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Characterization on Semicircle Distribution

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ABSTRACT

A weighted average of n independent continuous random variables $X_1, \cdots, X_n$ with random proportions is introduced. A formula between the Stieltjes transforms of the distribution functions of the weighted averages and $X_1, \cdots, X_n$ is established. We show that, among some other distributions, the Cauchy distribution and the power semicircle distribution can be characterized in a particular way by means of this construction.
KEYWORDS: Randomly weighted averages, Schwartz theory.

1INTRODUCTION

Van Assche (1987) on identifying the distribution of a random variable S uniformly distributed between two independent random variables X and Y, Soltani and Homei (2009) considered a randomly weighted average of independent random variables $X_1, \cdots, X_n$ defined by

$$S_n = R_1X_1 + \cdots + R_nX_n, \quad n \geq 2, (1.1)$$

where random proportions are $R_i = U_{(i)} - U_{(i-1)},\ i = 1, \cdots, n-1$ and $R_n = 1 - \sum_{i=1}^{n-1} R_i$, $U_{(1)}, \cdots, U_{(n-1)}$ order statistics of a random sample $U_1, \cdots, U_n$ from a uniform distribution on $[0,1]$, $U_{(0)} = 0$ and $U_{(n)} = 1$. We refer to $R_i,\ i = 1, \cdots, n$, as the cuts of $[0,1]$ by $U_{(1)}, \cdots, U_{(n)}$. Soltani and Homei (2009) express the $(n-1)-$th derivative of the Stieltjes transform of the distribution function of S_n as the product of the Stieltjes transforms of the distribution functions of $X_1, \cdots, X_n$. Their method is similar to the one of Van Assche (1987), using certain techniques in Schwartz distribution theory and the formulas for the distribution of random average of $x_1, \cdots, x_n$, given by Dempster and Kleyle (1968), where random proportions are cuts of $[0,1]$ by $U_{(1)}, \cdots, U_{(n-1)}$. For application refer to Soltani and Roozegar (2012). In this paper we give some examples.

2 Conditional directed power Distribution

The distribution of a linear combination of the random variables $R_1, \cdots, R_{n-1}$, say $\sum_{i=r}^{n-1} c_i R_i$ for constants c_i satisfying $c_1 > c_2 > \cdots > c_{n-1} > 0$, at a point x is given by

$$x^{n-1}\left[\prod_{i=1}^{n-1} c_i\right]^{-1} - \sum_{j=t+1}^{n-1} (x - c_j)^{n-1}\left[c_j \prod_{i \neq j} (c_i - c_j)\right]^{-1}, \qquad (2.1)$$

where $0 \leq x \leq c_1$ and t is the largest positive integer such that $x \leq c_t$, (Dempster and Kleyle, 1968).Let us apply (2.1) to derive the conditional distribution of S_n given $X_1 = x_1, \cdots, X_n = x_n$ at z, denoted by $K(z|x_1, \cdots, x_n)$, for $x_1 > x_2 > \cdots > x_n$ and $x_{n-i} < z \leq x_{n-i-1}, i = 0, \cdots, n-2$. We note that $\sum_{i=1}^{n} x_i R_i = \sum_{i=1}^{n-1} (x_i - x_n) R_i + x_n$. Thus by using (2.1) with $c_i = x_i - x_n,\ i = 1, \cdots, n-1$ and $t = n - i$, we obtain that for $x_{r+1} < z \leq x_r, r = 1, \cdots, n-1$, $K(z|x_1, \cdots, x_n)$, is equal to

$$(z - x_n)^{n-1}\left[\prod_{j=1}^{n-1} (x_j - x_n)\right]^{-1} - \sum_{j=r+1}^{n-1} (z - x_j)^{n-1}\left[(x_j - x_n) \prod_{k \neq j} (x_k - x_j)\right]^{-1}.$$

By changing variables, first $j^* = n - 1 - j$ and then $j = j^* + 1$ in the summation, the conditional distribution for $x_{r+1} < z \leq x_r,\ r = 1, \cdots, n-1$ will be equal to

$$(z - x_n)^{n-1}\left[\prod_{j=1}^{n-1} (x_j - x_n)\right]^{-1} - \sum_{j=1}^{n-r-1} (z - x_{n-j})^{n-1}\left[(x_{n-j} - x_n) \prod_{k \neq j} (x_k - x_{n-j})\right]^{-1}.$$

Now we let $i = n - 1 - r$, then

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$$\begin{cases} K(z|x_1,\cdots,x_n) = \sum_{j=0}^{i} \frac{(z-x_{n-j})^{n-1}}{C(x_{n-i};x_1,\cdots,x_n)}, \\ x_{n-i} < z \le x_{n-i-1}, \qquad i = 0,\cdots,n-2, \end{cases}$$

where for $j = 0,\cdots,n-1$,

$$C(x_{n-j};x_1,\cdots,x_n) = \prod_{k=1}^{n-j-1}(x_k - x_{n-j}) \prod_{k=n-j+1}^{n}(x_k - x_{n-j}).$$

By using the Heaviside function: $U(x) = 0,\ x < 0,\ = 1,\ x \ge 0$, we obtain that for any given distinct values $x_1,\cdots,x_n$, the conditional distribution is given by

$$K(z|x_1,\cdots,x_n) = \sum_{i=0}^{n-1} \frac{(z-x_{n-i})^{n-1}U(z-x_{n-i})}{C(x_{n-i};x_1,\cdots,x_n)}, \qquad (2.2)$$

For $z \in [min\{x_1,\cdots,x_n\}, max\{x_1,\cdots,x_n\}]$, together with $K(z|x_1,\cdots,x_n) = 0$ for $z < min\{x_1,\cdots,x_n\}$ and $= 1$, for $z > max\{x_1,\cdots,x_n\}$, Thus we arrive at the following result.

Theorem 2.1. Assume S_n is a randomly weighted average given by (1.1). Then the conditional distribution of S_n, for given distinct values $X_1 = x_1,\cdots,X_n = x_n$ at $z,\ -\infty < z < +\infty$ will be given by (2.2).

3 preliminaries and previous works

In this section we present the main results of this article. Let us first develop some basic tools. We first record the following partial fraction formula:

$$\frac{1}{(z-x_1)(z-x_2)\cdots(z-x_n)} = \sum_{i=1}^{n} \frac{a_i}{z-x_i}, \qquad (3.1)$$

where

$$a_i = \left[\prod_{j=1, j\neq n-i}^{n} (x_{n-i} - x_j)\right]^{-1}, \qquad i = 0,\cdots,n-1.$$

The second item is the following formula taken from the Schwartz distribution theory, namely,

$$\int_{-\infty}^{\infty} \varphi(x)\Lambda^{[n]}(dx) = \frac{(-1)^n}{n!}\int_{-\infty}^{\infty} \frac{d^n}{dx^n}\varphi(x)\Lambda(dx), \qquad (3.2)$$

Λ is a distribution function and $\Lambda^{[n]}$ is the n-th distributional derivative of Λ.

The conditional distribution $K(z|x_1,\cdots,x_n)$ given by (2.2) leads us to the following linear functional on complex-valued function f, defined on the set of real numbers $\mathbb{R}$;

$$K(f|x_1,\cdots,x_n) = \sum_{i=0}^{n-1} \frac{f(x_{n-i})}{C(x_{n-i};x_1,\cdots,x_n)}, \qquad f\colon \mathbb{R} \to \mathbb{C}.$$

It easily follows that

$$K(af + bg|x_1,\cdots,x_n) = aK(f|x_1,\cdots,x_n) + bK(g|x_1,\cdots,x_n) \qquad (3.3)$$

for any choice of complex-valued functions f, g and of complex constants a, b. We note that $K(z|x_1,\cdots,x_n) = K(f_z|x_1,\cdots,x_n)$, whenever $f_z(x) = (z-x)^{n-1}U(z-x)$. Also we note that $U(z-x) = (-1)^n(n-1)!\frac{d^{n-1}}{dx^{n-1}}f_z(x)$. Thus $P(S_n \le z) = \int_{\mathbb{R}} U(z-x)dF_{S_n}(x) = \int_{\mathbb{R}^n} K(z|x_1,\cdots,x_n)\prod_{i=1}^{n} F_{X_i}(dx_i)$ can be viewed as:

$$\int_{\mathbb{R}} \frac{d^{n-1}}{dx^{n-1}}f_z(x)dF_{S_n}(x) = \frac{(-1)^{n-1}}{(n-1)!}\int_{\mathbb{R}^n} K(f_z|x_1,\cdots,x_n)\prod_{i=1}^{n} F_{X_i}(dx_i). \qquad (3.4)$$

Therefore by using linear property (3.3) along with (3.4) and a standard argument in the integration theory, we obtain that

$$(-1)^{n-1}(n-1)!\int_{\mathbb{R}} \frac{d^{n-1}}{dx^{n-1}}f(x)dF_{S_n}(x) = \int_{\mathbb{R}^n} K(f|x_1,\cdots,x_n)\prod_{i=1}^{n} F_{X_i}(dx_i) \qquad (3.5)$$

for a suitable f. Now (3.5) together with (3.2) lead us to

$$\int_{\mathbb{R}} f(x)dF_{S_n}^{[n-1]}(x) = \int_{\mathbb{R}^n} K(f|x_1,\cdots,x_n)\prod_{i=1}^{n} F_{X_i}(dx_i), \qquad (3.6)$$

for a suitable f, where $F_{S_n}^{[n-1]}$ is the $(n-1)$–th distributional derivative of the distribution of S_n.
Let us denote the Stieltjes transform of a distribution H by

$$\mathcal{S}(H,z) = \int_{\mathbb{R}} \frac{1}{z-x} H(dx),$$

for every z in the set of complex numbers $\mathbb{C}$ which does not belong to the support of H, $z \in \mathbb{C} \bigcap (supp\ H)^c$. For more on the Stieltjes transform see Zayed (1996).

The following theorem indicates how the Stieltjes transforms of S_n and $X_1, \cdots, X_n$ are related.

Theorem 3.1. Under the assumption that $X_1, \cdots, X_n$ independent and continuous,

$$\frac{(-1)^{n-1}}{(n-1)!} \frac{d^{n-1}}{dz^{n-1}} \mathcal{S}(F_{S_n}, z) = \prod_{i=1}^{n} \mathcal{S}(F_{X_i}, z), \quad z \in \mathbb{C} \bigcap_{i=1}^{n} \left(supp\ F_{X_i}\right)^c.$$

Proof. It follows from (3.6) that

$$\mathcal{S}\left(F_{S_n}^{[n-1]}, z\right) = \int_{\mathbb{R}^n} K(g_z | x_1, \cdots, x_n) \prod_{i=1}^{n} F_{X_i}(dx_i),$$

for $g_z(x) = \frac{1}{z-x}$. But

$$K(g_z | x_1, \cdots, x_n) = \sum_{i=0}^{n-1} \frac{1/(z - x_{n-i})}{C(x_{n-i}; x_1, \cdots, x_n)} = \sum_{i=0}^{n-1} \frac{1/C(x_{n-i}; x_1, \cdots, x_n)}{(z - x_{n-i})}$$
$$= (-1)^{n-1} \sum_{i=1}^{n} \frac{a_i}{z - x_i} = (-1)^{n-1} \prod_{i=1}^{n} \frac{1}{z - x_i},$$

where the last equality follows from (3.1). Thus

$$\mathcal{S}\left(F_{S_n}^{[n-1]}, z\right) = (-1)^{n-1} \prod_{i=1}^{n} \mathcal{S}(F_{X_i}, z), \quad z \in \mathbb{C} \bigcap_{i=1}^{n} \left(supp\ F_{X_i}\right)^c.$$

Therefore

$$\frac{(-1)^{n-1}}{(n-1)!} \frac{d^{n-1}}{dz^{n-1}} \mathcal{S}(F_{S_n}, z) = \int_{\mathbb{R}} \frac{1}{(z-x)^n} F_{S_n}(dx) = \frac{1}{(n-1)!} \int_{\mathbb{R}} \frac{d^{n-1}}{dx^{n-1}} \frac{1}{z-x} F_{S_n}(dx)$$
$$= (-1)^{n-1} \int_{\mathbb{R}} \frac{1}{z-x} F_{S_n}^{[n-1]}(dx) = (-1)^{n-1} \mathcal{S}\left(F_{S_n}^{[n-1]}, z\right)$$
$$= \prod_{i=1}^{n} \mathcal{S}(F_{X_i}, z),$$

giving the result. The proof of the theorem is complete.

Now we are in a position to present the Cauchy characterization and the Arcsin result.

Theorem 3.2. Assume S_n is given by (1.1) and $X_1, \cdots, X_n$ are i.i.d. continuous random variables with a common distribution function F. Then S_n has distribution F if and only if F is a Cauchy distribution.

Proof. The "if" part is immediate. For the "only if" part we note that if F is also the distribution of S_n, then it will follow from Theorem 3.1 that

$$\frac{(-1)^n}{(n-1)!} \frac{d^{n-1}}{dz^{n-1}} \mathcal{S}(F, z) = [\mathcal{S}(F, z)]^n, \qquad z \in \mathbb{C} \quad (3.7)$$

By an argument similar to the one given by Van Assche (1987), the solution for $\mathcal{S}(F, z)$ in (3.7) is

$$\mathcal{S}(F, z) = \frac{1}{z - a + ib}, \quad Im(z) > 0, \quad b \neq 0,$$

which is the Stieltjes transform of the Cauchy distribution. The proof is complete.

Theorem 3.3. Under the assumption that $X_1, \cdots, X_n$ independent and continuous,

$$\frac{(-1)^{n^*-1}}{(n^*-1)!} \frac{d^{n^*-1}}{dz^{n^*-1}} \mathcal{S}(F, z) = \prod_{i=1}^{n} \frac{(-1)^{m_i-1}}{(m_i-1)!} \frac{d^{m_i-1}}{dz^{m_i-1}} \mathcal{S}(F_{X_i}, z), \quad z \in \mathbb{C} \bigcap_{i=1}^{n} \left(supp\ F_{X_i}\right)^c.$$

Lemma 3.4. Let Z_1 be a random variables that have a conditionally direct distribution. Suppose that random variables X_1 and X_2 are independent and continuous with distribution functions F_{X_1} and F_{X_2}, respectively. Then

$$\frac{1}{n} \mathcal{S}^{(n)}(F_{Z_1}, z) = -\mathcal{S}(F_{X_1}, z) \mathcal{S}^{(n-1)}(F_{X_2}, z), \quad z \in \mathbb{C} \bigcap_{i=1}^{2} \left(supp\ F_{X_i}\right)^c.$$

Theorem 3.4. Let X_1 and X_2 be i.i.d random variables on $[-1,1]$, then

(a) if X_1 has uniform distribution on $[-1,1]$, then Z_1 has semicircle distribution on $[-1,1]$ if and only if X_2 has Arcsin distribution on $[-1,1]$;

(b) if X_1 has uniform distribution on $[-1,1]$, then Z_1 has power semicircle distribution if and only if X_2 has power semicircle distribution, i.e.,

$$f(z) = \frac{3(1-z^2)}{4}, \quad -1 \le z \le 1;$$

(c) if X_1 has Beta $(1,1)$ distribution on $[0,1]$, then Z_1 has Beta $\left(\frac{3}{2},\frac{3}{2}\right)$ distribution if and only if X_2 has Beta$\left(\frac{1}{2},\frac{1}{2}\right)$distribution;

(d) if X_1 has uniform distribution on $[0,1]$, then Z_1 has Beta $(2,2)$ distribution if and only if X_2 has Beta$(2,2)$distribution.

Proof. (a) For the "if" part we note that the random variable X_1 has uniform distribution and X_2 has Arcsin distribution on $[-1,1]$; then

$$\mathcal{S}\left(F_{X_1}, z\right) = \frac{1}{2}\left(ln|z+1| - ln|z-1|\right).$$

and $\mathcal{S}\left(F_{X_2}, z\right) = \frac{1}{\sqrt{z^2-1}}$.

From Lemma 3.4 and substituting the corresponding Stieltjes transforms of distributions, we get

$$\mathcal{S}''\left(F_{Z_1}, z\right) = \frac{2}{(z^2-1)^{\frac{3}{2}}}.$$

The solution $\mathcal{S}\left(F_{Z_1}, z\right)$ is

$$\mathcal{S}\left(F_{Z_1}, z\right) = 2\left(z - \sqrt{z^2-1}\right),$$

which is the Stieltjes transform of the semicircle distribution on $[-1,1]$.

For the "only if" part we assume that the random variable Z_1 has semicircle distribution. Then it follows from lemma 3.4 that

$$\mathcal{S}\left(F_{X_2}, z\right)\frac{1}{1-z^2} = \frac{-1}{(z^2-1)^{\frac{3}{2}}}.$$

The proof is completed.

(b) By an argument similar to that given in (a) and solving the following differential equations,

$\mathcal{S}''(F_Z, z) = \frac{2}{(z^2-1)}\left(\frac{3z}{2} + \frac{3}{4}\left((1-z^2)(ln|z+1| - ln|z-1|)\right)\right)$, (for the "if" part), and

$\frac{1}{1-z^2}\mathcal{S}\left(F_{X_2}, z\right) = \frac{3}{4}\frac{2z+(1-z^2)(ln|z+1|-ln|z-1|)}{(1-z^2)}$, (for the "only if" part),

the proof can be completed.

(c) By Lemma 3.4, we have

$-\frac{1}{2}\mathcal{S}''(F_Z, z) = \frac{-1}{z(z-1)}\frac{1}{\sqrt{z(z-1)}}$, (for the "if" part), and

$\frac{-1}{z(z-1)\sqrt{z(z-1)}} = \frac{-1}{z(z-1)}\mathcal{S}\left(F_{X_2}, z\right)$, (for the "only if" part).

The proof can be completed by solving the above differential equations.

(d) By Lemma 3.4, we have

$\mathcal{S}''\left(F_{Z_1}, z\right) = \frac{-2}{z(z-1)}\left(6(z^2 - z_1)(ln|z| - ln|z-1|) - 6z + 3\right)$, (for the "if" part), and

$\mathcal{S}\left(F_{X_2}, z\right) = 6(z - z^2)(ln|z| - ln|z-1|) + 6z - 3$, (for the "only if" part).

Solving the differential equations, can complete the proof.

4 Some characterization

In this section, we also observe application of theorem 3.3, as:

Theorem4.1. Let $m_1 = 3$, $m_2 = 1$, $m_3 = 1$ and X_1, X_2 and X_3 be independent random variables on $[-1,1]$.ThenifX_2, X_3 have Arcsin distribution, thenX_3has semicircle distribution on $[-1,1]$if and only ifS_3has power semicircle distribution on $[-1,1]$, i.e.,

$$f(z) = \frac{8}{3\pi}(1-z^2)^{\frac{3}{2}}, \quad -1 \le z \le 1.$$

Proof. The random variableS_3has a power semicircle distribution on$[-1,1]$ andX_2, X_3 have Arcs in distribution on $[-1,1]$, then it follows from the theorem 3.3

$$\frac{1}{12}\frac{d^4}{dz^4}\mathcal{S}(F,z) = \mathcal{S}''(F_{X_1},z)\mathcal{S}(F_{X_2},z)\mathcal{S}(F_{X_3},z)$$

$$\frac{2}{(z^2-1)^{\frac{5}{2}}} = \mathcal{S}''(F_{X_1},z)\frac{1}{\sqrt{z^2-1}}\frac{1}{\sqrt{z^2-1}}.$$

The solution for$\mathcal{S}(F_{X_1},z)$is

$$\mathcal{S}(F_{X_1},z) = 2\left(z-\sqrt{z^2-1}\right),$$

Which is the Stie ltjes transform of the semi circled is distribution on$[-1,1]$.

Theorem4.2. Let $m_1 = 1$, $m_2 = 1$, $m_3 = 2$ and X_1, X_2 and X_3 be independent random variables on $[-1,1]$, thenX_1, X_2andX_3 have Arcs in distribution on$[-1,1]$if and only ifS_3has a semicircle distribution on$[-1,1]$.

Proof. The random variableS_3 has semicircle distribution on $[-1,1]$, then it follows from the theorem3.3

$$\mathcal{S}(F_{X_1},z)\mathcal{S}(F_{X_2},z)\mathcal{S}'(F_{X_3},z) = \frac{1}{6}\frac{d^3}{dz^3}\mathcal{S}(F,z) = \frac{-z}{(z^2-1)^{\frac{5}{2}}}.$$

Thesolutionfor$\mathcal{S}(F_{X_i},z)$is

$$\mathcal{S}(F_{X_i},z) = \frac{1}{\sqrt{z^2-1}}, \quad i = 1,2,3$$

Which is the Stie ltjes transform of the Arcs in distribution on $[-1,1]$.

REFERENCES

[1]Dempster. A. P., Kleyle, R. M., 1968. Distribution by cutting a simplex with hyperplanes. Ann. Math. Statist. 39, 1473-1478.

[2]Diniz, M. C., de Souza e Silva, E., Gail, H. R., 2002.Calculating the distribution of a linear combination of uniform order statistics. Informs J. Computing 14, 0124-0135.

[3] Homei, H., 2012. Randomly weightedaverageswithBetarandomproportions.Statist.Probab.Lett.82,1515-1520.

[4]Jagannathan. Ravi, Ma, Tongshu, 2002. Risk reduction in large portfolios: Why imposing the wrong constraints helps. NBER Working Paper No.W 8922.Available at SSRN.

[5]Johnson,N.L.,Kotz,S.,1990.Randomlyweightedaverages:someaspectsandextensions. Amer.Statist.44,245-249.

[6]Soltani, A. R., Roozegar, R., 2012. On distribution of randomly ordered uniform incremental weighted averages: Divided difference approach.Statist.Probab.Lett.,82,1012–1020.

[7]VanAssche,W.,1987.A random variable uniformly distributed between two independent random variables.SankhyāSer.A49,207-211.