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Adomian Decomposition Method with Laguerre Polynomials for Solving Ordinary Differential Equation

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ABSTRACT

In this paper, the Adomian Decomposition Method (ADM) based on orthogonal polynomials is employed for solving nonlinear ordinary differential equations. Laguerre polynomials are employed to improve the Adomian decomposition method. The results are compared with the method of using Taylor expansion. This method can be applied successfully to different types of ordinary and partial differential equations. The results show that the Laguerre polynomials based method is better than usual Adomian Decomposition Method.
KEYWORDS: Adomian decomposition method; Orthogonal polynomial; Laguerre polynomials

INTRODUCTION

A numerical method based on the Adomian decomposition method which has been developed by George Adomian [1] and is generally used for solving differential equations. In recent years, the methods like the homotopy perturbation and the variational iteration method have drawn the attention of scientists and engineers. The Adomian decomposition method is one of these, which has been shown [1-3] to solve effectively, easily and accurately a large class of linear and nonlinear problems, for instance differential equations both ordinary and partial equations, linear and nonlinear equations with approximate solutions which converge rapidly to accurate solutions. This paper focuses on the modification of ADM using orthogonal polynomials.

Recently, a lot of attention has been focused on the application of the Adomian decomposition method (ADM) to such diverse areas as chaos theory [4], heat and/or mass transfer [5,6], particle transport [7], nonlinear optics [8] and the fermentation process [9]. Hosseini [10] proposed the method of implementing ADM with chebyshev polynomials, where the reliability of the proposed scheme was verified to be applicable for both linear and nonlinear models. Just like Chebyshev polynomials, Yucheng Liu [11] employed Legendre polynomials to modify the ADM that interval of orthogonality of them is [-1,1].

The Laguerre polynomials are named after Edmond Laguerre (1834-1886) and they are solutions of the following Laguerre's differential equation:

$$xy'' + (1-x)y' + ny = 0, \qquad (1)$$

where n is a real number.

These polynomials, usually denoted by $l_0(x), l_1(x), \ldots$.They can be expressed by Rodrigue's formula

$$l_n(x) = \frac{e^x d^n}{n! dx^n}(e^{-x}x^n), \qquad (2)$$

which can be defined recursively by using the following recurrence relation

$$l_0(x) = 1,$$
$$l_1(x) = 1 - x,$$
$$l_{k+1}(x) = \frac{1}{k+1}\left((2k+1-x)l_k(x) - kl_{k-1}(x)\right), \quad k \geq 1. \qquad (3)$$

This paper applies Laguerre polynomials [12] to modify the ADM and compares with ADM on the basis of Taylor series expansion. We note that Laguerre polynomials are orthogonal on $(0,+\infty)$ with respect to the weight function $w(x) = e^{-x}$. The obtained results are studied to show the advantage and efficiency of this modified ADM.

In section 2, ADM is explained. ADM based on Laguerre polynomials is shown in section 3. The numerical examples and comparison of the solutions by different expansions of right hand side function is shown in section 4. Conclusion and the absolute errors between the exact solution and the approximate solution are shown in section 5.

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2. ADOMIAN DECOMPOSITION METHOD

We begin with the equation

$$Lu + R(u) + F(u) = g(t), \quad (4)$$

where L is the linear operator of the highest-ordered derivative with respect to t and R is the remainder of the linear operator. The nonlinear term is represented by $F(u)$. Thus we get

$$Lu = g(t) - R(u) - F(u). \quad (5)$$

The inverse operator L^{-1} is assumed to be an integral operator given by

$$L^{-1} = \int_0^t (.)dt\,. \quad (6)$$

Operating with the operator L^{-1} on both sides of the equation (5) we have

$$u = f_0 + L^{-1}\big(g(t) - R(u) - F(u)\big), \quad (7)$$

where f_0 is the solution of homogeneous equation $Lu = 0$.

The Adomian decomposition method assumes that the unknown function $u(x,t)$ can be expressed by an infinite series of the form

$$u(x,t) = \sum_{n=0}^{\infty} u_n(x,t), \quad (8)$$

and the nonlinear operator $F(u)$ can be decomposed by an infinite series of polynomials given by

$$F(u(x,t)) = Nu(x,t) = \sum_{n=0}^{\infty} A_n, \quad (9)$$

There appears to be no well-defined method for constructing a definite set of polynomials for arbitrary F, but rather slightly different approaches are used for different specific functions one possible set of polynomials is given by

$$A_0 = F(u_0), \quad (10)$$

$$A_1 = u_1F'(u_0), \quad (11)$$

$$A_2 = u_2F'(u_0) + \frac{1}{2!}u_1^2F''(u_0), \quad (12)$$

$$A_3 = u_3F'(u_0) + u_1u_2F''(u_0) + \frac{1}{3!}u_1^3F'''(u_0), \quad (13)$$

$\vdots$

We can obtain u_i for $i = 0,1,2,\ldots$ with the recurrence relation as followes

$$u_0 = L^{-1}g + \emptyset(x), \quad (14)$$

$$u_1 = -L^{-1}(Ru_0) - L^{-1}(Nu_0),$$

$$u_2 = -L^{-1}(Ru_1) - L^{-1}(Nu_1),$$

$$u_3 = -L^{-1}(Ru_2) - L^{-1}(Nu_2),$$

$\vdots$

and we can calculate the final u, $u = \sum_{n=0}^{\infty} u_n$, if the series converges.

3. ADM based on Laguerre polynomials

To solve differential equation by the Adomian decomposition method, for an arbitrary integer number, $g(x)$ can be expressed in the Taylor series and Laguerre series, that is pointed by $g_{T,m}(x)$ and $g_{l,m}(x)$, respectively, where

$$g(x) \approx g_{T,m}(x) = \sum_{n=0}^{m} \frac{g^{(n)}(0)}{n!}x^n, \quad (15)$$

$$g(x) \approx g_{l,m}(x) = \sum_{n=0}^{m} c_n l_n(x), \quad (16)$$

where $l_n(x)$ $(n = 0,1,2,\ldots)$ are the orthogonal Laguerre polynomials [4] with equation (2), (3) that we can obtain them as followes

n	$l_n(x)$	(17)
0	1	
1	$1 - x$	
2	$\frac{1}{2}(x^2 - 4x + 2)$	
3	$\frac{1}{6}(-x^3 + 9x^2 - 18x + 6)$	
4	$\frac{1}{24}(x^4 - 16x^3 + 72x^2 - 96x + 24)$	
5	$\frac{1}{120}(-x^5 + 25x^4 - 200x^3 + 600x^2 - 600x + 120)$	
…	….	

and

$$c_n = \int_0^{\infty} e^{-x}.\, l_n(x).\, g(x)dx, \quad n = 0,1,\ldots. \quad (18)$$

By substituting equation (16) in (14), we have

$u_0 = L^{-1}(c_0 l_0 + c_1 l_1 + \cdots + c_n l_n) + \emptyset(x),$ (19)
$u_1 = -L^{-1}(Ru_0) - L^{-1}(Nu_0),$
$u_2 = -L^{-1}(Ru_1) - L^{-1}(Nu_1),$
$u_3 = -L^{-1}(Ru_2) - L^{-1}(Nu_2),$
$\vdots$

Above equation is governing equation of modified ADM using Laguerre polynomials. The approximate $u(x)$ is obtained from (19) as $u(x) = \sum_{n=0}^{m} u_n$ which can be very close to the Laguerre-expansion of the exact solution $u(x)$ for appropriate m.

4. NUMERICAL RESULTS

In this section, we solve differential equations of second order by ADM based on Laguerre polynomials. In order to compare the precision of ADM on the basis of Taylor and Laguerre, their absolute errors are drown in figures 1 , 2.

Example 1: Consider the following differential equation with the exact solution $u(x) = e^{-x}$.
$u'' + u' + 2xu^3 = 2xe^{-3x}, \quad u(0) = 1, \quad u'(0) = -1,$ (20)
The operator from (20) respect to $Lu = g(t) - R(u) - F(u)$, is
$L = \dfrac{d^2u}{dx^2} = u'', F(u) = Nu = 2xu^3, R(u) = u', g(x) = 2xe^{-3x}.$
Then the inverse operator L^{-1} can be regarded as the definite integral in the following form
$L^{-1} = \int_0^x \int_0^x (.)dxdx.$
According (10)-(13) the Adomian polynomials are
$A_0 = 2xu_0^3$
$A_1 = 6xu_1 u_0^2$ (21)
$A_2 = 2x(3u_2 u_0^2 + 3u_0 u_1^2)$
$A_3 = 2x(3u_3 u_0^2 + 6u_2 u_1 u_0 + u_1^3$
$\vdots$
In this work we expand $g(x)$ with taylor series and Laguerre polynomials (15), (16), then we obtain u_i for $i = 0,1,2,\ldots$ by using (19) and $u(x) = \sum_{n=0}^{m} u_n$.
By Eq. (14), $u_T(x), u_l(x)$ can be evaluated based on g(x), u_i and Adomian polynomials A_n as
$u_0 = L^{-1}\big(g(x)\big) + \emptyset(x)$ (22)
$u_1 = -L^{-1}\left(\frac{du_0}{dx}\right) - L^{-1}(A_0)$
$u_2 = -L^{-1}\left(\frac{du_1}{dx}\right) - L^{-1}(A_1)$
$u_3 = -L^{-1}\left(\frac{du_2}{dx}\right) - L^{-1}(A_2)$
$\vdots$
$u_k = L^{-1}\left(\frac{du_{k-1}}{dx}\right) - L^{-1}(A_{k-1}).$

Case (A): Let m=10, we first expand $g(x)$ with Taylor series

$$g_{T,10}(x) \approx 2x - 6x^2 + 9x^3 + \frac{27}{4}x^5 - \frac{81}{20}x^6 + \frac{81}{40}x^7 - \frac{243}{280}x^8 + \frac{729}{2240}x^9 - \frac{243}{2240}x^{10} + O(x^{11}), \quad (23)$$

By using Eq. (22) we obtained $u_T(x)$ based on Eq. (23) as
$u_{T,10}(x) = \sum_{m=0}^{10} u_m = 1 - x + \frac{1}{2}x^2 - \frac{1}{6}x^3 + \frac{1}{24}x^4 - \frac{1}{120}x^5 + \frac{1}{720}x^6 - \frac{1}{5040}x^7 + \cdots.$

Case (B): By setting $m = 10$ and from recurrence relation (3) and Eqs. (16), (17), (18) we can have
$g(x) \approx \sum_{n=0}^{10} c_n l_n(x), \quad 0 \le x \le 2,$ (24)
then

$$\begin{aligned} g_{l,10} \approx\ & 0.077432 + 0.83150x - 1.5266x^2 + 1.0316x^3 - 0.35411x^4 + 0.069105x^5 \\ & -0.0080389x^6 + 0.00056267x^7 - 0.000023038x^8 + 0.50436 \times 10^{-6}x^9 \\ & -0.45263 \times 10^{-8}x^{10}, \end{aligned} \quad (25)$$

Similarly, placing (25) in g(x) at (22), the approximate solution based on Laguerre polynomials is

$$u_{l,10}(x) = \sum_{m=0}^{10} u_m = 1 + 0.53872x^2 - x - 0.374323\,x^3 + 0.466364x^4 - 0.503357x^5 + 0.429098x^6 - .351646x^7 + 0.312085x^8 - \cdots.$$

The accuracy of the $u_{l,10}(x)$ is validated by comparing to the exact u(x).

Example 2: Consider differential equation

$$u'' + u' - uu' = (-2 + 4x^2 - 2x)e^{(-x^2)} + 2xe^{(-2x^2)}, \qquad (26)$$

with initial values $u(0) = 1, u'(0) = 0$.

The exact solution of this equation is $u(x) = e^{(-x^2)}$.

An operator form of the above equation is

$$L(u) + R(u) + N(u) = g(x), \qquad 0 \le x \le 2,$$

that

$$L(u) = \frac{d^2}{dx^2},\ \ R(u) = u',\ \ N(u) = -uu',\ \ g(x) = (-2 + 4x^2 - 2x)e^{(-x^2)} + 2xe^{(-2x^2)},$$

Then the inverse operator L^{-1} can be regarded as the definite integral in the following form

$$L^{-1} = \int_0^x \int_0^x (.)dxdx.$$

According (10)-(13) the Adomian polynomials are

$$A_0 = -u_0'u_0$$
$$A_1 = -u_0'u_1 - u_1'u_0$$
$$A_2 = -u_0'u_2 - u_1'u_1 - u_2'u_0$$
$$A_3 = -u_0'u_3 - u_1'u_2 - u_2'u_1 - u_3'u_0$$
$$\vdots$$

Similar to example 1 we expand $g(x)$ with taylor series and Laguerre polynomials (15), (16), then we obtain u_i for $i = 0,1,2,\ldots$ by using (19) and $u(x) = \sum_{n=0}^{m} u_n$.

By Eq. (14), $u_T(x), u_l(x)$ can be evaluated based on g(x), u_i and Adomian polynomials A_n as

$$u_0 = L^{-1}(g(x)) + \emptyset(x) \qquad (27)$$
$$u_1 = -L^{-1}\left(\frac{du_0}{dx}\right) - L^{-1}(A_0)$$
$$u_2 = -L^{-1}\left(\frac{du_1}{dx}\right) - L^{-1}(A_1)$$
$$u_3 = -L^{-1}\left(\frac{du_2}{dx}\right) - L^{-1}(A_2)$$
$$\vdots$$
$$u_k = L^{-1}\left(\frac{du_{k-1}}{dx}\right) - L^{-1}(A_{k-1}).$$

Case (A): We first expand g(x) with Taylor series and let m=10

$$g_{T,10} = -2 + 6x^2 - 2x^3 - 5x^4 + 3x^5 + \frac{7}{3}x^6 - \frac{7}{3}x^7 - \frac{3}{4}x^8 + \frac{5}{4}x^9 + \frac{11}{60}x^{10} + O(x^{11}),$$

Case (B): We also expand g(x) using the laguerre polynomials as

$$g_{l,10} = -2.230016 + 2.415372x + 1.597006x^2 - 2.602548x^3 + 1.173732x^4 + \cdots + 0.213048 \times 10^{-7}x^{10},$$

Similar to explanations of previous example and replacing g(x) with $g_{T,10}$ and $g_{l,10}$ at (27), approximate solutions are obtained as followes

$$u_{T,10} = 1 - x^2 + 0.5x^4 - 0.166667x^6 + 0.041666x^8 - 0.0083333x^{10} - \cdots.$$

and

$$u_{l,10} = 1 - 1.115008x^2 + 0.402562x^3 + 0.133084x^4 - 0.005803x^5 - 0.035686x^6 - \cdots.$$

5. Conclusion

This paper illustrates applying Laguerre polynomials and Taylor expansion for the Adomian decomposition method. Examples verified that the proposed method can be used to solve both linear and nonlinear problems. Through analyses and comparisons, it is concluded that Laguerre polynomials can be used to improve the ADM and the obtained approximated u(x) is more accurate than the one obtained through regular ADM. Considering presented examples and their figures, we conclude that solutions of Adomian decomposition method on the basis of orthogonal polynomials expansion (Laguerre polynomials) is better than Taylor expansion, especially when the approximate interpolation is wider than [0,1] (we have considered [0,2]).

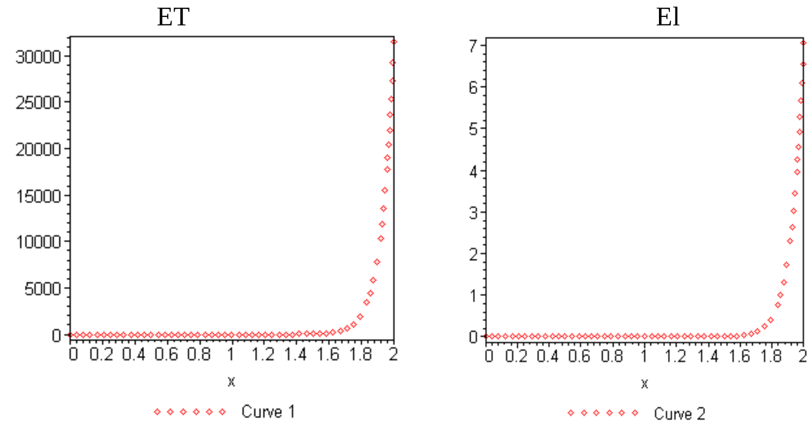

Figure 1: Absolute errors of ADM by Taylor and Laguerre polynomials of example 1.

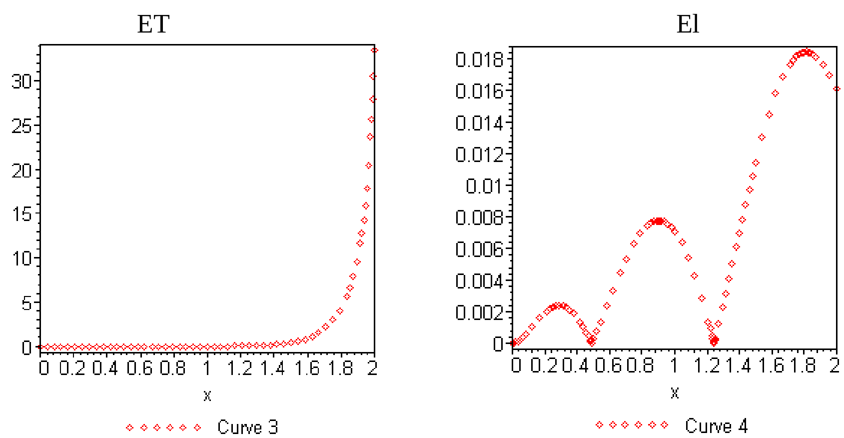

Figure 2: Absolute errors of ADM by Taylor and Laguerre polynomials of example 2.
ET = $|u_{exact} - u_T|$ and El =$|u_{exact} - u_l|$.

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