We have μ preserves congruence for if $\hat{\sigma_1}\tau\hat{\sigma_2}$ $(\hat{\sigma_1}, \hat{\sigma_2} \in S^{sap})$ then we can choose nets $\{u_\alpha\}$, $\{v_\alpha\}$ in S such that $\lim_\alpha \varepsilon_S(u_\alpha) = \hat{\sigma_1}$, $\lim_\alpha \varepsilon_S(v_\alpha) = \hat{\sigma_2}$ and there exists $e \in E(S)$ such that $\hat{\sigma_1}\varepsilon_S(e) = \hat{\sigma_2}\varepsilon_S(e)$ and since S^{sap} is a topological group we have $\hat{\sigma_1} = \hat{\sigma_2}\varepsilon_S(e)(\varepsilon_S(e))^{-1} = \hat{\sigma_2}\varepsilon_S(e^2)$. Now

$$\begin{aligned}\mu(\hat{\sigma_1}) &= \mu(\hat{\sigma_2}\varepsilon_S(e^2)) = \mu(\sigma_2)\mu(\varepsilon_S(e^2)) = \mu(\sigma_2)\varepsilon_{G_S} \circ \pi(e^2) \\ &= \mu(\hat{\sigma_2})\end{aligned}$$

Thus there exists continuous homomorphism $\psi : \frac{S^{sap}}{\tau} \longrightarrow G_S^{sap}$ such that $\psi \circ \omega(p) = \mu(p)\,(p \in S^{sap})$ where $\omega : S^{sap} \to \frac{S^{sap}}{\tau}$ is the quotient map. It is remains to show that $\theta : G_S^{sap} \to \frac{S^{sap}}{\tau}$ is isomorphism. Let $\omega(t) \in \frac{S^{sap}}{\tau}$, so there exists a net $\{s_\alpha\}$ in S such that $\lim_\alpha \varepsilon_S(s_\alpha) = t$. By the above relations we have

$$\begin{aligned}\theta \circ \psi(\omega(t)) &= \theta \circ \mu(t) = \lim_\alpha \theta \circ \mu(\varepsilon_S(s_\alpha)) = \lim_\alpha \theta \circ \varepsilon_{G_S} \circ \pi(s_\alpha) \\ &= \lim_\alpha \hat{\varepsilon_S} \circ \pi(s_\alpha) \lim_\alpha \omega(\varepsilon_S(s_\alpha)) = \omega(\lim_\alpha \varepsilon_S(s_\alpha)) = \omega(t).\end{aligned} \tag{1}$$

This implies that $\theta \circ \psi = id_{\frac{S^{sap}}{\tau}}$. By the similar calculation we can conclude that $\psi \circ \theta = id_{G_S^{sap}}$. Thus θ is isomorphism $G_S^{sap} \simeq \frac{S^{sap}}{\tau}$.

By using the analogous method of Theorems 3.3 we can obtain the similar result in general.

Corollary 3.4. With the assumptions of the preceding theorem, let $(\varepsilon_{G_S}, G_S^{\mathcal{P}})$, $(\varepsilon_S, S^{\mathcal{P}})$ be the universal t group $\mathcal{P}$ -compactifications of G_S and S, respectively. Then $G_S^{\mathcal{P}} \simeq \frac{S^{\mathcal{P}}}{\tau}$ if $\mathcal{P}$ has joint continuity property.

Conclusion

Characterization of the function spaces of topological semigroup was considered by many researchers , see [1, 4, 5, 15, 16] for example. On the other hand, It is known that $\frac{S}{\rho} \simeq G_S$ where S is an inverse semigroup and G_S is a maximal subgroup S [12]. In this paper we showed that G_S is a topological group where S is a topological inverse semigroup and then by using the suitable congruence on the compactification space S we characterized the strongly almost periodic compactification and universal $\mathcal{P}$ -compactification of S where $\mathcal{P}$ is property of compactification.

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REFERENCES

[1] J.F. Berglund, H.D. Junhenn and P. Milnes, 1989. Analysis on Semigroups: Functions spaces, Compactifications, Representations, John Wiley \& Sons, New York.

[2] A. Cannas da Silva, A. Weinstein, 1999. Lectures on Geometric Models for Noncommutative Algebras, Berkeley Math. Lect. 10, Amer. Math. Soc.

[3] A. H. Clifford and J. B. Preston, 1961. The Algebraic Theory of Semigroups I , American Mathematical Society Surveys 7.

[4] S. Ferri and D. Strauss, 2004. A note on the WAP-compactification and the LUC-compactification of a topological group, Semigroup Forum, pp. 87-101.

[5] H. D. Junghenn, P. Milnes, 2002. Almost periodic compactifications of group extensions, Czechoslovak Mathematical Journal, Vol. 52, No. 2, pp. 237-254.

[6] P.J. Higgins, 1971. Categories and Groupoids, Reprints in Theory and Applications of Categories. No. 7, pp. 1-195.

[8] J.M. Howie, 1995. Fundamentals of semigroup theory, Clarendon Press, Oxford.

[9] J.D. Lawson, 1992. Flows and compactifications, J. London Math. Soc. (2) 46, 349-363.

[10] M.V. Lawson, 1998. Inverse Semigroups : The Theory of Partial Symmetries, World Sci.

[11] K. Mackenzie, 1987. Lie Groupoids and Lie Algebroids in Dikerential Geometry, London Math. Soc. Lect. Note Ser. 124, Cambridge University Press.

[12] D. Munn and R. Penrose, 1955. A note on inverse semigroups,Proc. Cambridge Philos. Soc. 51, 396-399

[13] W. D. Munn, 1961. A class of irreducible matrix representations of an arbitrary inverse semigroup, Proc. Glasgow Math. Assoc. 5, 41-18.

[14] A.L.T. Paterson,1999. Groupoids, Inverse Semigroups, and their Operator Algebras, Birkh.auser.

[15] H. Rahimi, 2011. Function spaces on tensor product of semigroups, Iranian Journal of Sci.Tech. A3, 223-228

[16] H. Rahimi, 2012. Function spaces of Ress matrix semigroups, Bulletin of the Iranian MathSoc. Vol. 38 No. 1, pp 27-38

[17] A. Weil, 1935. Sur les fonctions presque periodiques de von Neumann,C. R. Acad.Sci. Paris 200,38-40.

[18] A. Weil, 1937. Sur les espaces a structure uniforme et sur la topologie generale, Her.mann, Paris.