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The Quadratic Trigonometric Bézier Curve with Single Shape Parameter

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ABSTRACT

In this work, we have constructed a quadratic trigonometric Bézier curve with single shape parameter which is analogous to quadratic Bézier curve. We have adjusted the shape of the curve as desired, by simply altering the values of shape parameter, without changing the control polygon. The quadratic trigonometric Bézier curve can be made closer to the quadratic Bézier curve or nearer to the control polygon than quadratic Bézier curve due to shape parameter. The representation of ellipse is more accurate and exact by using quadratic trigonometric Bézier curve. The smoothness of curve is C^k.

KEYWORDS: Trigonometric polynomials, quadratic trigonometric basis functions, quadratic trigonometric Bézier curves, shape parameters.

1. INTRODUCTION

Curves and surfaces design is an important topic of CAGD (Computer Aided Geometric Design) and computer graphics. The significance of trigonometric polynomials in diverse areas, namely electronics or medicine is well known [1]. The parametric representation of curves and surfaces specially in polynomial form is most suitable for design, as the planer curves cannot deal with infinite slopes and are axis dependent too. Lately, The parametric representation of curves and surfaces with shape parameters has gained much attention of the designers and several new trigonometric splines have been proposed for geometric modeling in CAGD. Han [2] constructed quadratic trigonometric polynomial curves with a shape parameter which are C^1 continuous and similar to quadratic B-spline curves. Wu. X., et al [3] presented quadratic trigonometric spline curves with multiple shape parameters. Han, X. [4] discussed piecewise quadratic trigonometric polynomial curves with C^2 continuity. Han, X. [5] studied Cubic trigonometric polynomial curves with a shape parameter. These papers [4,5] describe the trigonometric polynomial curves with global shape parameters.

The theory of Bézier curves uphold a key position in CAGD. These are considered as ideal geometric standard for the representation of piecewise polynomial curves. In recent years trigonometric polynomial curves like those of Bézier type are considerably in discussion. Wang, L.,et al [6] discussed the adjustable quadratic trigonometric polynomial Bézier curves with a shape parameter. Han, X.A.,et al [7] introduced cubic trigonometric Bézier curves with two shape parameters. In the light of [7], Liu, H., et al [8] discussed study on a class of cubic trigonometric Bézier curves with two shape parameters. Wei [9] constructed quadratic TC- Bézier curves with shape parameter in a special space.

In this paper, we present quadratic trigonometric Bézier curves with single shape parameter. The curve is constructed based on new quadratic trigonometric polyonomials. It is analogous to ordinary quadratic Bézier curve. This curve takes over the existing curve in the manner that it exactly expresses some quadratic trigonometric curves, arc of a circle and arc of an ellipse, under sufficient conditions. The proposed curve attains C^k continuity.

The present work is organised as follows: In section 2, the basis functions of the quadratic trigonometric Bézier curve with single shape parameter are constructed and the properties of the basis functions are shown. In section 3, quadratic trigonometric Bézier curves and their properties are discussed. The effect on the shape of the curve by altering the value of shape parameter is given in section 4 empowered by the graphical representation. In section 5 the representation of ellipses is presented. The approximation of the quadratic trigonometric Bézier curve to the ordinary quadratic Bézier curve is given in section 6. Finally, conclusion of the work and future road map is given in section 7.

2. QUADRATIC TRIGONOMETRIC BASIS FUNCTIONS

In this section, definition and some properties of quadratic trigonometric basis functions are given.

Definition 2.1: For $u \in [0,1]$, the quadratic trigonometric basis functions with single shape parameter m, $m \in [-1,1]$ are defined as:

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$$\begin{cases} f_0(u) = (1-\sin\frac{\pi}{2}u)(1-m\sin\frac{\pi}{2}u) \\ f_1(u) = (1+m)(\cos\frac{\pi}{2}u+\sin\frac{\pi}{2}u-1) \\ f_2(u) = (1-\cos\frac{\pi}{2}u)(1-m\cos\frac{\pi}{2}u) \end{cases} \tag{1}$$

For $m=-1$, the basis functions are linear trigonometric polynomials.

Theorem 2.1: The basis functions (1) have the following properties:

(a) Nonnegativity: $f_i(u) \geq 0$, $i = 0,1,2$

(b) Partition of unity: $\sum_{i=0}^{2} f_i(u) = 1$

(c) Monotonicity: For the given value of the parameter *m*, $f_0(u)$ is momotonically decreasing and $f_2(u)$ is monotoniclly increasing.

(d) Symmetry: $f_i(u;m) = f_{2-i}(1-u;m)$, $i=0,1,2$

Proof:

(a) For $u \in [0,1]$ and $m \in [-1,1]$, $(1-\sin\frac{\pi}{2}u) \geq 0$, $(1-m\sin\frac{\pi}{2}u) \geq 0$, $(1-\cos\frac{\pi}{2}u) \geq 0$, $(1-m\cos\frac{\pi}{2}u) \geq 0$, $\cos\frac{\pi}{2}u \geq 0$, $\sin\frac{\pi}{2}u \geq 0$ and $(1+m) \geq 0$. It is obvious that $f_i(u) \geq 0$, $i = 0,1,2$

(b) $\sum_{i=0}^{2} f_i(u) = (1-\sin\frac{\pi}{2}u)(1-m\sin\frac{\pi}{2}u) + (1+m)(\cos\frac{\pi}{2}u+\sin\frac{\pi}{2}u-1) + (1-\cos\frac{\pi}{2}u)(1-m\cos\frac{\pi}{2}u) = 1$

(c) Monotonicity of functions is seen in Fig.1.

(d) For $i = 2$,

$$\begin{aligned} f_2(u;m) &= (1-\cos\frac{\pi}{2}u)(1-m\cos\frac{\pi}{2}u) \\ &= (1-\sin\frac{\pi}{2}(1-u))(1-m\sin\frac{\pi}{2}(1-u)) \\ &= f_0(1-u;m) \end{aligned}$$

The quadratic trigonometric basis functions for $m=1$ (dashed) and for $m=-1$ (solid) are shown in Fig.1.

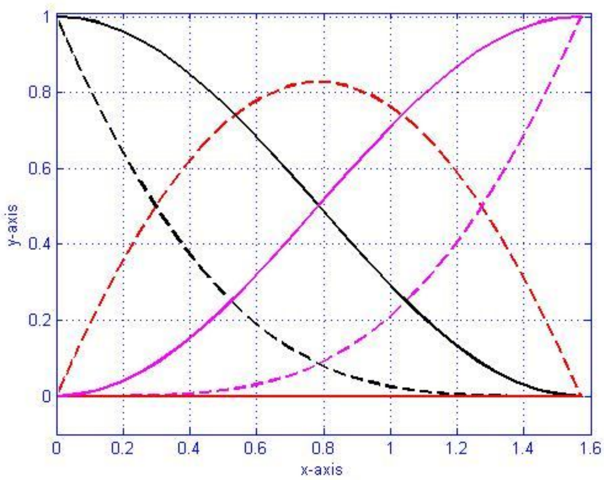

Fig .1: The quadratic trigonometric basis functions

3. QUADRATIC TRIGONOMETRIC BÉZIER CURVE

We construct the Quadratic Trigonometric Bézier (i.e.QT-Bézier) curve with single shape parameter as follows:

Definition 3.1: Given the control points $P_i\,(i = 0,1,2)$ in $\mathbb{R}^2$, we define the QT- Bézier curve with single shape parameter as:

$$f(u)=\sum_{i=0}^{2} f_i(u)\,P_i\,, \quad u\in[0,1], \quad m\in[-1,1] \tag{2}$$

The curve defined by (2) possesses some geometric properties which can be obtained easily from the properties of the basis functions.

Theorem 3.1: The QT- Bézier curve have the following properties:

(a) End point properties:

$f(0)=P_0\,,\quad f(1)=P_2$

$f'(0)=(1+m)(P_1-P_0)\,,\qquad f'(1)=(1+m)(P_2-P_1)$

$f''(0)=2m\,P_0-(1+m)P_1+(1-m)\,P_2\,,\qquad f''(1)=2m\,P_2-(1+m)P_1+(1-m)\,P_0$

$f'''(0)=-(1+m)(P_1-P_0)\,,\qquad f'''(1)=-(1+m)(P_2-P_1)$

$f^{(iv)}(0)=-8m\,P_0+(1+m)P_1+(7m-1)\,P_2\,,\qquad f^{(iv)}(1)=-8m\,P_2+(1+m)P_1+(7m-1)\,P_0$

Remark

(i) $f^{(2k+1)}(0)=(-1)^k f'(0)$ and $f^{(2k+1)}(1)=(-1)^k f'(1)$, $\quad k=1,\ 2,\ 3,\ \ldots,\ \mathrm{n}$

(ii) Inverting the order of the control points, all derivatives of order $2k$, $k=1,\ 2,\ 3,\ \ldots,\ \mathrm{n}$ coincide.

(b) Symmetry: P_0,P_1,P_2 and P_2,P_1,P_0 define the same curve in different parametrizations, that is:

$$f(u;m,P_0,P_1,P_2)=f(1-u;m,P_2,P_1,P_0),\quad u\in[0,1],\ m\in[-1,1]$$

(c) Geometric invarience: The shape of the curve (2) is independent of the choice of coordinates, i.e., it satisfies the following two equations:

$f(u;m,P_0+q,P_1+q,P_2+q)=f(u;m,P_0,P_1,P_2)+q$

$f(u;m,P_0*T,P_1*T,P_2*T)=f(u;m,P_0,P_1,P_2)*T,\quad u\in[0,1],\ m\in[-1,1]$

Where q is any arbitrary vector in $\mathbb{R}^2$ and T is an arbitrary $d\times d$ matrix, $d=2$

(d) Convex hull Property: From the non-negativity and partition of unity of basis functions, it follows that the whole curve is located in the convex hull generated by its control points.

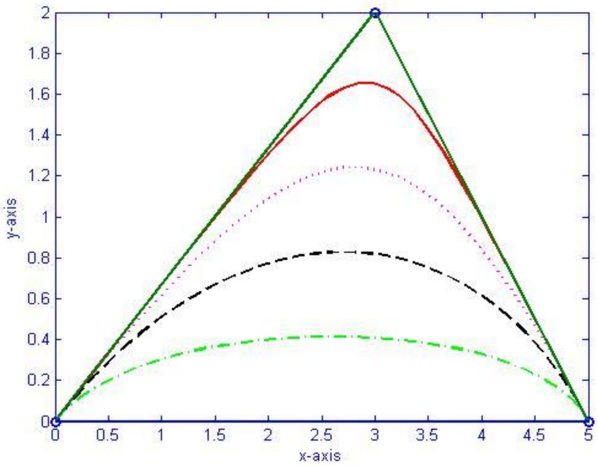

Fig 2. The effect on the shape of the QT- Bézier curve for different values of *m*

4. SHAPE CONTROL OF THE QT- BÉZIER CURVE

The parameter *m* controls the shape of the curve (2). The QT- Bézier curve *f(u)* gets closer to the control polygon as the value of the parameter increases gradually in $[-1,1]$.

In Fig 2, the curves are generated by setting the values of *m* as $m=-0.5$ (green dashed dotted), $m=0$ (black dashed), $m=0.5$ (pink dotted), and $m=1$ (red solid). The curve goes back to the straight line when $m=-1$ (blue solid).

5. THE REPRESENTATION OF ELLIPSE

Theorem 5.1: Let P_0,P_1,P_2 be three control points on an ellipse with semi major and minor axes as $\sqrt{2}\,a$ and $\sqrt{2}\,b$ respectively. By the proper selection of coordinates, their coordinates can be written in the form

$$P_0 = \begin{pmatrix} -a \\ 0 \end{pmatrix} \quad P_1 = \begin{pmatrix} 0 \\ b \end{pmatrix} \quad P_2 = \begin{pmatrix} a \\ 0 \end{pmatrix}$$

Then for the value of shape parameter $m=0$ and local domain $u\in[0,4]$, the curve (2) represents arc of an ellipse with

$$\begin{cases} x(u)=a(\cos\frac{\pi}{2}u-\sin\frac{\pi}{2}u) \\ y(u)=b(\cos\frac{\pi}{2}u+\sin\frac{\pi}{2}u-1) \end{cases} \tag{3}$$

Proof:

If we plug $m=0$ and $P_0 = \begin{pmatrix} -a \\ 0 \end{pmatrix}$ $P_1 = \begin{pmatrix} 0 \\ b \end{pmatrix}$ $P_2 = \begin{pmatrix} a \\ 0 \end{pmatrix}$ into (2), then the coordinates of QT- Bézier curve are

$$\begin{cases} x(u)=a(\cos\frac{\pi}{2}u-\sin\frac{\pi}{2}u) \\ y(u)=b(\cos\frac{\pi}{2}u+\sin\frac{\pi}{2}u-1) \end{cases}$$

This gives the intrinsic equation

$$(\frac{x}{\sqrt{2}a})^2+(\frac{y+b}{\sqrt{2}b})^2=1$$

It is an eqution of the ellipse centered at (0,-b). □

For $u\in[0,4]$, equation (3) represents the whole ellipse. Fig.3 shows the representation of ellipse with QT- Bézier curves

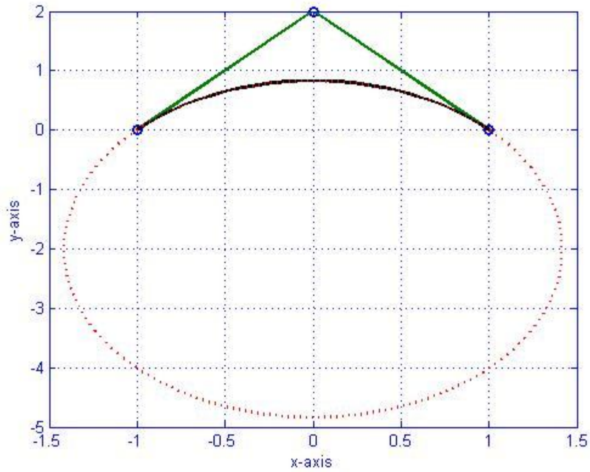

Fig 3: The representation of ellipses with QT- Bézier curves

6. APPROXIMABILITY

Control polygons provide an important tool in geometric modeling. It is an advantage if the curve being modelled tends to preserve the shape of its control polygon.

The QT- Bézier curve with single shape parameter is analogous to ordinary quadratic Bézier curve with same control points.

Theorem 6.1: Suppose P_0, P_1, P_2 are not collinear; the relationships between QT- Bézier curve *f*(*u*) and the quadratic Bézier curve $B(t)=\sum_{i=0}^{2}P_i\binom{2}{i}(1-t)^{2-i}t^i$, $t\in[0,1]$ with the same control points $P_i\left(i=0,1,2\right)$ are as follows:

$$\begin{cases} f(0)=B(0) \\ f(1)=B(1) \end{cases}$$

$$f(\frac{1}{2})-P_1=2(\sqrt{2}-1)(\sqrt{2}-m)(B(\frac{1}{2})-P_1) \tag{4}$$

Proof:

By simple computations,

$f(0)=P_0=B(0)$, $\quad f(1)=P_2=B(1)$

Since

$B(t)=(1-t)^2P_0+2(1-t)tP_1+t^2P_2$

So,

$$B(\frac{1}{2})-P_1=\frac{1}{4}(P_0-2P_1+P_2)$$

and

$$\begin{aligned} f(\frac{1}{2})-P_1 &= \frac{1}{2}(\sqrt{2}-1)(\sqrt{2}-m)P_0-(\sqrt{2}-1)(\sqrt{2}-m)P_1+\frac{1}{2}(\sqrt{2}-1)(\sqrt{2}-m)P_2 \\ &= \frac{1}{2}(\sqrt{2}-1)(\sqrt{2}-m)(P_0-2P_1+P_2) \\ &= 2(\sqrt{2}-1)(\sqrt{2}-m)(B(\frac{1}{2})-P_1) \end{aligned}$$

□

Corollary 6.1: The QT- Bézier curve is closer to the control polygon than the quadratic Bézier curve if and only if $\frac{\sqrt{2}-1}{2}\le m\le 1$.

Corollary 6.2: When $m=\frac{\sqrt{2}-1}{2}$, the QT- Bézier curve is close to quadratic Bézier curve, i.e. $f(\frac{1}{2})=B\left(\frac{1}{2}\right)$.

Fig. 4 shows the relationship between the QT- Bézier curves and quadratic Bézier curves. The QT- Bézier curve(black dotted) with shape parameter $m=\frac{\sqrt{2}-1}{2}$ is analogous to ordinary quadratic Bézier curve (pink solid)

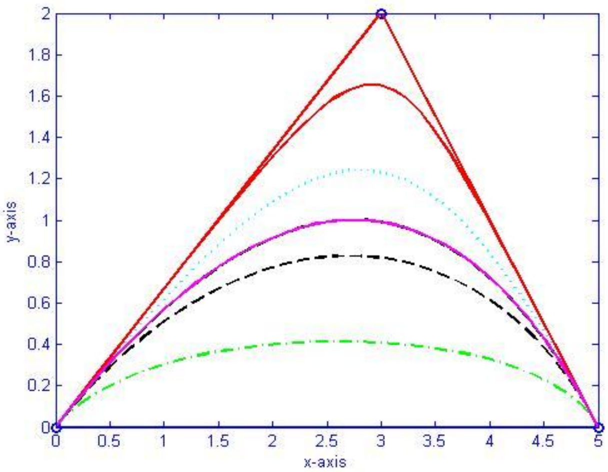

Fig 4: The relationship between the QT- Bézier curves and ordinary quadratic Bézier curves

7. CONCLUSION

In this paper, we have presented the QT- Bézier curve with single shape parameter. All physical properties of QT-Bézier curve are similar to the ordinary quadratic Bézier curve. Due to the shape parameter, It is more useful in font designing as compared to ordinary quadratic Bézier curve. We can deal precisely with circular arcs with the help of QT-Bézier curve. The curve exactly represents the arc of an ellipse, the arc of a circle under certain conditions. The curve acheives C^k continuity. Further more, it is analogous in structure to quadratic Bézier curve. It is not difficult to adapt a QT-Bézier curve to CAD/CAM system because the quadratic Bézier curve is already in use. In future, we would extend it to cubic Bézier curve and its surfaces.

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