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A Comparison of Homotopy Perturbation Method (HPM),Adomian's Decomposition Method (ADM) and Homotopy Analysis Method (HAM)in Solving Gardner Equation

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ABSTRACT

The adiabatic parameter dynamics of solitons due to The Gardner's equation is very widely studied in different areas of Physics: Plasma Physics, Fluid Dynamics Quantum Field Theory, Solid State Physics and others. The Gardner's equation is famous as the mixed KdV-mKdV equation. The objective and goal of this paper is to present the analytical Solution of Gardner equation, one of the newest, powerfuland easy-to-use analytical methods is the homotopy perturbation method (HPM), which isapplied in this paper to solve Gardner equation with high nonlinearity order. Then, we solve Gardner equation with the homotopy analysismethod (HAM) and Adomian's decomposition method (ADM) and obtain analytical solution. In the end, we compared the results with each other.The homotopy analysis method (HAM)contains the auxiliary parameter ħ that provides us to adjust and control the convergence region of solution Series. .The study has highlighted the efficiency and capability of aforementioned methods in solving Gardner equation which has risen from a number of important physical phenomenon's.

KEYWORDS:homotopy perturbation Method, Adomian's Decomposition Method, homotopy analysis method, Gardner equation, HPM, HAM, ADM,KdV-mKdV equation

1. INTRODUCTION

The mathematical theory of nonlinear evolution equations were starting from theKorteweg-de Vries (KdV) equation and the modified Korteweg-de Vries (mKdV)equation. These theorieswere an importantfield of research for researcher in the past decades [1-10]. Korteweg and Vries formodeling Russell's phenomenon of solitons derived KdV equation [11] like shallow water waves with small but finite amplitudes [12]. Solitons are localized waves that propagate without change of its formation and velocity properties and stable versusreciprocalimpact [13]. It has been used to qualify important of physical phenomena such as magneto hydrodynamics waves in warm plasma and ion-acoustic waves [14]. The Gardner's equation, that is known as the mixed KdV-mKdV equationis studied in various areas of Physics includes Plasma Physics,Fluid Dynamics, Quantum Field Theory, Solid State Physics and others. This Gardner equation shows up, particularly, in the basis of internal gravitywaves in a density-stratified ocean. This is usually described by theKdV equations and its versions with small nonlinearity. However, there aresituations when waves with powerful nonlinearity are experienced, similarto the case ofCoastal Ocean Probe Experiment during 1995 in the Oregon Bay, the problemof creating ansufficient theoretical model was imagined necessary. This lead to the study of the Gardner equation [1],the dimensionless form of Gardner equationwill be studied in this paper, for real $a$and b, is given by

$$\frac{\partial U}{\partial t}+ a\times u\frac{\partial u}{\partial x}+b\times u^2\ \frac{\partial u}{\partial x}+\frac{\partial_u^3}{\partial x^3}=0 \quad (1)$$

Many standard methods for solving nonlinear partial differential equations are presented [15]; for example, Backland transformation method [16], Lie group method [17] and Adomian's decomposition method [18], inverse scattering method [19], Hirota's bilinear method [20], homogeneous balance method [21], He's homotopy perturbation method (HPM) [22-27, 28]and VIM [29, 30, 31, 32];Many researchers chose the similar kinds of problems for study in other applications [33-37] and many strong methods have been proposed to obtain the exact solutions of nonlinear differential equations; for example, Backlund transformation [38],[39], Darboux transformation[40] and the inverse scattering method [41].

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The homotopy perturbation method (HPM)was established by He [42-45]. The method is a strong and effective technique to find the solutions of Non-linear and linear equations. The homotopy perturbation method is coupling of the perturbation and homotopy methods. This method can take the advantages of the conventional perturbation method while deleting its restrictions. HPM has been used by many authors [46-49] solve many types of linear and non-linear equations in science and engineering.In this paper, homotopy perturbation method (HPM) [50-52]and Liao's homotopy analysis method (HAM) [53] are used to conduct an analytic study on the Gardner equation and then compared with Adomian's decomposition method (ADM) in order to show all the Methods above are capable and useful in solving a large number of Linear or nonlinear differential equations, also all the aforementioned methods give rapidly convergent successive approximations, approximations can be used for numerical purposes.

2. Basic idea of homotopy perturbation method

The homotopy perturbation method[54, 55, 56, 57, 58, 59, 60] is combination of the classical perturbation technique and homotopy technique. To explain the basic idea of the HPM for solving nonlinear differential equations, we consider the following nonlinear differential equation:

$$A(u) - f(r) = 0, \; r \in \Omega, \qquad (2)$$

Subject to boundary condition

$$B(u, \partial u/\partial n) = 0, \; r \in \Gamma, (3)$$

Where **A** is a general differential operator, **B** a boundary operator, $\mathbf{f(r)}$ is a known analytical function, $\mathbf{\Gamma}$ is the boundary of domain $\mathbf{\Omega}$ and $\mathbf{\partial u/\partial n}$ denotes differentiation along the normal drawn outwards from$\mathbf{\Omega}$. The operator **A** can, generally speaking, be divided into two parts: a linear part **L** and a nonlinear part**N**. Eq. (3) therefore can be rewritten as follows:

$$L(u) + N(u) - f(r) = 0. \; (4)$$

In case the nonlinear Eq. (2) has no "small parameter",We can construct the following homotopy,

$$H(v, p) = L(v) - L(u_0) + pL(u_0) + p\,(N(v) - f(r)) = 0 (5)$$

$p$is called homotopy parameter.According to the homotopy perturbation method, the approximation solution of Eq. (5) can be expressed as a series of the power of **p**, i.e.

$$v = p^0 v_0 + p^1 v_1 + p^2 v_2 + \cdots, \qquad (6)$$
$$v = \lim_{p\to 1} v = v_0 + v_1 + v_2 + \cdots. \qquad (7)$$

When Eq. (5) corresponds to Eq. (2) and Eq. (7) becomes the approximate solution of Eq. (2).
In our case, for GardnerEq. (1) we obtain:

$$\frac{\partial U}{\partial t} + au\frac{\partial u}{\partial x} + bu^2 \frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3} = 0 \qquad (8)$$

Where:
aAnd **b** are constant

$$L(V) = \frac{\partial v}{\partial t} \; (9)$$
$$N(V) = av\frac{\partial v}{\partial x} + bv^2 \frac{\partial v}{\partial x} + \frac{\partial^3 v}{\partial x^3} \; (10)$$
$$f(r) = 0 (11)$$

Therefore:

$$H(V, P) = \frac{\partial v}{\partial t} - \frac{\partial u_0}{\partial t} + p\left(\frac{\partial u_0}{\partial t} + a\times v\frac{\partial v}{\partial x} + b\times v^2 \frac{\partial v}{\partial x} + \frac{\partial^3 v}{\partial x^3} - 0\right) \; (12)$$

Changing from u_0 to $\mathbf{u(r)}$. We consider **v**, as the following:

$$V = p^0 v_0 + p^1 v_1 + p^2 v_2 + p^3 v_3 + \cdots (13)$$

And the best approximation for the solution is:

$u = \lim_{p\to 1} (V)$

$V = v_0 + v_1 + v_2 + v_3 + \cdots$ (14)

Comparison of the expressions with the same powers of the parameter **p** gives the following equations:

$\mathbf{p^1}$: $\frac{\partial v_1}{\partial t} - \frac{\partial u_0}{\partial t} + a\times v_0 \frac{\partial v_0}{\partial x} + b\times (v_0)^2 \frac{\partial v_0}{\partial x} + \frac{\partial^3 v_0}{\partial x^3}$ (15)

$\mathbf{p^2}$: $\frac{\partial v_2}{\partial t} + a\times(v_0 \frac{\partial v_1}{\partial x} + v_1 \frac{\partial v_0}{\partial x}) + b\times (v_0^2 \frac{\partial v_1}{\partial x} + 2\times v_0 \times v_1 \times \frac{\partial v_0}{\partial x}) + \frac{\partial^3 v_1}{\partial x^3}$ (16)

$\mathbf{p^3}$: $\frac{\partial v_3}{\partial t} + a\times(v_0 \frac{\partial v_2}{\partial x} + v_1 \frac{\partial v_1}{\partial x} + v_2 \frac{\partial v_0}{\partial x}) + b\times (v_0^2 \frac{\partial v_2}{\partial x} + v_1^2 \frac{\partial v_0}{\partial x} + 2\times v_0 \times v_1 \times \frac{\partial v_1}{\partial x} + 2\times v_0 \times v_2 \times \frac{\partial v_0}{\partial x}) + \frac{\partial^3 v_2}{\partial x^3}$ (17)

$\mathbf{p^4}$: $\frac{\partial v_4}{\partial t} + a\times(v_0 \frac{\partial v_3}{\partial x} + v_1 \frac{\partial v_2}{\partial x} + v_2 \frac{\partial v_1}{\partial x} + v_3 \frac{\partial v_0}{\partial x}) + b\times (v_0^2 \frac{\partial v_3}{\partial x} + v_1^2 \frac{\partial v_1}{\partial x} + 2\times v_0 \times v_1 \times \frac{\partial v_2}{\partial x} + 2\times v_0 \times v_2 \times \frac{\partial v_1}{\partial x} + 2\times v_0 \times v_3 \times \frac{\partial v_0}{\partial x} + 2\times v_1 \times v_2 \times \frac{\partial v_0}{\partial x}) + \frac{\partial^3 v_3}{\partial x^3}$ (18)

$\mathbf{p^5}$: $\frac{\partial v_5}{\partial t} + a\times(v_0 \frac{\partial v_4}{\partial x} + v_1 \frac{\partial v_3}{\partial x} + v_2 \frac{\partial v_2}{\partial x} + v_3 \frac{\partial v_1}{\partial x} + v_4 \frac{\partial v_0}{\partial x}) + b\times (v_0^2 \frac{\partial v_4}{\partial x} + v_1^2 \frac{\partial v_2}{\partial x} + v_2^2 \frac{\partial v_0}{\partial x} + 2\times v_0 \times v_1 \times \frac{\partial v_3}{\partial x} + 2\times v_0 \times v_2 \times \frac{\partial v_2}{\partial x} + 2\times v_0 \times v_3 \times \frac{\partial v_1}{\partial x} + 2\times v_0 \times v_4 \times \frac{\partial v_0}{\partial x} + 2\times v_1 \times v_2 \times \frac{\partial v_1}{\partial x}) + \frac{\partial^3 v_4}{\partial x^3}$ (19)

$\mathbf{p^6}$: $\frac{\partial v_6}{\partial t} + a\times(v_0 \frac{\partial v_5}{\partial x} + v_1 \frac{\partial v_4}{\partial x} + v_2 \frac{\partial v_3}{\partial x} + v_3 \frac{\partial v_2}{\partial x} + v_4 \frac{\partial v_1}{\partial x} + v_5 \frac{\partial v_0}{\partial x}) + b\times (v_0^2 \frac{\partial v_5}{\partial x} + v_1^2 \frac{\partial v_3}{\partial x} + v_2^2 \frac{\partial v_1}{\partial x} + 2\times v_0 \times v_1 \times \frac{\partial v_4}{\partial x} + 2\times v_0 \times v_2 \times \frac{\partial v_3}{\partial x} + 2\times v_0 \times v_3 \times \frac{\partial v_2}{\partial x} + 2\times v_0 \times v_4 \times \frac{\partial v_1}{\partial x} + 2\times v_0 \times v_5 \times \frac{\partial v_0}{\partial x} + 2\times v_1 \times v_2 \times \frac{\partial v_2}{\partial x} + 2\times v_2 \times v_3 \times \frac{\partial v_0}{\partial x}) + \frac{\partial^3 v_5}{\partial x^3}$ (20)

$\mathbf{p^7}$: $\frac{\partial v_7}{\partial t} + a\times(v_0 \frac{\partial v_6}{\partial x} + v_1 \frac{\partial v_5}{\partial x} + v_2 \frac{\partial v_4}{\partial x} + v_3 \frac{\partial v_3}{\partial x} + v_4 \frac{\partial v_2}{\partial x} + v_5 \frac{\partial v_1}{\partial x} + v_6 \frac{\partial v_0}{\partial x}) + b\times (v_0^2 \frac{\partial v_6}{\partial x} + v_1^2 \frac{\partial v_4}{\partial x} + v_2^2 \frac{\partial v_2}{\partial x} + v_3^2 \frac{\partial v_0}{\partial x} + 2\times v_0 \times v_1 \times \frac{\partial v_5}{\partial x} + 2\times v_0 \times v_2 \times \frac{\partial v_4}{\partial x} + 2\times v_0 \times v_3 \times \frac{\partial v_3}{\partial x} + 2\times v_0 \times v_4 \times \frac{\partial v_2}{\partial x} + 2\times v_0 \times v_5 \times \frac{\partial v_1}{\partial x} + 2\times v_0 \times v_6 \times \frac{\partial v_0}{\partial x} + 2\times v_1 \times v_2 \times \frac{\partial v_3}{\partial x} + 2\times v_2 \times v_3 \times \frac{\partial v_1}{\partial x}) + \frac{\partial^3 v_6}{\partial x^3}$ (21)

$\mathbf{p^8}$: $\frac{\partial v_8}{\partial t} + a\times(v_0 \frac{\partial v_7}{\partial x} + v_1 \frac{\partial v_6}{\partial x} + v_2 \frac{\partial v_5}{\partial x} + v_3 \frac{\partial v_4}{\partial x} + v_4 \frac{\partial v_3}{\partial x} + v_5 \frac{\partial v_1}{\partial x} + v_6 \frac{\partial v_1}{\partial x}) + b\times (v_0^2 \frac{\partial v_7}{\partial x} + v_1^2 \frac{\partial v_5}{\partial x} + v_2^2 \frac{\partial v_3}{\partial x} + v_3^2 \frac{\partial v_1}{\partial x} + 2\times v_0 \times v_1 \times \frac{\partial v_6}{\partial x} + 2\times v_0 \times v_2 \times \frac{\partial v_5}{\partial x} + 2\times v_0 \times v_3 \times \frac{\partial v_4}{\partial x} + 2\times v_0 \times v_4 \times \frac{\partial v_3}{\partial x} + 2\times v_0 \times v_5 \times \frac{\partial v_2}{\partial x} + 2\times v_0 \times v_6 \times \frac{\partial v_1}{\partial x} + 2\times v_0 \times v_7 \times \frac{\partial v_0}{\partial x} + 2\times v_1 \times v_2 \times \frac{\partial v_4}{\partial x} + 2\times v_2 \times v_3 \times \frac{\partial v_2}{\partial x} + 2\times v_3 \times v_4 \times \frac{\partial v_0}{\partial x}) + \frac{\partial^3 v_7}{\partial x^3}$ (22)

The above partial differential equations must be supplemented by conditions ensuring a uniqueness of the solution. For above equations we assume the following conditions in the Example 1:

Application of Homotopy Perturbation Method

Example1

Consider the following Gardner equation with initial value problem [24]:

$\frac{\partial U}{\partial t} + a\, u\frac{\partial u}{\partial x} + bu^2 \frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3} = 0$ (23)

$u_0(x,0) = x$

$u_1(x,0) = 0$

$u_2(x,0) = 0$ (24)

$u_3(x,0) = 0$

.

.

$u_m(x,0) = 0 \quad m>0, m = n$

nis the order of **p**in Equation[13].
By Assuming**a=-1/6, b=-1/9, n =8**and above initial conditions a solution for equationssystem [15-22] is as follows:

Important Point: We assume **n=8** because after **8** iteration the sum of homotopy perturbation sentences converged

$u_1(x,y) = \frac{3}{18}xt + \frac{2}{18}x^2t$(25)

$u_2(x,y) = \frac{xt^2}{36} + \frac{x^2t^2}{18} + \frac{2x^3t^2}{81}$(26)

$u_3(x,y) = -\frac{4t^3}{81} + \frac{xt^3}{216} + \frac{x^2t^3}{54} + \frac{5x^3t^3}{243} + \frac{5x^4t^3}{729}$(27)

$u_4(x,y) = -\frac{8t^4}{243} - \frac{503xt^4}{11664} + \frac{5x^2t^4}{972} + \frac{5x^3y^4}{486} + \frac{35x^4t^4}{4374} + \frac{14x^5t^4}{6561}$(28)

$u_5(x,y) = -\frac{10t^5}{729} - \frac{15187xt^5}{349920} - \frac{2915x^2t^5}{104976} + \frac{14x^3t^5}{3645} + \frac{167x^4t^5}{32805} + \frac{20x^5t^5}{6561} + \frac{40x^6t^5}{59049}$(29)

$u_6(x,y) = -\frac{2251152}{510183360}t^6 - \frac{12415113}{510183360}xt^6 - \frac{18702252}{510183360}x^2t^6 - \frac{7739712}{510183360}x^3t^6 + \frac{1260360}{510183360}x^4t^6 + \frac{1253376}{510183360}x^5t^6 + \frac{611520}{510183360}x^6t^6 + \frac{116480}{510183360}x^7t^6$(30)

$u_7(x,y) = \frac{549257760}{42855402240}t^7 - \frac{449539308}{42855402240}xt^7 - \frac{1142072055}{42855402240}x^2t^7 - \frac{1111945806}{42855402240}x^3t^7 - \frac{335335032}{42855402240}x^4t^7 + \frac{59457024}{42855402240}x^5t^7 + \frac{48018432}{42855402240}x^6t^7 + \frac{19921920}{42855402240}x^7t^7 + \frac{3320320}{42855402240}x^8t^7$(31)

$u_8(x,y) = \frac{40720702968}{2057059307520}t^8 + \frac{47645361972}{2057059307520}xt^8 - \frac{28564465077}{2057059307520}x^2t^8 - \frac{46584544608}{2057059307520}x^3t^8 - \frac{33408580140}{2057059307520}x^4t^8 - \frac{7696898208}{2057059307520}x^5t^8 + \frac{1611614592}{2057059307520}x^6t^8 + \frac{1072800768}{2057059307520}x^7t^8 + \frac{383028480}{2057059307520}x^8t^8 + \frac{56744960}{2057059307520}x^9t^8$(32)

Having u_i, i = 0, 1,…., 8, the solution $u(x,t)$ is as:

U(x,t)=$x+\frac{3}{18}xt + \frac{2}{18}x^2t+\frac{xt^2}{36} + \frac{x^2t^2}{18} + \frac{2x^3t^2}{81} - \frac{4t^3}{81} + \frac{xt^3}{216} + \frac{x^2t^3}{54} + \frac{5x^3t^3}{243} + \frac{5x^4t^3}{729} - \frac{8t^4}{243} - \frac{503xt^4}{11664} + \frac{5x^2t^4}{972} + \frac{5x^3y^4}{486} + \frac{35x^4t^4}{4374} + \frac{14x^5t^4}{6561} - \frac{10t^5}{729} - \frac{15187xt^5}{349920} - \frac{2915x^2t^5}{104976} + \frac{14x^3t^5}{3645} + \frac{167x^4t^5}{32805} + \frac{20x^5t^5}{6561} + \frac{40x^6t^5}{59049} - \frac{2251152}{510183360}t^6 - \frac{12415113}{510183360}xt^6 - \frac{18702252}{510183360}x^2t^6 - \frac{7739712}{510183360}x^3t^6 + \frac{1260360}{510183360}x^4t^6 + \frac{1253376}{510183360}x^5t^6 + \frac{611520}{510183360}x^6t^6 + \frac{116480}{510183360}x^7t^6 + \frac{549257760}{42855402240}t^7 - \frac{449539308}{42855402240}xt^7 - \frac{1142072055}{42855402240}x^2t^7 - \frac{1111945806}{42855402240}x^3t^7 - \frac{335335032}{42855402240}x^4t^7 + \frac{59457024}{42855402240}x^5t^7 + \frac{48018432}{42855402240}x^6t^7 + \frac{19921920}{42855402240}x^7t^7 + \frac{3320320}{42855402240}x^8t^7 + \frac{40720702968}{2057059307520}t^8 + \frac{47645361972}{2057059307520}xt^8 - \frac{28564465077}{2057059307520}x^2t^8 - \frac{46584544608}{2057059307520}x^3t^8 - \frac{33408580140}{2057059307520}x^4t^8 - \frac{7696898208}{2057059307520}x^5t^8 + \frac{1611614592}{2057059307520}x^6t^8 + \frac{1072800768}{2057059307520}x^7t^8 + \frac{383028480}{2057059307520}x^8t^8 + \frac{56744960}{2057059307520}x^9t^8$(33)

3. Basic idea of Homotopy analysis method (HAM)

In this section we employ the homotopy analysis method [61-66] to the discussed problem. To show the basic idea,
Let us consider the following differential equations

$N[u(\tau)] = 0$(34)

- Where**N** is a nonlinear operator, $\boldsymbol{\tau}$ denotes independent variable, $\mathbf{u}(\boldsymbol{\tau})$ is an unknown function, respectively. For
- Simplicity, we ignore all boundary or initial conditions, which can be treated in the similar way. By means of

Generalizing the traditional homotopy method, Liao [67] constructs the so-called zero-order deformation equation

$(1-p)L[\phi(\tau;p)-u_0(\tau)]=p\hbar H(\tau)N[\phi(\tau;p)]$(35)

Where$p\in[0,1]$ is the embedding parameter, $\hbar\neq 0$ is a non-zero auxiliary parameter, $H(\tau)\neq 0$ is an auxiliary Function, L is an auxiliary linear operator, $u_0(\tau)$ is an initial guess of u (τ),$\phi(\tau;p)$is a unknown function, Respectively, it is important that one has great freedom to choose auxiliary things in HAM. Obviously, when$p=0$ And$p=1$, it holds

$\phi(\tau;0)=u_0(\tau)$
$\phi(\tau;1)=u(\tau)$(36)

Respectively, Thus as ρ increases from 0to1, the solution $\phi(\tau;p)$ varies from the initial guess $u_0(\tau)$ to the solution $u(\tau)$.Expanding $\phi(\tau;p)$in Taylor series with respect to**p**, one has

$\phi(\tau;p)=u_0(\tau)+\sum_{m=1}^{+\infty}u_m(\tau)p^m$(37)

$u_m(\tau)=\frac{1}{m!}\frac{\partial^m\phi(\tau;p)}{\partial p^m}\Big|_{p=0}$(38)

If the auxiliary linear operator, the initial guess, the auxiliary parameter**ħ**,and the auxiliary function are so properly Chosen, the series (2) converges at p =1, one has

$u(\tau)=u_0(\tau)+\sum_{m=1}^{+\infty}u_m(\tau)$(39)

Which must be one of solutions of original nonlinear equation, as proved by Liao [67],As$\hbar=-1$and$H(\tau)=1$, Eq. (35) Becomes

$(1-p)L[\phi(\tau;p)-u_0(\tau)]+pN[\phi(\tau;p)]=0$(40)

This is mostly used in HPM, whereas the solution can be obtained directly without using Taylor series [68, 69]. According to Eq. (41), the governing equation can be deduced from the zero-order deformation equation (35). The vector is defined as

$\underset{u_n}{\rightarrow}=\{u_0(\tau),u_1(\tau),\dots,u_n(\tau)\}$(41)

Differentiating Eq. (35) m times with respect to the embedding parameter p, and then setting p = 0 and Finally dividing them by m!We will have the so-called mth-order deformation equation as

$L[u_m(\tau)-x_m u_{m-1}(\tau)]=\hbar H(\tau)R_m(u_{m-1})$(42)

Where

$R_m(u_{m-1})=\frac{1}{(m-1)!}\frac{\partial^{m-1}N[\phi(\tau;p)]}{\partial p^{m-1}}\Big|_{p=0}$ (43)

And

$x_m=\begin{cases}0 & m\leq 1\\ 1 & m>1\end{cases}$(44)

It should be emphasized that $u_m(\tau)$ for m ≥ 1 is governed by the linear equation (38) with the linear boundary Conditions coming from the original problem, which can be easily solved using the symbolic computation Software.

Application of Homotopy analysis Method

In the following, we apply HAM to solve Gardner equation in the example 1:

$L(V)=\frac{\partial\theta}{\partial t}$(45)

$N(V) = a\times \theta \frac{\partial\theta}{\partial\tau} + b\theta^2 \frac{\partial\theta}{\partial\tau} + \frac{\partial_\theta^3}{\partial\tau^3}$ (46)
Review: a=-1/6, b=-1/9
Initial conditions:

Θ_0 (τ, 0)= τ
Θ_1 (τ, 0)=0
Θ_2(τ, 0)=0(47)
Θ_3 (τ, 0)=0
.
.
Θ_m (τ, 0)=0
To obey both the rule of solution expression and the rule of the coefficient ergodicity, the corresponding Auxiliary function can be determined uniquely $H(\tau) = 1$.Then

$(1-p)L[\phi(\tau;p) - u_0(\tau)] = p\hbar N[\phi(\tau;p)]$ (48)

Following the homotopy analysis method

$\Theta_1(\tau,t)=\frac{1}{18}(-3h\tau - 2h\tau^2)t$ (49)

$\Theta_2(\tau,t)=\frac{1}{36}h^2\tau t^2 + \frac{1}{18}h^2\tau^2 t^2 + \frac{2}{81}h^2\tau^3 t^2$ (50)

$\Theta_3(\tau,t)=\frac{288}{5832}h^3t^3 - \frac{27}{5832}\tau h^3t^3 - \frac{108}{5832}\tau^2 h^3t^3 - \frac{120}{5832}\tau^3h^3t^3 - \frac{40}{5832}\tau^4 h^3t^3$ (51)

$\Theta_4(\tau,t)= -\frac{3456}{104976}h^4t^4 - \frac{4527}{104976}\tau h^4t^4 + \frac{540}{104976}\tau^2 h^4t^4 + \frac{1080}{104976}\tau^3 h^4t^4 + \frac{840}{104976}\tau^4 h^4t^4 + \frac{224}{104976}\tau^5 h^4t^4$ (52)

$\Theta_5(\tau,t)= -h^5(\frac{3}{5}(-\frac{43200}{1889568} - \frac{135531}{1889568}\tau - \frac{86022}{1889568}\tau^2 + \frac{11520}{1889568}\tau^3 + \frac{14976}{1889568}\tau^4 + \frac{8800}{1889568}\tau^5 + \frac{1920}{1889568}\tau^6)t^5 + \tau^2(-\frac{864}{1889568} - \frac{495}{1889568}\tau + \frac{378}{1889568}\tau^2 + \frac{576}{1889568}\tau^3 + \frac{360}{1889568}\tau^4 + \frac{80}{1889568}\tau^5)t^5\tau)$ (53)

$\Theta_6(\tau,t)=h^6(\frac{1}{6}(-\frac{719280}{28343520} - \frac{3906819}{28343520}\tau - \frac{5792202}{28343520}\tau^2 - \frac{2347248}{28343520}\tau^3 + \frac{409320}{28343520}\tau^4 + \frac{404256}{28343520}\tau^5 + \frac{196000}{28343520}\tau^6 + \frac{37120}{28343520}\tau^7)t^6 + 2(\frac{4455}{28343520} - \frac{13608}{28343520}\tau - \frac{52488}{28343520}\tau^2 - \frac{65871}{28343520}\tau^3 - \frac{25140}{28343520}\tau^4 + \frac{1242}{28343520}\tau^5 + \frac{1302}{28343520}\tau^6 + \frac{640}{28343520}\tau^7 + \frac{120}{28343520}\tau^8)t^6\tau)$ (54)

$\Theta_7(\tau,t)= -h^3t^6(15\tau(-\frac{82944}{42855402240} - \frac{163701}{42855402240}\tau - \frac{53154}{42855402240}\tau^2 + \frac{67464}{42855402240}\tau^3 + \frac{100800}{42855402240}\tau^4 + \frac{69760}{42855402240}\tau^5 + \frac{17920}{42855402240}\tau^6)t + 4h^4((\frac{124797672}{42855402240} - \frac{108953019}{42855402240}\tau - \frac{274781160}{42855402240}\tau^2 - \frac{266017572}{42855402240}\tau^3 - \frac{79848996}{42855402240}\tau^4 + \frac{14048208}{42855402240}\tau^5 + \frac{11332944}{42855402240}\tau^6 + \frac{4720640}{42855402240}\tau^7 + \frac{794240}{42855402240}\tau^8)t + 7(\frac{7127433}{42855402240} + \frac{10796652}{42855402240}\tau - \frac{1905120}{42855402240}\tau^2 - \frac{4051701}{42855402240}\tau^3 - \frac{3475530}{42855402240}\tau^4 - \frac{985254}{42855402240}\tau^5 + \frac{89100}{42855402240}\tau^6 + \frac{71832}{42855402240}\tau^7 + \frac{29360}{42855402240}\tau^8 + \frac{4800}{42855402240}\tau^9)t\tau))$ (55)

After trying higher iterations with the unique and proper assignment of**ħ**, the results will be converged.

$\Theta(\tau,t)=\Theta_0(\tau,t)+\Theta_1(\tau,t)+\Theta_2(\tau,t)+\Theta_3(\tau,t)+\ldots+\Theta_m(\tau,t)$ (56)

$$\Theta(\tau,t)= \tau + \frac{1}{18}(-3h\tau - 2h\tau^2)t + \frac{1}{36}h^2\tau t^2 + \frac{1}{18}h^2\tau^2 t^2 + \frac{2}{81}h^2\tau^3 t^2 + \frac{288}{5832}h^3 t^3 - \frac{27}{5832}\tau h^3 t^3 - \frac{108}{5832}\tau^2 h^3 t^3 - \frac{120}{5832}\tau^3 h^3 t^3 - \frac{40}{5832}\tau^4 h^3 t^3 - \frac{3456}{104976}h^4 t^4 - \frac{4527}{104976}\tau h^4 t^4 + \frac{540}{104976}\tau^2 h^4 t^4 + \frac{1080}{104976}\tau^3 h^4 t^4 + \frac{840}{104976}\tau^4 h^4 t^4 + \frac{224}{104976}\tau^5 h^4 t^4 - h^5\left(\frac{3}{5}\left(-\frac{43200}{1889568} - \frac{135531}{1889568}\tau - \frac{86022}{1889568}\tau^2 + \frac{11520}{1889568}\tau^3 + \frac{14976}{1889568}\tau^4 + \frac{8800}{1889568}\tau^5 + \frac{1920}{1889568}\tau^6\right)t^5 + \tau^2\left(-\frac{864}{1889568} - \frac{495}{1889568}\tau + \frac{378}{1889568}\tau^2 + \frac{576}{1889568}\tau^3 + \frac{360}{1889568}\tau^4 + \frac{80}{1889568}\tau^5\right)t^5\tau\right) + h^6\Big(\frac{1}{6}\Big(-\frac{719280}{28343520} - \frac{3906819}{28343520}\tau - \frac{5792202}{28343520}\tau^2 - \frac{2347248}{28343520}\tau^3 + \frac{409320}{28343520}\tau^4 + \frac{404256}{28343520}\tau^5 + \frac{196000}{28343520}\tau^6 + \frac{37120}{28343520}\tau^7\Big)t^6 + 2\Big(\frac{4455}{28343520} - \frac{13608}{28343520}\tau - \frac{52488}{28343520}\tau^2 - \frac{65871}{28343520}\tau^3 - \frac{25140}{28343520}\tau^4 + \frac{1242}{28343520}\tau^5 + \frac{1302}{28343520}\tau^6 + \frac{640}{28343520}\tau^7 + \frac{120}{28343520}\tau^8\Big)t^6\tau\Big) - h^3 t^6\Big(15\tau\Big(-\frac{82944}{42855402240} - \frac{163701}{42855402240}\tau - \frac{53154}{42855402240}\tau^2 + \frac{67464}{42855402240}\tau^3 + \frac{100800}{42855402240}\tau^4 + \frac{69760}{42855402240}\tau^5 + \frac{17920}{42855402240}\tau^6\Big)t + 4h^4\Big(\Big(\frac{124797672}{42855402240} - \frac{108953019}{42855402240}\tau - \frac{274781160}{42855402240}\tau^2 - \frac{266017572}{42855402240}\tau^3 - \frac{79848996}{42855402240}\tau^4 + \frac{14048208}{42855402240}\tau^5 + \frac{11332944}{42855402240}\tau^6 + \frac{4720640}{42855402240}\tau^7 + \frac{794240}{42855402240}\tau^8\Big)t + 7\Big(\frac{7127433}{42855402240} + \frac{10796652}{42855402240}\tau - \frac{1905120}{42855402240}\tau^2 - \frac{4051701}{42855402240}\tau^3 - \frac{3475530}{42855402240}\tau^4 - \frac{985254}{42855402240}\tau^5 + \frac{89100}{42855402240}\tau^6 + \frac{71832}{42855402240}\tau^7 + \frac{29360}{42855402240}\tau^8 + \frac{4800}{42855402240}\tau^9\Big)t\tau\Big)\Big) \quad (57)$$

4. Description of the Adomian decomposition method

For the sake of generality, the Adomian's method is described as applied to a nonlinear differential equation $Fu = g$, Where F represents a nonlinear differential operator. The technique consists on decomposing the linear part of F into $L + R$, where L is an operator easily invertible and R is the remaining part. Representing the nonlinear term by N, the equation in canonical form is

$$Lu + Ru + Nu = g \quad (58)$$

Representing the inverse of the operator L as L^{-1}, one gets the following equivalent equation:

$$L^{-1}Lu = g - L^{-1}Ru - L^{-1}Nu \quad (59)$$

Being L the operator derivative of order n, one represents L^{-1} as the n-fold integration operator. Thus,

$$L^{-1}Lu = u + a \quad (60)$$

Where a is the term emerging from the integration and one gets

$$u = g - a - L^{-1}Ru - L^{-1}Nu \quad (61)$$

A series solution $u = \sum_{n=0}^{\infty} u_n$ is looked for. Identifying u_0 as $g - a$, the rest of the terms u_n, $n > 0$ will further be settled by a recursive relation. The key of the method is to decompose the nonlinear term Nu in the equation (61), into a particular series of polynomials $Nu = \sum_{n=0}^{\infty} A_n$, being A_n the so-called Adomian's polynomials. Each polynomial A_n depends only on $u_0, u_1, \ldots, u_n$. Adomian introduced formulae to generate these polynomials for all kinds of Nonlinearities [70, 71, 72, 73]. It has also been shown that the sum of the Adomian's polynomials is a generalization of the Taylor series in a neighborhood of a function u_0 rather than a point

$$Nu = \sum_{n=0}^{\infty} A_n = \sum_{n=0}^{\infty} \frac{1}{n!}(u - u_0)^n N^{(n)}(u_0) \quad (62)$$

Tending the general term of the series to zero very fast, as $\frac{1}{(mq)!}$, according to the optimal choice of the initial term, for m terms and q the order of the linear operator L [15] and [27].

Substituting $\mathbf{u} = \sum_{\mathbf{n=0}}^{\infty} \mathbf{u_n}$ and $\mathbf{Nu} = \sum_{\mathbf{n=0}}^{\infty} \mathbf{A_n}$ into Eq. (61) one gets

$$\sum_{n=0}^{\infty} u_n = g - a - L^{-1}R\sum_{n=0}^{\infty} u_n - L^{-1}\sum_{n=0}^{\infty} A_n \quad (63)$$

To determine the components $u_n(x,t)$, $n = 0,1,2,\ldots$, one can employ the recursive relation

$u_0 = g - a,$

$u_{1=}-L^{-1}Ru_0-L^{-1}A_0$(64)
$u_{2=}-L^{-1}Ru_1-L^{-1}A_1$
.
.
.
$u_{n+1=}-L^{-1}Ru_n-L^{-1}A_n$

Adomian's polynomials were formally introduced in [74, 75, 71, 72, 73], and expressed as

$A_n(u_0,u_1,\dots,u_n)=\frac{1}{n!}\left[\frac{d^n}{d\lambda^n}N(\sum_{i=0}^{\infty}\lambda^i u_i)\right]_{\lambda=0}$(65)
This formula is obtained by introducing, for the sake of convenience, the parameter$\boldsymbol{\lambda}$, and writing

$u(\lambda)=\sum_{n=0}^{\infty}\lambda^n u_n$ (66)

$N(u(\lambda))=\sum_{n=0}^{\infty}\lambda^n A_n$ (67)

Expanding in a Taylor series $\mathbf{N(u(\lambda))}$ in a neighborhood of $\boldsymbol{\lambda}$ =0 one gets

$N(u(\lambda))=\sum_{n=0}^{\infty}\frac{1}{n!}\left[\frac{d^n}{d\lambda^n}N(u(\lambda))\right]_{\lambda=0}\lambda^n$(68)

$=\sum_{n=0}^{\infty}\frac{1}{n!}\left[\frac{d^n}{d\lambda^n}N(\sum_{i=0}^{\infty}\lambda^i u_i)\right]_{\lambda=0}\lambda^n$(69)

From which Eq. (65) follows immediately.
Other methods have been developed for the calculation of Adomian's polynomials $\mathbf{A_n}$ [74, 76–78]. The next theorem [74, 79], allows the infinite series representing the Adomian's polynomials,$\mathbf{A_n}$, to be substituted by a finite sum, fact that allows its computation.
Theorem Adomian's polynomials A_n may be computed by the formula

$A_n=\frac{1}{n!}\left[\frac{d^n}{d\lambda^n}N(\sum_{i=0}^{\infty}\lambda^i u_i)\right]_{\lambda=0}$(70)

Application of Adomian decomposition method to Gardner equation

$\frac{\partial u}{\partial t}+a\,u\frac{\partial u}{\partial x}+bu^2\,\frac{\partial u}{\partial x}+\frac{\partial^3 u}{\partial x^3}=0$, $u(x,0)=u_0$ (x) =x(71)

Following Adomian, the linear operators expressed by Eqs. (72) Are defined.

$L_t(0)=\frac{\partial}{\partial t}(0),L_{xx}(0)=\frac{\partial^3}{\partial x^3}(0)$(72)
Applying the inverse operator of $L_t(0)=\frac{\partial}{\partial t}(0)$,$L_t{}^{-1}(0)=\int_0^t(0)\,dt$, to both sides of Gardner equation one obtains
$u(x,t)=u_0$ (x)+ $L_t{}^{-1}(-a\,u\frac{\partial u}{\partial x}-bu^2\,\frac{\partial u}{\partial x}-\frac{\partial^3 u}{\partial x^3})$(73)

According to Adomian's method, one assumes that the unknown function $u(x,t)$ can be expressed by an infinite Sum of components of the form

$u(x,t)=\sum_{n=0}^{\infty}u_n(x,t)$(74)

The nonlinear term $u\frac{\partial u}{\partial x}$ into an infinite series of Adomian's polynomials

$u\frac{\partial u}{\partial x}=\sum_{n=0}^{\infty}A_n$(75)
And the nonlinear term $u^2\,\frac{\partial u}{\partial x}$into an infinite series of Adomian's polynomials

$u^2 \frac{\partial u}{\partial x} = \sum_{n=0}^{\infty} B_n$(76)

Substituting Eqs.(74) And (75) and (76) into Eq. (73) one obtains

$\sum_{n=0}^{\infty} u_n = u_0(x) + {L_t}^{-1}(-\frac{\partial^3}{\partial x^3}\sum_{n=0}^{\infty} u_n - a \times \sum_{n=0}^{\infty} A_n - b \times \sum_{n=0}^{\infty} B_n)$(77)

To determine the components of $u_n(x,t)$, n = 0, 1, 2,…, Adomian's technique can employ the recursive relationdefined by

$u_0 = u_0$ (x),
$u_1 = {L_t}^{-1}(-\frac{\partial^3}{\partial x^3} u_0 - aA_0 - bB_0)$
$u_2 = {L_t}^{-1}(-\frac{\partial^3}{\partial x^3} u_1 - aA_1 - bB_1)$ (78)
.
.
.
$u_n = {L_t}^{-1}(-\frac{\partial^3}{\partial x^3} u_{n-1} - aA_{n-1} - bB_{n-1})$

The Adomian's polynomials depend on the particular nonlinearity. In this case, the A_n polynomials are given by

$$A_0 = u_0 \frac{\partial u_0}{\partial x}$$

$A_1 = u_1 \frac{\partial u_0}{\partial x} + u_0 \frac{\partial u_1}{\partial x}$
$A_2 = u_2 \frac{\partial u_0}{\partial x} + u_1 \frac{\partial u_1}{\partial x} + u_0 \frac{\partial u_2}{\partial x}$
$A_3 = u_3 \frac{\partial u_0}{\partial x} + u_2 \frac{\partial u_1}{\partial x} + u_1 \frac{\partial u_2}{\partial x} + u_0 \frac{\partial u_3}{\partial x}$
$A_4 = u_4 \frac{\partial u_0}{\partial x} + u_3 \frac{\partial u_1}{\partial x} + u_2 \frac{\partial u_2}{\partial x} + u_1 \frac{\partial u_3}{\partial x} + +u_0 \frac{\partial u_4}{\partial x}$
$A_5 = u_5 \frac{\partial u_0}{\partial x} + u_4 \frac{\partial u_1}{\partial x} + u_3 \frac{\partial u_2}{\partial x} + u_2 \frac{\partial u_3}{\partial x} + u_1 \frac{\partial u_4}{\partial x} + u_0 \frac{\partial u_5}{\partial x}$ (79)
$A_6 = u_6 \frac{\partial u_0}{\partial x} + u_5 \frac{\partial u_1}{\partial x} + u_4 \frac{\partial u_2}{\partial x} + u_3 \frac{\partial u_3}{\partial x} + u_2 \frac{\partial u_4}{\partial x} + u_1 \frac{\partial u_5}{\partial x} + u_0 \frac{\partial u_6}{\partial x}$
$A_7 = u_7 \frac{\partial u_0}{\partial x} + u_6 \frac{\partial u_1}{\partial x} + u_5 \frac{\partial u_2}{\partial x} + u_4 \frac{\partial u_3}{\partial x} + u_3 \frac{\partial u_4}{\partial x} + u_2 \frac{\partial u_5}{\partial x} + u_1 \frac{\partial u_6}{\partial x} + u_0 \frac{\partial u_7}{\partial x}$
$A_8 = u_8 \frac{\partial u_0}{\partial x} + u_7 \frac{\partial u_1}{\partial x} + u_6 \frac{\partial u_2}{\partial x} + u_5 \frac{\partial u_3}{\partial x} + u_4 \frac{\partial u_4}{\partial x} + u_3 \frac{\partial u_5}{\partial x} + u_2 \frac{\partial u_6}{\partial x} + u_1 \frac{\partial u_7}{\partial x} + u_0 \frac{\partial u_8}{\partial x}$

The Adomian's polynomials depend on the particular nonlinearity. In this case, the B_n polynomials are given by
$B_0 = {u_0}^2 \frac{\partial u_0}{\partial x}$
$B_1 = {u_0}^2 \frac{\partial u_1}{\partial x}$
$B_2 = {u_1}^2 \frac{\partial u_0}{\partial x} + {u_0}^2 \frac{\partial u_2}{\partial x}$
$B_3 = {u_1}^2 \frac{\partial u_1}{\partial x} + {u_0}^2 \frac{\partial u_3}{\partial x}$ (80)
$B_4 = {u_2}^2 \frac{\partial u_0}{\partial x} + {u_1}^2 \frac{\partial u_2}{\partial x} + {u_0}^2 \frac{\partial u_4}{\partial x}$
$B_5 = {u_2}^2 \frac{\partial u_1}{\partial x} + {u_1}^2 \frac{\partial u_3}{\partial x} + {u_0}^2 \frac{\partial u_5}{\partial x}$
$B_6 = {u_3}^2 \frac{\partial u_0}{\partial x} + {u_2}^2 \frac{\partial u_2}{\partial x} + {u_1}^2 \frac{\partial u_4}{\partial x} + {u_0}^2 \frac{\partial u_6}{\partial x}$

$B_7 = {u_3}^2 \frac{\partial u_1}{\partial x} + {u_2}^2 \frac{\partial u_3}{\partial x} + {u_1}^2 \frac{\partial u_5}{\partial x} + {u_0}^2 \frac{\partial u_7}{\partial x}$

$$B_8 = {u_4}^2 \frac{\partial u_0}{\partial x} + {u_3}^2 \frac{\partial u_2}{\partial x} + {u_2}^2 \frac{\partial u_4}{\partial x} + {u_1}^2 \frac{\partial u_6}{\partial x} + {u_0}^2 \frac{\partial u_8}{\partial x}$$

In the following, for the example 1:
a=-1/6
b=-1/9
$u(x,0) = u_0$=x
Following the Adomian decomposition method

$$u_1 = t(\frac{x}{6} + \frac{x^2}{9}) \quad (81)$$

$$u_2 = \frac{t^2x}{36} + \frac{t^2x^2}{27} + \frac{t^2x^3}{81} \quad (82)$$

$$u_3 = -\frac{2t^3}{81} + \frac{t^3x}{216} + \frac{11t^3x^2}{972} + \frac{2t^3x^3}{243} + \frac{4t^3x^4}{2187} \quad (83)$$

$$u_4 = -\frac{13t^4}{972} - \frac{119t^4x}{11664} + \frac{11t^4x^2}{3888} + \frac{59t^4x^3}{17496} + \frac{43t^4x^4}{26244} + \frac{11t^4x^5}{39366} \quad (84)$$

$$u_5 = -\frac{t^5}{216} - \frac{1009t^5x}{116640} - \frac{613t^5x^2}{209952} + \frac{103t^5x^3}{87480} + \frac{2t^5x^4}{2187} + \frac{97t^5x^5}{295245} + \frac{79t^5x^6}{1771470} \quad (85)$$

$$u_6 = -\frac{2122848}{1530550080}t^6 \pm \frac{6672051}{1530550080}xt^6 - \frac{2850066}{1530550080}2x^2t^6 - \frac{512109}{1530550080}2x^3t^6 + \frac{75627}{1530550080}24x^4t^6 + \frac{10809}{1530550080}32x^5t^6 + \frac{3081}{1530550080}32t^6x^6 + \frac{347}{1530550080}32t^6x^7 \quad (86)$$

$$u_7 = \frac{33446520}{64283103360}t^7 - \frac{109894077}{64283103360}xt^7 - \frac{78178203}{64283103360}2x^2t^7 - \frac{20772909}{64283103360}4x^3t^7 - \frac{242703}{64283103360}32x^4t^7 + \frac{120978}{64283103360}64x^5t^7 + \frac{54375}{64283103360}64x^6t^7 + \frac{12793}{64283103360}64x^7t^7 + \frac{1228}{64283103360}x^8t^7 \quad (87)$$

$$u_8 = \frac{6790919310}{6942575162880}t^8 + \frac{1579694886}{6942575162880}xt^8 - \frac{8334930375}{6942575162880}x^2t^8 - \frac{7382730987}{6942575162880}x^3t^8 - \frac{1376185707}{6942575162880}2x^4t^8 - \frac{25133841}{6942575162880}4x^5t^8 + \frac{14591718}{6942575162880}16x^6t^8 + \frac{5387067}{6942575162880}16x^7t^8 + \frac{538677}{6942575162880}32x^8t^8 + \frac{45026}{6942575162880}32x^9t^8 \quad (88)$$

Havingu_i,i = 0, 1,...8,the solution u (x, t) is as:

$$\mathbf{u(x,t)} = x + t(\frac{x}{6} + \frac{x^2}{9}) + \frac{t^2x}{36} + \frac{t^2x^2}{27} + \frac{t^2x^3}{81} - \frac{2t^3}{81} + \frac{t^3x}{216} + \frac{11t^3x^2}{972} + \frac{2t^3x^3}{243} + \frac{4t^3x^4}{2187} - \frac{13t^4}{972} - \frac{119t^4x}{11664} + \frac{11t^4x^2}{3888} + \frac{59t^4x^3}{17496} + \frac{43t^4x^4}{26244} + \frac{11t^4x^5}{39366} - \frac{t^5}{216} - \frac{1009t^5x}{116640} - \frac{613t^5x^2}{209952} + \frac{103t^5x^3}{87480} + \frac{2t^5x^4}{2187} + \frac{97t^5x^5}{295245} + \frac{79t^5x^6}{1771470} - \frac{2122848}{1530550080}t^6 + -\frac{6672051}{1530550080}xt^6 - \frac{2850066}{1530550080}2x^2t^6 - \frac{512109}{1530550080}2x^3t^6 + \frac{75627}{1530550080}24x^4t^6 + \frac{10809}{1530550080}32x^5t^6 + \frac{3081}{1530550080}32t^6x^6 + \frac{347}{1530550080}32t^6x^7 + \frac{33446520}{64283103360}t^7 - \frac{109894077}{64283103360}xt^7 - \frac{78178203}{64283103360}2x^2t^7 - \frac{20772909}{64283103360}4x^3t^7 - \frac{242703}{64283103360}32x^4t^7 + \frac{120978}{64283103360}64x^5t^7 + \frac{54375}{64283103360}64x^6t^7 + \frac{12793}{64283103360}64x^7t^7 + \frac{1228}{64283103360}x^8t^7 + \frac{6790919310}{6942575162880}t^8 + \frac{1579694886}{6942575162880}xt^8 - \frac{8334930375}{6942575162880}x^2t^8 - \frac{7382730987}{6942575162880}x^3t^8 - \frac{1376185707}{6942575162880}2x^4t^8 - \frac{25133841}{6942575162880}4x^5t^8 + \frac{14591718}{6942575162880}16x^6t^8 + \frac{5387067}{6942575162880}16x^7t^8 + \frac{538677}{6942575162880}32x^8t^8 + \frac{45026}{6942575162880}32x^9t^8 \quad (89)$$

RESULT AND CONCLUSION

In the figure 1, two-dimensional plot for the comparison of HPM, HAM and ADM for the solution u(x, t) for different values of x and t=.1, ħ=-.65is shown,then in the figure 2and 3 we change time and ħ from ħ=-.65 to ħ=-.96

and for t=.4 s, t=.8 showthe comparison of HPM, HAM and ADM.In the end, Figure4 shown Three-dimensional plot for the solution u(x, t) for 0≤t≤1, 0≤x ≤1, ħ=-.65obtained by (a) HPM, (b) ADM, (c) HAM

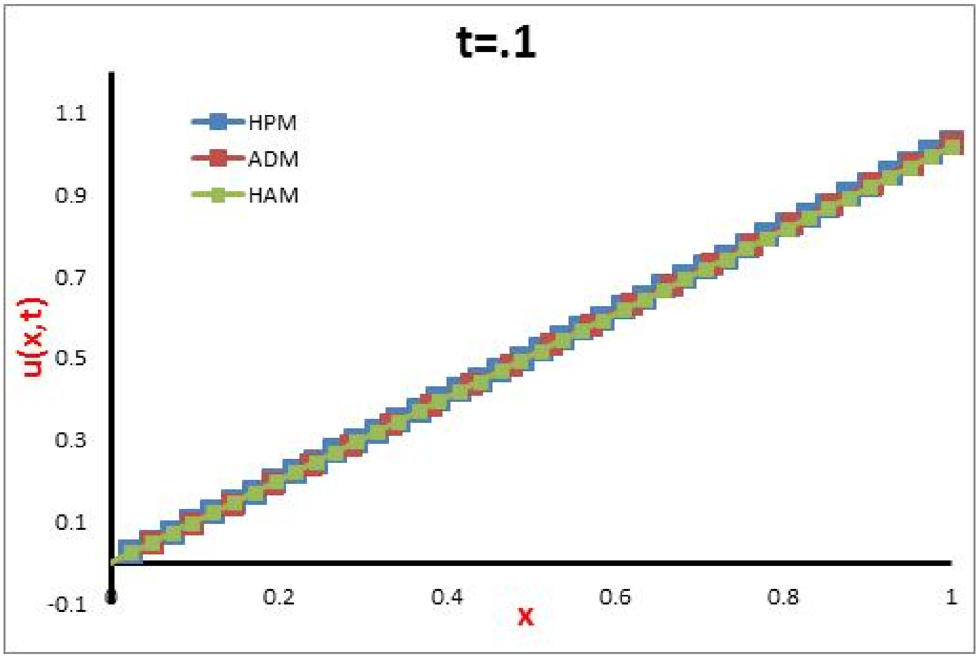

Figure1. The comparison of HPM, HAM and ADM for the solution u(x, t) for different values of x and t=.1, *ħ=-.65*

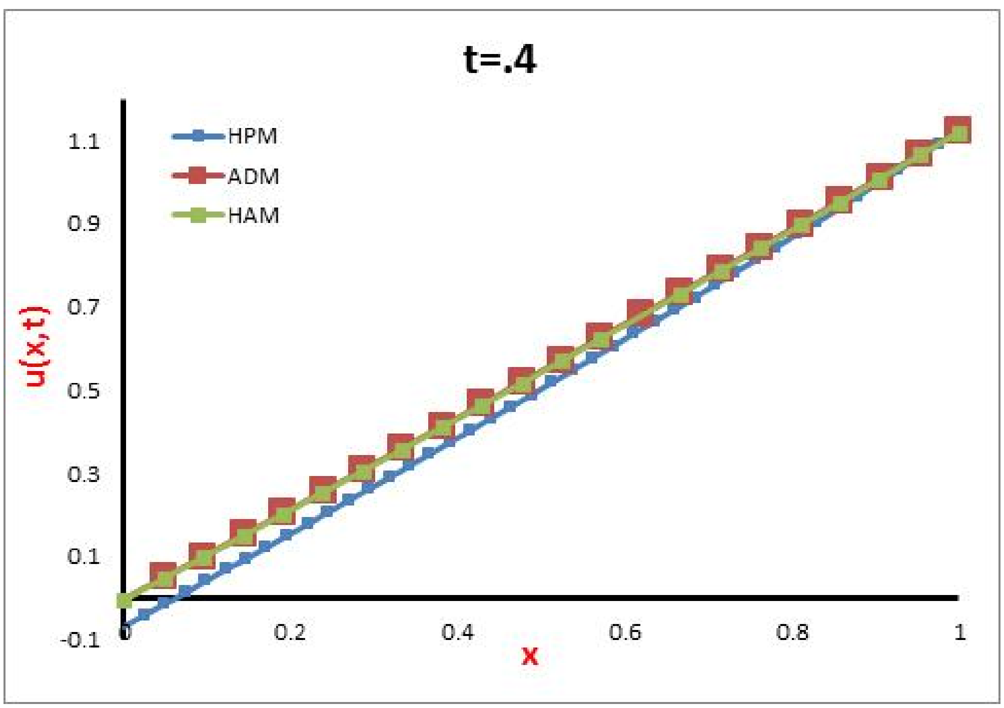

Figure2. The comparison of HPM, HAM and ADM for the solution u(x, t) for different values of x and t=.4, *ħ=-.96*

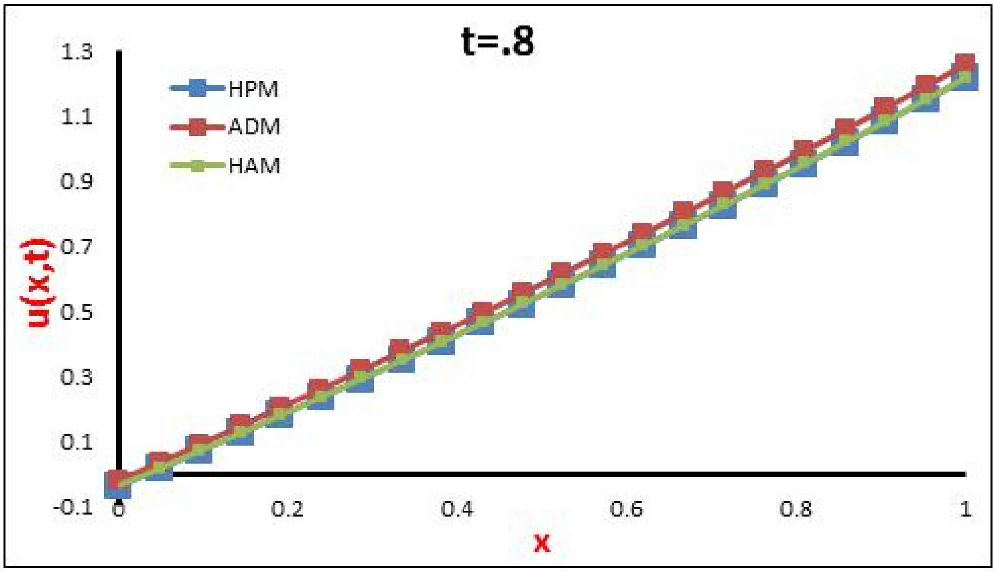

Figure3. The comparison of HPM, HAM and ADM for the solution u(x, t) for different values of x and t=.8, $\hbar$=-*.96*

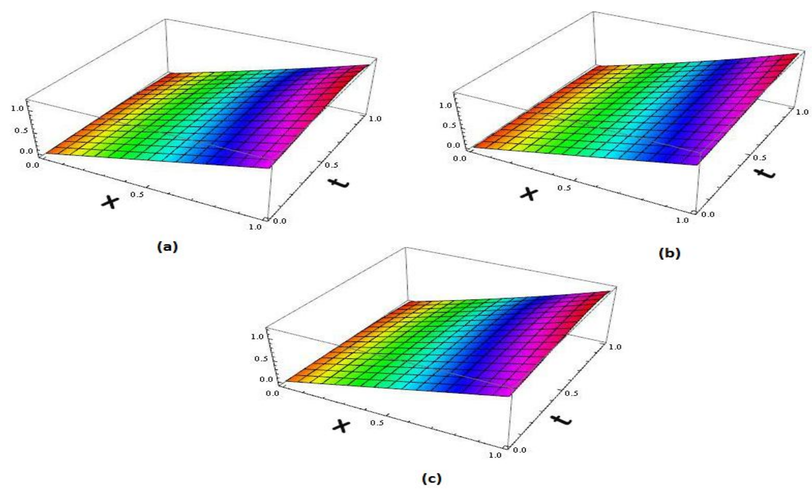

Figure4:Three-dimensional plot for the solution obtained by (a) HPM, (b) ADM, (c) HAM

In this paper, HPM has been successfully applied to finding the solutions of Gardner Equation. The obtained solution is compared with Adomian's decomposition method. The homotopy perturbation method had a little difference With Adomian's decomposition method, so we Solve Gardner Equation BY homotopy analysis methodand by changing ħ, we could control error and observed that the accuracy of ham is more than hpm for solution of Gardner equation. All the figures show that the results of the homotopy analysis method arein approximate agreement with ADM.

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