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INVESTIGATION OF BOUNDARY LAYER FOR A SECOND ORDER EQUATION UNDER LOCAL AND NON – LOCAL BOUNDARY CONDITIONS

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ABSTRACT

Arise or absence of a boundary layer a perturbed ordinary linear differential equation of second order is investigated in the paper,
In the first two examples, a boundary strip arises in the left or right and, respectively. In the third strip, there is no strip, but in the fourth example it appears in the both ends. Finally, at the last example, arise of a layer is investigated under one non- local boundary condition.
KEYWORDS: perturbed equation; non-local boundary condition; adjoint equation; fundamental solution; necessary condition; boundary layer.

1. INTRODUCTION

It is known that mathematical model of boundary layer arise is reduced to the problem for a differential equation with coefficient of a higher order derivative contains a small parameter. If in the boundary value problem the conditions are local and limiting state of the solution of the problem (when a small parameter tends to zero) doesn't satisfy one of the given conditions, then there arises a boundary layer at the point where boundary conditions are given, Prandtl (1905) first studied this problem. In sequel, mathematical models of other natural phenomena were reduced to these problems. Main investigators of these problems were R.E.O'Malleyr (1991) and Tikhonov (1952). Duolan et al. (1980) said that this boundary value problem remains open for the second order ordinary differential equation.

Here, at first an equation adjoint to the given equation is constructed, and then fundamental solution of this adjoint equation is found. The necessary condition that satisfies arbitrary solution of the given equation is determined by means of this fundamental solution. With the help of this necessary condition, the boundary conditions of the stated problem is localized. The case when under non-local condition, the perturbed problem remains open is shown in the paper Duolan et al. (1980). By means of necessary conditions obtained while investigating this problem the boundary conditions are localized and the place of arising of boundary strip is defined.

Up to now in investigation of boundary value problems is stated for the second order equations Aliev (1989), Jahanshahi (2000), Aliev et al. (1989) , Pashaev et al. (1995) and Aliev et al (2009) it was succeeded to obtain final result namely in this paper.

So, for arising (or not arising) of a boundary layer the conditions are reduced on data. A problem for the nonlinear system of the second order is considered in Jahanshahi et al. (2004).

2. Investigation of Arise of a Boundary Layer for Second Order Equation Under Local Conditions.

Now, let's consider the following problem:

$$\varepsilon^2 y''_\varepsilon(x) + y'_\varepsilon(x) - 2y_\varepsilon(x) = 0, x \in (0.1), \tag{1}$$

$$y_\varepsilon(0) = 1, \qquad y_\varepsilon(1) = 2. \tag{2}$$

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Obviously, the general solution of Eq.(1) is of the form:

$$y_\varepsilon(x) = \sum_{k=1}^{2} C_k e^{\theta_k(\varepsilon)x} \tag{3}$$

Here, $C_k (k = 1, 2)$ are arbitrary constants, $\theta_k(\varepsilon)$ are the roots of the characteristic equation

$$\varepsilon^2\theta^2 + \theta - 2 = 0,$$

Of the from

$$\theta_k(\varepsilon) = \frac{-1 + (-1)^k \sqrt{1 + 8\varepsilon^2}}{2\varepsilon^2}, (k = 1, 2). \tag{4}$$

Using conditions (2) we determine arbitrary constants contained in the general solution.

$$\begin{cases} C_1 + C_2 = 1, \\ C_1 e^{\theta_1(\varepsilon)} + C_2 e^{\theta_2(\varepsilon)} = 2 \end{cases}$$

We put the values

$$C_1 = \frac{e^{\theta_2(\varepsilon)} - 2}{e^{\theta_2(\varepsilon)} - e^{\theta_1(\varepsilon)}}, C_2 = \frac{2 - e^{\theta_1(\varepsilon)}}{e^{\theta_2(\varepsilon)} - e^{\theta_1(\varepsilon)}},$$

Obtained from the obtained system to Eq.(3) and find expressions for the solution of problem Eqs.(1) and (2):

$$y_\varepsilon(x) = \frac{e^{\theta_2(\varepsilon)} - 2}{e^{\theta_2(\varepsilon)} - e^{\theta_1(\varepsilon)}} e^{\theta_1(\varepsilon)x} + \frac{2 - e^{\theta_1(\varepsilon)}}{e^{\theta_2(\varepsilon)} - e^{\theta_1(\varepsilon)}} e^{\theta_2(\varepsilon)x}. \tag{5}$$

For determining limit state of this solution, at first we determine limit roots in Eq. (4). Since

$$\theta_1(0) = \lim_{\varepsilon\to 0} \theta_1(\varepsilon) = -\infty,$$

$$\theta_2(0) = \lim_{\varepsilon\to 0} \theta_2(\varepsilon) = \lim_{\varepsilon\to 0} \frac{-1 + \sqrt{1 + 8\varepsilon^2}}{2\varepsilon^2} = \lim_{\varepsilon\to 0} \frac{1 - (1 + 8\varepsilon^2)}{2\varepsilon^2\left(-1 - \sqrt{1 + 8\varepsilon^2}\right)} = 2.$$

The from Eq.(5) we get

$$y_0(x) = \frac{e^2 - 2}{e^2 - 0}.0 + \frac{2 - 0}{e^2 - 0} e^{2x} = 2e^{2(x-1)}. \tag{6}$$

Since Eq. (6) satisfies the second condition from Eq.(2), but doesn't satisfy the first one, a boundary layer arises at the end.

Theorem2.1: For boundary value problem Eqs.(1) and (2), a boundary layer arises on the boundary $x = 0$.

Now, consider the second boundary value problem:

$$\varepsilon^2 y''_\varepsilon(x) - y_\varepsilon(x) = 0, x \in (0, 1) \tag{7}$$

$$y_\varepsilon(0) = 0, \qquad y_\varepsilon(1) = 1 \tag{8}$$

General solution of Eq.(7) is of the form:

$$y_\varepsilon(x) = C_1 e^{-\frac{1}{\varepsilon}} + C_2 e^{\frac{1}{\varepsilon}}, \tag{9}$$

By means of boundary conditions (8), we determine the arbitrary constants C_1 and C_2 contained in Eq.(9):

$$\begin{cases} C_1 + C_2 = 0, \\ C_1 e^{-\frac{1}{\varepsilon}} + C_2 e^{\frac{1}{\varepsilon}} = 1, \end{cases}$$

$$C_1\left(e^{-\frac{1}{\varepsilon}} - e^{\frac{1}{\varepsilon}}\right) = 1,\ C_1 = \frac{1}{e^{-\frac{1}{\varepsilon}} - e^{\frac{1}{\varepsilon}}}, C_2 = -C_1 = \frac{1}{e^{\frac{1}{\varepsilon}} - e^{-\frac{1}{\varepsilon}}}.$$

Then the solution of problem Eqs.(6) and (8) is obtained on the form:

$$y_\varepsilon(x) = \frac{e^{-\frac{x}{e}}}{e^{-\frac{1}{\varepsilon}} - e^{\frac{1}{\varepsilon}}} + \frac{e^{-\frac{x}{e}}}{e^{\frac{1}{\varepsilon}} - e^{-\frac{1}{\varepsilon}}}, \tag{10}$$

Whence for $x \in (0,1)$ we find

$$y_0(x) = 0 \tag{11}$$

Since this function satisfies the first condition from (8), but doesn't satisfy the second one a boundary layer arises at the right end.

Theorem 2.2 For boundary value problem Eqs.(7) and (8) a boundary strip arises on the boundary $x = 1$.

Consider the following third problem:

$$\varepsilon\, y''_\varepsilon(x) + y'_\varepsilon(x) = x, \tag{12}$$

$$y_\varepsilon(0) = \frac{1}{2}, \qquad y_\varepsilon(1) = 1 \tag{13}$$

We can easily show that general solution of Eq.(12) is of the from:

$$y_\varepsilon(x) = C_1 + C_2 e^{-\frac{x}{\varepsilon}} + \frac{x^2}{2} - \varepsilon\, x \tag{14}$$

Here, the first two terms is a general solution of a homogeneous equation corresponding to Eq.(12), and the sum of two last terms is a particular solution of Eq.(12). Using conditions (13), we find constants C_1 and C_2 contained in the solution (14):

$$\begin{cases} C_1 + C_2 = \frac{1}{2}, \\ C_1 + C_2 e^{-\frac{1}{\varepsilon}} + \frac{1}{2} - \varepsilon = 1, \end{cases}$$

$$C_2\left(e^{-\frac{1}{\varepsilon}} - 1\right) = 1 - \frac{1}{2} + \varepsilon - \frac{1}{2}$$

$$C_2 = \frac{\varepsilon}{e^{-\frac{1}{\varepsilon}} - 1},$$

$$C_1 = \frac{1}{2} - \frac{\varepsilon}{e^{-\frac{1}{\varepsilon}} - 1} = \frac{e^{-\frac{1}{\varepsilon}} - 1 - 2\varepsilon}{2(e^{-\frac{1}{\varepsilon}} - 1)}.$$

Having substituted the values of the constants C_1 and C_2 in Eq.(14) for the solution of Eqs.(12) and (13) we get the expression:

$$y_\varepsilon(x) = \frac{e^{-\frac{1}{\varepsilon}} - 1 - 2\varepsilon}{2\left(e^{-\frac{1}{\varepsilon}} - 1\right)} + \frac{\varepsilon e^{-\frac{x}{\varepsilon}}}{e^{-\frac{1}{\varepsilon}} - 1} + \frac{x^2}{2} - \varepsilon x \tag{15}$$

Passing to limit as $\varepsilon \to 0$, from Eq.(15) we find

$$y_0(x) = \frac{x^2}{2} + \frac{-1}{-2} = \frac{1 + x^2}{2} \tag{16}$$

Since the obtained function (16) satisfies conditions (13), a boundary layer is not to be found anywhere.

Theorem 2.3 A boundary layer doesn't arise for Eqs.(12) and (13).
At last, we consider the following problem:

$$\varepsilon^2 y_\varepsilon''(x) - y_\varepsilon(x) = x, x \in (0, 1) \tag{17}$$

$$y_\varepsilon(0) = 1, \qquad y_\varepsilon(1) = 0 \tag{18}$$

General solution of Eq.(17) is of the form:

$$y_\varepsilon(x) = C_1 e^{-\frac{x}{\varepsilon}} + C_2 e^{\frac{x}{\varepsilon}} - x. \tag{19}$$

From the systems

$$\begin{cases} C_1 + C_2 = 1, \\ C_1 e^{-\frac{1}{\varepsilon}} + C_2 e^{\frac{1}{\varepsilon}} = 1, \end{cases}$$

We determine the constants C_1 and C_2 as

$$C_1 = \frac{1 - e^{\frac{1}{\varepsilon}}}{e^{-\frac{1}{\varepsilon}} - e^{\frac{1}{\varepsilon}}}, \; C_2 = \frac{1 - e^{-\frac{1}{\varepsilon}}}{e^{\frac{1}{\varepsilon}} - e^{-\frac{1}{\varepsilon}}},$$

And having substituted then in (19), for the solution of problem Eqs.(13) and (18) we get the expression:

$$y_\varepsilon(x) = \frac{1 - e^{\frac{1}{\varepsilon}}}{e^{-\frac{1}{\varepsilon}} - e^{\frac{1}{\varepsilon}}} e^{-\frac{x}{\varepsilon}} + \frac{1 - e^{-\frac{1}{\varepsilon}}}{e^{\frac{1}{\varepsilon}} - e^{-\frac{1}{\varepsilon}}} e^{-\frac{x}{\varepsilon}} - x,$$

Where

$$y_0(x) = -x.$$

Since this function doesn't satisfy conditions (18), a boundary layer arises at both ends.

Theorem 2.4 For the boundary value Eqs.(17) and (18) a boundary layer arises at both boundaries.

3. Investigation of Arise of a Boundary Layer for a Second Order Equation Under Non-Local Boundary Condition.

Definition of arise or absence of a boundary layer under non-local boundary conditions is not difficult. Indeed, if limit state of the solution of the given problem doesn't satisfy then of one condition, than there arises a boundary layer. But it in difficult to define a boundary a which a boundary layer arises.
In this connection we consider the following problem:

$$\ell_\varepsilon y_\varepsilon(x) \equiv \varepsilon\, y''_\varepsilon(x) - y'_\varepsilon(x) + 3y_\varepsilon(x) = 0, x \in (0,1) \tag{20}$$

$$y_\varepsilon(0) + 2y_\varepsilon(1) = 0,$$

$$y'_\varepsilon(0) - 4y'_\varepsilon(1) = 1. \tag{21}$$

Solve this problem by ordinary method.
General solution of Eq. (20) is of the form:

$$y_\varepsilon(x) = C_1 e^{\theta_1(\varepsilon)x} + C_2 e^{\theta_2(\varepsilon)x} \tag{22}$$

Here C_1 and C_2 are arbitrary constants, $\theta_k(\varepsilon)$ are the roots of the characteristic equation

$$\varepsilon\,\theta^2 - \theta + 3 = 0,$$

That is of the form:

$$\theta_k(\varepsilon) = \frac{1 + (-1)^k \sqrt{1 - 12\varepsilon}}{2\varepsilon}, k = 1, 2. \tag{23}$$

For using the boundary condition, we differentiate (22) sleet put it into (21) and get:
If in (22) we write the found constants $C_k\,(k = 1, 2)$, we get an expression for the solution of Eqs.(20) and (21) in the form:

$$y_\varepsilon(x) = -\frac{1 + 2e^{\theta_2(\varepsilon)}}{\Delta(\varepsilon)} e^{\theta_1(\varepsilon)x} + -\frac{1 + 2e^{\theta_1(\varepsilon)}}{\Delta(\varepsilon)} e^{\theta_2(\varepsilon)x}. \tag{24}$$

So, that

$$\Delta(\varepsilon) = \begin{vmatrix} 1 + 2e^{\theta_1(\varepsilon)} & 1 + 2e^{\theta_2(\varepsilon)} \\ \theta_1(\varepsilon)\left[1 - 4e^{\theta_1(\varepsilon)}\right] & \theta_2(\varepsilon)\left[1 - 4e^{\theta_2(\varepsilon)}\right] \end{vmatrix} \neq 0 \tag{25}$$

In order to pass to limit (as $\varepsilon \to 0$) in expression (24) obtained for the solution of the problem, at first we determine expressions for the roots of characteristic Eq.(23) as $\varepsilon \to 0$.

$$\theta_1(0) = \lim_{\varepsilon \to 0} \frac{1 - \sqrt{1 - 12\varepsilon}}{2\varepsilon} = 3,$$

$$\theta_2(0) = \lim_{\varepsilon \to 0} \frac{1 - \sqrt{1 - 12\varepsilon}}{2\varepsilon} = \infty,$$

Consequently from (24) we get:

$$y_0(x) = \lim_{\varepsilon \to 0} y_\varepsilon(x) = 0 \tag{26}$$

Since the obtained function (26) satisfies the first condition but doesn't satisfy the second condition from (21), then there arises a boundary layer for Eqs.(20) and (21). But it is not mown on which boundary it arises.
It is easy to see that Vladimirov (1976) the equation adjoint to Eq.(20) is of the form:

$$\ell^*_\varepsilon z_\varepsilon(x) \equiv \varepsilon\, z''_\varepsilon(x) + z'_\varepsilon(x) + 3z_\varepsilon(x) = g(x),$$

Where $g(x)$ is an arbitrary continuous function. The fundamental solution of this adjoint equation is of the form Aliev and Jahanshahi (1997):

$$z_\varepsilon(x - \xi) = \theta(x - \xi)\frac{e^{\chi_2(\varepsilon)(x - \xi)} - e^{\chi_1(\varepsilon)(x - \xi)}}{\sqrt{1 - 12\varepsilon}} \tag{27}$$

Here, $\theta(t)$ is Heaviside's unit function, $\chi_k(\varepsilon)$ are the roots of the characteristic equation

$$\varepsilon\chi^2 + \chi + 3 = 0$$

That are of the form:

$$\chi_k(\varepsilon) = \frac{-1 + (-1)^k \sqrt{1 - 12\varepsilon}}{2\varepsilon}, k = 1,\ 2, \tag{28}$$

We can easily see that

$$\begin{cases} \chi_1(0) = -\infty, \\ \chi_2(0) = -3. \end{cases} \tag{29}$$

Having scalary multiplied the obtained fundamental solution and its derivative by the both sides of Eq.(21), and integrating by part, similar to Aliev and Jahanshahi (2002) and Aliev and Hosseini (2003) and Aliev and Ashrafi (2009) we get following necessary conditions:

$$y_\varepsilon(0) = -\frac{e^{\chi_2(\varepsilon)} - e^{\chi_1(\varepsilon)}}{\sqrt{1 - 12\varepsilon}} \varepsilon\, y'_\varepsilon(1) + \frac{[\varepsilon\chi_2(\varepsilon) + 1]e^{\chi_2(\varepsilon)} - [\varepsilon\chi_1(\varepsilon) + 1]e^{\chi_1(\varepsilon)}}{\sqrt{1 - 12\varepsilon}} y_\varepsilon(1) \tag{30}$$

$$y'_\varepsilon(0) = \varepsilon\, y'_\varepsilon(1) \frac{\chi_2(\varepsilon) e^{\chi_2(\varepsilon)} - \chi_1(\varepsilon)\, e^{\chi_1(\varepsilon)}}{\sqrt{1 - 12\varepsilon}} + \varepsilon\, y_\varepsilon(1) \frac{e^{\chi_2(\varepsilon)} - e^{\chi_1(\varepsilon)}}{\sqrt{1 - 12\varepsilon}} \tag{31}$$

Passing to limit in necessary Eq.(30) and Eq.(31) and given boundary Eq.(21), we get a system of four linear algebraic equations. We solve this system and find, :

$$y_0(0) = y_0(1) = y'_0(0) = 0, y'_0(1) = -\frac{1}{4}. \tag{32}$$

Since the obtained Eq.(7) doesn't satisfy the last condition from (32), then a boundary layer arises at the point $x = 1$

Theorem 3.1. In Eqs.(20) and (21) with non-local boundary condition, a boundary layer arises at the point $x = 1$.

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