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Investigation of Arising of Boundary a Layer in a Boundary Value Problem for the Fourth Order Ordinary Differential Equation

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ABSTRACT

Mathematical model of boundary layer arise is reduced to the problems for differential equation with coefficient at the higher order derivative contains a small parameter.
Investigation of such problems is reduced to the fact that some equations of the obtained system of the first order, contains a small parameter in the coefficient of the derivative. The other equations of this system have no such property. In some cases the trajectories velocity is ordinary, in some trajectories it is instantaneous.
In this paper the given non-local conditions of the problem are localized by means of necessary conditions obtained while investigating the same problem and the process of arising or (not arising) of boundary layer is researched by the ordinary way.
KEYWORDS: boundary layer, instantaneous velocity, small parameter, adjoint equation, fundamental solution, necessary conditions, asymptotic.

INTRODUCTION

It is known that mathematical model of boundary layer arise is reduced to the problem for a differential equation with coefficient of a higher order derivative contains a small parameter.
If in the boundary value problem the conditions are local and limiting state of the solution of the problem (when a small parameter tends to zero) doesn't satisfy one of the given conditions, then a boundary layer arises at the point where boundary conditions are given, Prandtl [1] first studied this problem. Then, mathematical models of other natural phenomena were reduced to these problems. Main investigators of these problems were R.E.O'Malleyr[5] and Tikhonov [4]. P. Duolan, J.J.H. Milleranal and L.H.A. Shilders [6] noted that this boundary value problem remains open for the fourth order ordinary differential equation.
Here we consider a problem for the fourth order ordinary linear differential equation with the coefficient of the higherst order derivative depends on a small parameter under non-local boundary conditions.
So, limiting state of the solution of the problem (for $\varepsilon > 0$) and non-fulfillment of one or some boundary conditions don't define at which end boundary layer arises.
Therefore, by means of necessary conditions obtained while investigating this problem the boundary conditions are localized and the place of arising of boundary strip is defined.
Up to now in investigation of boundary value problems is stated for the fourth order equations [7], [8],[9],[10],[12] it was succeeded to obtain final result namely in this paper.
So, for arising (or not arising) of a boundary layer the conditions are reduced on data.
A problem for the nonlinear system of the second order is considered in [14].

Problem statement and equation of adjoint problem

Let's consider the following boundary value problem

$$\ell_\varepsilon y_\varepsilon \equiv \varepsilon^2 y_\varepsilon^{IV}(x) + y_\varepsilon^{II}(x) + a y_\varepsilon^{I}(x) + b y_\varepsilon(x) = f(x), \quad (1)$$
$$x \in (0,1),\ 0 < \varepsilon << 1,$$

$$\ell_j y_\varepsilon \equiv \sum_{k=0}^{3}\left[\alpha_{jk} y_\varepsilon^{(k)}(0) + \beta_{jk} y_\varepsilon^{(k)}(1)\right] = 0,\ j = \overline{1,4}, \quad (2)$$

here $\varepsilon > 0$ is a small parameter, a, b, α_{jk} and β_{jk}, $j = \overline{1,4}$; $k = \overline{0,3}$ are the given real constants, $f(x)$ is a real valued function. Boundary conditions are linearly independent,
the conditions imposed on data will be given below.

If $y(x)$ and $z(x)$ are complex-valued functions, their scalar product is given in the form of

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$$(y,Z)\overset{def}{=}\int_0^1 y(x)\overline{Z(x)}dx\,, \qquad (3)$$

here $\overline{Z}(x)$ denotes the complex adjoint of $Z(x)$.

Now we use the scheme of [2] and [15], to get the expression in the form

$$\ell^*_\varepsilon Z_\varepsilon \equiv \varepsilon^2 Z^{IV}_\varepsilon(x) + Z''_\varepsilon(x) - aZ'_\varepsilon(x) + bZ_\varepsilon(x),$$

for the equation adjoint to the equation (1).

Construction of fundamental solution

Let's define the fundamental solution [2],[11],[13] of the equation

$$\ell^*_\varepsilon Z_\varepsilon \equiv \varepsilon^2 Z^{IV}_\varepsilon(x) + Z''_\varepsilon(x) - aZ'_\varepsilon(x) + bZ_\varepsilon(x) = g(x), \qquad (4)$$

adjoint to the equation (1). Here $g(x)$ is an arbitrary continuous function. It is easy to obtain for the special solution of the equation (4) the expression

$$Z_\varepsilon(x) = \sum_{k=1}^{4}(-1)^k \frac{W^{(4,k)}_\varepsilon}{\varepsilon^2 W_\varepsilon}\int_0^x e^{\theta_k(\varepsilon)(x-\xi)}g(\xi)d\xi\,, \qquad (5)$$

here

$$W_\varepsilon = \begin{vmatrix} 1 & 1 & 1 & 1 \\ \theta_1(\varepsilon) & \theta_2(\varepsilon) & \theta_3(\varepsilon) & \theta_4(\varepsilon) \\ \theta_1^2(\varepsilon) & \theta_2^2(\varepsilon) & \theta_3^2(\varepsilon) & \theta_4^2(\varepsilon) \\ \theta_1^3(\varepsilon) & \theta_2^3(\varepsilon) & \theta_3^3(\varepsilon) & \theta_4^3(\varepsilon) \end{vmatrix} \neq 0,$$

$W^{(4,k)}_\varepsilon$ are minors of the elements of the 4-th line and $k-$th column of the Vandermond determinant W_ε.

Thus, for the fundamental solution of equation (4) we obtain the expression:

$$Z_{1\varepsilon}(x-\xi) = -\sum_{k=1}^{4}(-1)^k \frac{W^{(4,k)}_\varepsilon}{\varepsilon^2 W_\varepsilon}\eta(x-\xi)e^{\theta_k(\varepsilon)(x-\xi)}\,, \qquad (6)$$

here

$$\eta(t) = \begin{cases} 1, & t>0, \\ 1/2, & t=0, \\ 0, & t<0, \end{cases} \qquad (7)$$

is a Heaviside unique function [11], $\theta_k(\varepsilon),\ k=\overline{1,4}$ are the solutions of the characteristic equation

$$\varepsilon^2\theta^4 + \theta^2 - a\theta + b = 0,$$

corresponding to (4).

Obtaining of the necessary conditions

If in the Lagrange formula we take the fundamental solution (6) we get [15]

$$\begin{aligned} &-\varepsilon^2\Big[y'''_\varepsilon(x)\overline{Z}_{1\varepsilon}(x-\xi) - y''_\varepsilon(x)\overline{Z}'_{1\varepsilon}(x-\xi) + y'_\varepsilon(x)\overline{Z}''_{1\varepsilon}(x-\xi) - \\ &- y_\varepsilon(x)\overline{Z}'''_{1\varepsilon}(x-\xi)\Big]\Big|_{x=0}^{1} - \Big[y^I_\varepsilon(x)\overline{Z}_{1\varepsilon}(x-\xi) - y_\varepsilon(x)\overline{Z}'_{1\varepsilon}(x-\xi)\Big]\Big|_{x=0}^{1} - \qquad (8) \\ &- ay_\varepsilon(x)\overline{Z}_{1\varepsilon}(x-\xi)\Big|_{x=0}^{1} + \int_0^1 f(x)\overline{Z}_{1\varepsilon}(x-\xi)dx = \begin{cases} y_\varepsilon(\xi), & \xi\in(0,1), \\ \dfrac{1}{2}y_\varepsilon(\xi), & \xi=0,\ \xi=1. \end{cases} \end{aligned}$$

In the same way, multiplying scalarly the equation (1) by the derivatives of the fundamental solution (6), similar to (8) [5],[6] one can get

$$\varepsilon^2\left[y'''_\varepsilon(x)\bar{Z}'_{1\varepsilon}(x-\xi)-y''_\varepsilon(x)\bar{Z}''_{1\varepsilon}(x-\xi)+y'_\varepsilon(x)\bar{Z}'''_{1\varepsilon}(x-\xi)\right]_{x=0}^{1}+$$

$$+y'_\varepsilon(x)\bar{Z}'_{1\varepsilon}(x-\xi)\Big|_{x=0}^{1}+by_\varepsilon(x)\bar{Z}_{1\varepsilon}(x-\xi)\Big|_{x=0}^{1}-\int_0^1 f(x)\bar{Z}'_{1\varepsilon}(x-\xi)dx=$$

$$=\begin{cases} y'_\varepsilon(\xi), & \xi\in(0,1),\\ \frac{1}{2}y'_\varepsilon(\xi), & \xi=0,\ \xi=1, \end{cases} \tag{9}$$

$$-\varepsilon^2\left[y'''_\varepsilon(x)\bar{Z}''_{1\varepsilon}(x-\xi)-y''_\varepsilon(x)\bar{Z}'''_{1\varepsilon}(x-\xi)\right]_{x=0}^{1}-ay'_\varepsilon(x)\bar{Z}'_{1\varepsilon}(x-\xi)\Big|_{x=0}^{1}-$$

$$-b\left[y_\varepsilon(x)\bar{Z}'_{1\varepsilon}(x-\xi)-y'_\varepsilon(x)\bar{Z}_{1\varepsilon}(x-\xi)\right]_{x=0}^{1}+\int_0^1 f(x)\bar{Z}''_{1\varepsilon}(x-\xi)dx= \tag{10}$$

$$=\begin{cases} y''_\varepsilon(\xi), & \xi\in(0,1),\\ \frac{1}{2}y''_\varepsilon(\xi), & \xi=0,\ \xi=1, \end{cases}$$

$$\varepsilon^2 y'''_\varepsilon(x)\bar{Z}'''_{1\varepsilon}(x-\xi)\Big|_{x=0}^{1}+y''_\varepsilon(x)\bar{Z}''_{1\varepsilon}(x-\xi)\Big|_{x=0}^{1}+$$

$$+a\left[y'_\varepsilon(x)\bar{Z}''_{1\varepsilon}(x-\xi)-y''_\varepsilon(x)\bar{Z}'_{1\varepsilon}(x-\xi)\right]\Big|_{x=0}^{1}+$$

$$+b\left[y_\varepsilon(x)\bar{Z}''_{1\varepsilon}(x-\xi)-y'_\varepsilon(x)\bar{Z}'_{1\varepsilon}(x-\xi)+y''_\varepsilon(x)\bar{Z}_{1\varepsilon}(x-\xi)\right]\Big|_{x=0}^{1}- \tag{11}$$

$$-\int_0^1 f(x)\bar{Z}'''_{1\varepsilon}(x-\xi)dx=\begin{cases} y'''_\varepsilon(\xi), & \xi\in(0,1),\\ \frac{1}{2}y'''_\varepsilon(\xi), & \xi=0,\ \xi=1. \end{cases}$$

Remark 1. The second expressions of the obtained above relations (8) –(11) are necessary conditions for the equation (1) i.e. any function defined in $(0,\,1)$ and satisfying the equation (1) satisfies the relations (8) –(11). Hence we see that in order to obtain necessary conditions it is sufficient to know only the fundamental solution of the adjoint equation. Necessary conditions are independent of (2).

Properties of fundamental solution

We can easily see that the following expressions are true for the derivatives of the fundamental solution (6)

$$Z_{1\varepsilon}^{(j)}(x-\xi)=\sum_{k=1}^{4}(-1)^k\frac{W_\varepsilon^{(4,k)}}{\varepsilon^2 W_\varepsilon}\eta(x-\xi)\theta_k^j(\varepsilon)e^{\theta_k(\varepsilon)(x-\xi)},\quad j=\overline{0,3}, \tag{12}$$

$$Z_{1\varepsilon}^{IV}(x-\xi)=\sum_{k=1}^{4}(-1)^k\frac{W_\varepsilon^{(4,k)}}{\varepsilon^2 W_\varepsilon}\delta(x-\xi)\theta_k^3(\varepsilon)+$$

$$+\sum_{k=1}^{4}(-1)^k\frac{W_\varepsilon^{(4,k)}}{\varepsilon^2 W_\varepsilon}\eta(x-\xi)\theta_k^4(\varepsilon)e^{\theta_k(\varepsilon)(x-\xi)}, \tag{13}$$

here the properties

$$\eta'(x-\xi)=\delta(x-\xi),$$

and

$$\delta(x-\xi)F(x)=\delta(x-\xi)F(\xi),$$

are used, $\delta(x-\xi)$ is Dirac's delta function.

Remark 2. As is seen from the properties of algebraic complement of the elements of determinant we take into account

$$\sum_{k=1}^{4}(-1)^k \frac{W_\varepsilon^{(4,k)}}{\varepsilon^2 W_\varepsilon}\theta_k^j(\varepsilon)=\begin{cases}0, & j=\overline{0,\ 2},\\ 1, & j=3,\end{cases}$$

as well.

Thus, the relation

$$\varepsilon^2 Z_{1\varepsilon}^{IV}(x-\xi)+Z_{1\varepsilon}''(x-\xi)-aZ_{1\varepsilon}'(x-\xi)+bZ_{1\varepsilon}(x-\xi)=\delta(x-\xi), \qquad (14)$$

is obtained for the fundamental solution.

Remark 3. Considering (7) in the expression (12) we can see that

$$Z_{1\varepsilon}^{(j)}(-1)=0, \quad j=0,3. \qquad (15)$$

Remark 4. In the same expression (12) we take into account the results given in Remark 2 and get

$$Z_{1\varepsilon}^{(j)}(0)=\begin{cases}0, & j=\overline{0,\ 2},\\ 1/2\varepsilon^2, & j=3,\end{cases} \qquad (16)$$

Considering notations above the necessary conditions will take the form

$$\begin{aligned} y_\varepsilon(0)=\int_0^1 f(x)\bar{Z}_{1\varepsilon}(x)dx-\varepsilon^2\left[y_\varepsilon'''(1)\bar{Z}_{1\varepsilon}(1)-y_\varepsilon''(1)\bar{Z}_{1\varepsilon}'(1)+y_\varepsilon''(1)\bar{Z}_{1\varepsilon}''(1)-y_\varepsilon(1)\bar{Z}_{1\varepsilon}'''(1)\right]-\\ -\varepsilon^2 y_\varepsilon(0)\cdot\frac{1}{2\varepsilon^2}-\left[y_\varepsilon^{I}(1)\bar{Z}_{1\varepsilon}(1)-y_\varepsilon(1)\bar{Z}_{1\varepsilon}'(1)\right]-ay_\varepsilon(1)\bar{Z}_{1\varepsilon}(1), \end{aligned} \qquad (17)$$

$$\begin{aligned} y_\varepsilon'(0)=-\int_0^1 f(x)\bar{Z}_{1\varepsilon}'(x)dx+\varepsilon^2\left[y_\varepsilon'''(1)\bar{Z}_{1\varepsilon}'(1)-y_\varepsilon''(1)\bar{Z}_{1\varepsilon}''(1)+y_\varepsilon'(1)\bar{Z}_{1\varepsilon}'''(1)\right]-\\ -\varepsilon^2 y_\varepsilon'(0)\cdot\frac{1}{2\varepsilon^2}+y_\varepsilon'(1)\bar{Z}_{1\varepsilon}'(1)+by_\varepsilon(1)\bar{Z}_{1\varepsilon}(1), \end{aligned} \qquad (18)$$

$$\begin{aligned} y_\varepsilon''(0)=\int_0^1 f(x)\bar{Z}_{1\varepsilon}''(x)dx-\varepsilon^2\left[y_\varepsilon'''(1)\bar{Z}_{1\varepsilon}''(1)-y_\varepsilon''(1)\bar{Z}_{1\varepsilon}''(1)\right]-\\ -\varepsilon^2 y_\varepsilon''(0)\cdot\frac{1}{2\varepsilon^2}-ay_\varepsilon'(1)\bar{Z}_{1\varepsilon}'(1)-b\left[y_\varepsilon(1)\bar{Z}_{1\varepsilon}'(1)-y_\varepsilon'(1)\bar{Z}_{1\varepsilon}(1)\right], \end{aligned} \qquad (19)$$

$$\begin{aligned} y_\varepsilon'''(0)=-\int_0^1 f(x)\bar{Z}_{1\varepsilon}'''(x)dx+\varepsilon^2 y_\varepsilon'''(1)\bar{Z}_{1\varepsilon}'''(1)-\varepsilon^2 y_\varepsilon'''(0)\cdot\frac{1}{2\varepsilon^2}+y_\varepsilon''(1)\bar{Z}_{1\varepsilon}''(1)+\\ +a\left[y_\varepsilon'(1)\bar{Z}_{1\varepsilon}''(1)-y_\varepsilon''(1)\bar{Z}_{1\varepsilon}'(1)\right]+b\left[y_\varepsilon(1)\bar{Z}_{1\varepsilon}''(1)-y_\varepsilon'(1)\bar{Z}_{1\varepsilon}'(1)+y_\varepsilon''(1)\bar{Z}_{1\varepsilon}(1)\right]. \end{aligned} \qquad (20)$$

Remark 5. In the same way in (8)–(11) necessary conditions corresponding to $\xi=1$ turn into identity. So, we obtain four independent necessary conditions in the form of (17) –(20).

Theorem 1. If a and b are real constants, $\varepsilon>0$ is a small parameter, $f(x)$ is a real- valued continuous function, then any solution of the equation (1) determined in $(0,\ 1)$ satisfies the necessary conditions (17)– (20).

Asymptotic expressions of the roots of the characteristic equation

As obtained above the characteristic equation corresponding to the equation (4) is in the form

$$\varepsilon^2\theta^4+\theta^2-a\theta+b=0, \qquad (21)$$

at first we are engaged in the solution of this equation [20] expressed in the form

$$\theta(\varepsilon)=\sum_{j=0}^{m}\theta_j\varepsilon^j+O(\varepsilon^{m+1}), \qquad (22)$$

Here θ_j are unknown constants. In order to determine them we write expression (22) and the expressions

$$\theta^2(\varepsilon)=\sum_{j=0}^{m}A_j\varepsilon^j\,,\qquad A_j=\sum_{p=0}^{j}\theta_p\theta_{j-p}\,,$$

$$\theta^4(\varepsilon)=\sum_{j=0}^{m}B_j\varepsilon^j\,,\qquad B_j=\sum_{s=0}^{j}\sum_{p=0}^{s}\theta_p\theta_{s-p}\sum_{q=0}^{j-s}\theta_q\theta_{j-s-q}\,,$$

in (21), compare the same degrees of the parameter ε and get

$$\theta_{ok}=\frac{a+(-1)^k\sqrt{a^2-4b}}{2},\ \ k=1,\,2,\ \ \theta_{01}\neq\theta_{02}\,. \tag{23}$$

If we accept the condition

$$\theta_{1k}=0\,,\quad k=1,\,2 \tag{24}$$

for the other coefficient we get

$$\theta_{2k}=(-1)^{k+1}\frac{\left[a+(-1)^k\sqrt{a^2-4b}\right]^4}{10\sqrt{a^2-4b}},\quad k=1,\,2\,. \tag{25}$$

Having continued this process we can find the remaining terms of the asymptotic expression of (22). Now let's find the solution of

$$\theta(\varepsilon)=\theta_{-1}\varepsilon^{-1}+\sum_{j=0}^{m}\theta_j\varepsilon^j+O(\varepsilon^{m+1})\,, \tag{26}$$

being the asymptotic expression of characteristic equation (21). Similar to the case above case we write the expressions (26) and

$$\theta^2(\varepsilon)=\theta_{-1}^2\varepsilon^{-2}+2\theta_{-1}\varepsilon^{-1}\sum_{j=0}^{m}\theta_j\varepsilon^j+\sum_{j=0}^{m}A_j\varepsilon^j\,,$$

$$\theta^4(\varepsilon)=\theta_{-1}^4\varepsilon^{-4}+4\theta_{-1}^2\varepsilon^{-2}\sum_{j=0}^{m}A_j\varepsilon^j+\sum_{j=0}^{m}B_j\varepsilon^j+$$

$$+4\theta_{-1}^3\varepsilon^{-3}\sum_{j=0}^{m}\theta_j\varepsilon^j+2\theta_{-1}^2\varepsilon^{-2}\sum_{j=0}^{m}A_j\varepsilon^j+4\theta_{-1}\varepsilon^{-1}\sum_{j=0}^{m}\theta_j\varepsilon^j\sum_{j=0}^{m}A_j\varepsilon^j,$$

in (21), grouping with respect to the degrees of ε and for the unknown coefficients of (26) we obtain

$$\theta_{-1k}=(-1)^k i\,,\quad k=3,\,4\,, \tag{27}$$

$$\theta_{0k}=-\frac{a}{2},\quad k=3,\,4\,, \tag{28}$$

$$\theta_{1k}=(-1)^k i\frac{3a^2-4b}{8},\quad k=3,\,4\,. \tag{29}$$

Continuing this process one can find the other coefficients of (26).

Theorem 2. Under the conditions of Theorem 1, if $a^2\neq 4b$ then there exist the solutions

$$\theta_k(\varepsilon)=\sum_{j=0}^{m}\theta_{jk}\varepsilon^j+O(\varepsilon^{m+1}),\qquad k=1,\,2\,,$$

and

$$\theta_k(\varepsilon)=\theta_{-1k}\varepsilon^{-1}+\sum_{j=0}^{m}\theta_{jk}\varepsilon^j+O(\varepsilon^{m+1}),\quad k=3,\,4$$

corresponding to the asymptotic expressions (22) and (26) of the roots of equation (21).

Fundamental solution and asymptotic expressions of its derivatives

Let's consider expressions obtained for the fundamental solution and its derivatives. As is seen from the asymptotics obtained for the roots of the characteristic function [3].

$$\left.\frac{W_\varepsilon^{(4,1)}}{\varepsilon^2 W_\varepsilon}\right|_{\varepsilon=0}=\left.\frac{W_\varepsilon^{(4,2)}}{\varepsilon^2 W_\varepsilon}\right|_{\varepsilon=0}=\frac{1}{\sqrt{a^2-4b}},$$

$$\left.\frac{W_\varepsilon^{(4,1)}}{\varepsilon^2 W_\varepsilon}\right|_{\varepsilon=0}=\left.\frac{W_\varepsilon^{(4,2)}}{\varepsilon^2 W_\varepsilon}\right|_{\varepsilon=0}=\frac{1}{\sqrt{a^2-4b}},$$

$$Z_{1\varepsilon}(x-\xi)\big|_{\varepsilon=0}=\frac{\eta(x-\xi)}{\sqrt{a^2-4b}}\left[e^{\theta_{02}(x-\xi)}-e^{\theta_{01}(x-\xi)}\right], \tag{30}$$

$$\varepsilon^j Z_{1\varepsilon}^{(j)}(x-\xi)\big|_{\varepsilon=0}=0,\quad j=\overline{1,3}. \tag{31}$$

Hence we see that the following statements are true.

Theorem 3. Under the conditions of Theorem 2 for the existence of the integrals for $\varepsilon=0$ in the expressions (17) – (20) it is sufficient fulfillment the conditions

$$f^{(k)}(0)=f^{(k)}(1)=0,\quad k=\overline{0,2},$$

$$f(x)\in C^{(3)}(0,1)\cap C^{(2)}[0,1].$$

Theorem 4. Under the conditions of Theorem 3 the condition

$$y_0^{(k+2)}(1)+ay_0^{(k+1)}(1)+by_0^{(k)}(1)=0,\quad k=0,1,$$

is satisfied.

Theorem 5. Since under the conditions of Theorem 4

$$\left[\varepsilon^2\overline{Z}_{1\varepsilon}'''(1)+\overline{Z}_{1\varepsilon}'(1)\right]\Big|_{\varepsilon=0}=\frac{\theta_{02}e^{\theta_{02}}-\theta_{01}e^{\theta_{01}}}{\sqrt{a^2-4b}},$$

as we see from the expressions (17) and (18) and the results of Theorem 4 the expressions (19) and (20) exist for $\varepsilon=0$.

Asymptotic expressions of necessary conditions

If we pass to limit in the expressions (17)–(20) obtained for the above mentioned necessary conditions as $\varepsilon\to 0$

$$y_0(0)=\int_0^1 f(x)\overline{Z}_{10}(x)dx-y_0'(1)\overline{Z}_{10}(1)-ay_0(1)\overline{Z}_{10}(1)+y_0(1)\sum_{k=1}^{2}(-1)^k\frac{\theta_{0k}}{\sqrt{a^2-4b}}e^{\theta_{0k}}, \tag{32}$$

$$y_0'(0)=\int_0^1 f'(x)\overline{Z}_{10}(x)dx+by_0(1)\overline{Z}_{10}(1)+y_0'(1)\sum_{k=1}^{2}(-1)^k\frac{\theta_{0k}}{\sqrt{a^2-4b}}e^{\theta_{0k}}, \tag{33}$$

$$y_0''(0)=\int_0^1 f''(x)\overline{Z}_{10}(x)dx+by_0'(1)\overline{Z}_{10}(1)-\left[ay_0'(1)+by_0(1)\right]\sum_{k=1}^{2}(-1)^k\frac{\theta_{0k}}{\sqrt{a^2-4b}}e^{\theta_{0k}}, \tag{34}$$

$$y_0'''(0)=\int_0^1 f'''(x)\overline{Z}_{10}(x)dx+by_0'(1)\overline{Z}_{10}(1)-\left[ay_0''(1)+by_0'(1)\right]\sum_{k=1}^{2}(-1)^k\frac{\theta_{0k}}{\sqrt{a^2-4b}}e^{\theta_{0k}}. \tag{35}$$

Remark 8. We can easily see that the expressions (32) and (33) are independent necessary conditions on the interval $(0,1)$ for the equation obtained as $\varepsilon\to 0$ in the equation (1).

But the expressions (34) and (35) are obtained by transforming the expressions obtained after differentiation of the solution of equation for $\varepsilon=0$.

Now, let's return to the problem (1)–(2) stated before. If we write independent necessary conditions (17)-(20) obtained for this equation in the boundary conditions (2), we get the system:

$$\sum_{k=1}^{4}B_{jk}(\varepsilon)y_\varepsilon^{(k-1)}(1)=\int_0^1 F_j(x)\overline{Z}_{1\varepsilon}(x)dx,\quad j=\overline{1,4}. \tag{36}$$

Here

$$\begin{cases} B_{j1}(\varepsilon)=\beta_{jo}+b\alpha_{j3}\overline{Z}_{1\varepsilon}(1)-b\alpha_{j2}\overline{Z}'_{1\varepsilon}(1)+b\alpha_{j1}\overline{Z}_{1\varepsilon}(1)+\alpha_{j0}\left[\varepsilon^2\overline{Z}'''_{1\varepsilon}(1)+\overline{Z}'_{1\varepsilon}(1)-a\overline{Z}_{1\varepsilon}(1)\right], \\ B_{j2}(\varepsilon)=\beta_{j1}-\alpha_{j3}\left[b\overline{Z}'_{1\varepsilon}(1)-a\overline{Z}''_{1\varepsilon}(1)\right]-\alpha_{j2}\left[a\overline{Z}'_{1\varepsilon}(1)-b\overline{Z}_{1\varepsilon}(1)\right]+\alpha_{j1}\left[\varepsilon^2\overline{Z}'''_{1\varepsilon}(1)+\overline{Z}'_{1\varepsilon}(1)\right]- \\ -\alpha_{j0}\left[\varepsilon^2\overline{Z}''_{1\varepsilon}(1)+\overline{Z}_{1\varepsilon}(1)\right], \\ B_{j3}(\varepsilon)=\beta_{j2}+\alpha_{j3}\left[\overline{Z}''_{1\varepsilon}(1)-a\overline{Z}'_{1\varepsilon}(1)+b\overline{Z}_{1\varepsilon}(1)\right]+\alpha_{j2}\varepsilon^2\overline{Z}'''_{1\varepsilon}(1)-\alpha_{j1}\varepsilon^2\overline{Z}''_{1\varepsilon}(1)+ \\ +\alpha_{j0}\varepsilon^2\overline{Z}'_{1\varepsilon}(1), \\ B_{j4}(\varepsilon)=\beta_{j3}+\alpha_{j3}\varepsilon^2\overline{Z}''_{1\varepsilon}(1)-\alpha_{j2}\varepsilon^2\overline{Z}''_{1\varepsilon}(1)+\alpha_{j1}\varepsilon^2\overline{Z}'_{1\varepsilon}(1)-\alpha_{j0}\varepsilon^2\overline{Z}_{1\varepsilon}(1), \\ F_j(x)=-\alpha_{j0}f(x)-\alpha_{j1}f'(x)-\alpha_{j2}f''(x)-\alpha_{j3}f'''(x). \end{cases} \quad (37)$$

Let's denote the determinant of the system (36) by $B(\varepsilon)$. Assume

$$B(\varepsilon)\neq 0, \quad (38)$$

then form (36) we get:

$$y_\varepsilon^{(s)}(1)==\int_0^1\sum_{k=1}^4(-1)^{k+s+1}\frac{B^{(k,s+1)}(\varepsilon)}{B(\varepsilon)}F_k(x)\overline{Z}_{1\varepsilon}(x)dx,\quad s=\overline{0,\,3}, \quad (39)$$

here by $B^{(k,s+1)}(\varepsilon)$ we denote a minor element of the element arranged in the intersection of the k-th row and $(s+1)$-th column of the determinant $B(\varepsilon)$.

If we write the expressions (39) in necessary conditions (17)–(20) we can determine the values of derivatives of $y_\varepsilon(x)$ up to third order in $x=0$. Finally, if we write these values in (8) we find the expression

$$\begin{aligned} y_\varepsilon(\xi)=&\int_0^1 f(x)\overline{Z}_{1\varepsilon}(x-\xi)dx-\varepsilon^2\overline{Z}_{1\varepsilon}(1-\xi)y'''_\varepsilon(1)+\varepsilon^2\overline{Z}_{1\varepsilon}(1-\xi)y''_\varepsilon(1)- \\ &-\left[\varepsilon^2\overline{Z}''_{1\varepsilon}(1-\xi)+\overline{Z}_{1\varepsilon}(1-\xi)\right]y'_\varepsilon(1)+\left[\varepsilon^2\overline{Z}'''_{1\varepsilon}(1-\xi)+\overline{Z}'_{1\varepsilon}(1-\xi)-a\overline{Z}_{1\varepsilon}(1-\xi)\right]y_\varepsilon(1), \end{aligned} \quad (40)$$

$$\xi\in(0,\,1),$$

for the unknown function $y_\varepsilon(x)$ in $x\in(0,\,1)$.

If we pass to limit as $\varepsilon\to 0$,we get the relation

$$\begin{aligned} y_0(\xi)=&\int_\xi^1 f(x)\frac{e^{\theta_{02}(x-\xi)}-e^{\theta_{01}(x-\xi)}}{\sqrt{a^2-4b}}dx-\frac{y'_0(1)}{\sqrt{a^2-4b}}\sum_{k=1}^2(-1)^k e^{\theta_{0k}(1-\xi)}+ \\ &+\frac{y_0(1)}{\sqrt{a^2-4b}}\left[\theta_{02}e^{\theta_{01}(1-\xi)}-\theta_{01}e^{\theta_{02}(1-\xi)}\right]. \end{aligned} \quad (41)$$

We write the values of $y_0(1)$ and $y'_0(1)$ in (39) and get

$$\begin{aligned} y_0(\xi)=&\int_\xi^1\frac{e^{\theta_{02}(x-\xi)}-e^{\theta_{01}(x-\xi)}}{\sqrt{a^2-4b}}f(x)dx-\frac{\theta_{01}e^{\theta_{02}(1-\xi)}-\theta_{02}e^{\theta_{01}(1-\xi)}}{\sqrt{a^2-4b}}\times \\ &\sum_{k=1}^4\int_0^1(-1)^{k+1}\frac{B^{(k,1)}(0)}{B(0)}F_k(x)\overline{Z}_{10}(x)dx- \\ &-\frac{e^{\theta_{02}(1-\xi)}-e^{\theta_{01}(1-\xi)}}{\sqrt{a^2-4b}}\int_0^1\sum_{k=1}^4(-1)^k\frac{B^{(k,2)}(0)}{B(0)}F_k(x)\overline{Z}_{10}(x)dx. \end{aligned} \quad (42)$$

Thus,

$$B(0)=\lim_{\varepsilon\to 0}B(\varepsilon),$$

$$B^{(k,j)}(0)=\lim_{\varepsilon\to 0}B^{(k,j)}(\varepsilon).$$

the elements of these determinants are the expressions obtained in the expression (37) for $\varepsilon = 0$.

In the expression (39) passing to limit as $\varepsilon \to 0$

$$y_0^{(s)}(1) = \lim_{\varepsilon \to 0} y_\varepsilon^{(s)}(1)$$

we get the values of the expressions for the unknown function and its derivatives in the right hand side for $\varepsilon = 0$. They are expressed directly by the data. If we write the given expressions in (32)-(35) we get data dependent expressions for the unknown function and its derivatives at the left hand side in $\varepsilon = 0$. So, we complete the first stage.

Now, let's pass to the second stage, i.e. define the end values of the expressions of the solution and its derivatives in $\varepsilon = 0$. For this it is sufficient to differentiate the expression (41) and write $\xi = 0$ or $\xi = 1$.

Theorem 6. Under the conditions of Theorem 5 the solution of the boundary value problem (1)-(2) and the right hand side values of this solution and its derivatives up to third order is given by the expression (38).

Theorem 7. Under the conditions of Theorem 6, using (39), the left hand side values of the solution of boundary value problem (1)–(2) and its derivatives up to third order are obtained from the expression (32) –(35).

Remark 9. If in the expressions obtained in Theorem 6 and Theorem 7 we pass to limit as $\varepsilon \to 0$ we get expressions for the solution of boundary value problem (1)–(2) and for the values of end formulae of its derivatives up to third order in $\varepsilon = 0$.

Theorem 8. Under the conditions of Theorem 7 the expression (42) is true for the value of the solution of boundary value problem (1)–(2) in $\varepsilon = 0$.

Remark 10. Using the expression given in Theorem 8 the values of the solution of boundary value problem (1)–(2) and its derivatives up to third order in $\varepsilon = 0$ may be directly obtained by means of differentiation of (42).

Remark 11. These expressions may be obtained from expressions (9)–(11) as well.

Taking $\xi = 0$ and $\xi = 1$ in the expressions obtained in Theorem 8 and Remark 10 we get end values for the solution of problem (1)–(2) and its derivatives in $\varepsilon = 0$.

REFERENCES

[1] Prandtl Uber Flussigleeits –bowengung bei kleiner Reibeng verhandlungen, III int. Math. Kongresses, Tuebner, Leipzig 1905.

[2] E.A. Caddington and Levison. Theory of ordinary Differential equations McGraw –Hill, New York, 1955.

[3] G.D.Birkhoff. On the asymptotic character of the studies of certain linear differential equations containing a parameter, Trans. Amer. Math. Soc., 9, 219 -231, 1908.

[4] A.N.Tikhonov. System of differential equations containing small parameters at the derivatives, Math. sb. 31, No 3, 1952.

[5] R.E.O'Malley Jr. Singular perturbation methods for ordinary differential equations, Springer- Verlag, 1991.

[6] E.P.Dualan, J.J.H.Milleranal and L.H.A.Schilders. Uniform numerical methods for problems with initial and boundary layers, Boole PRESS, Dublin, 1980.

[7] N.Aliev and N.Suleimanov. Boundary value problems with small parameter in boundary condition, Baku State University, Sb. Nauch. Trud., Azerbaijan, Baku, 1989.

[8] M Jahanshahi. Investigation of boundary layers in a singular perturbation problem 31-th Annual Iranian Mathematical Conference Tehran, Iran, September, 2000.

[9] N.Aliev and Y.M.Salmanov. Research of uniformal correctness of boundary value problem for 4th order O.D.E, Dep. AZNIINTI, 1227, No 48, 1989.

[10] R.T.Pashaev and D.N.Aliev. Uniformal correctness of B.V.P for the second order O.D.E, News of Baku State University, NI, 1995.

[11] V.S.Vladimirov. Equations of Mathematical Physics, Nauka, Moscow, 1976. (in Russian)

[12] B. Mehri and M.Jahanshahi. Linear Differentail Equation with Perturb Boundary condition, Proceedings of 29th Annual Iranian Mathematical conference, 28-31 March, 1998, Tehran, Iran.

[13] R. Courant. Partial Differential Equations, New York, London, 1962.

[14] M.Jahanshahi, B.Mehri, N.Aliev, K.Sakai Asymptotic expansions of Periodic solutions for a singular perturbation problem including nonlinear dynamical system with two boundary layers Southeast Asian Bulletin of Mathematics, 2004,28, pp 41-57.

[15] M.A.Naymark Linear differential operators, Gostekhizdat, Moscow, 1954.