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# Solution of Eighth-Order Boundary-Value Problems Using Non-Polynomial Spline in off Step Points

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**ABSTRACT**

We use non-polynomial spline in off step points to develop numerical methods for the solution of the eighth-order linear boundary-value problems. End conditions of the spline are derived. We compare our results with the results produced by decomposition method and spline method. However, it is observed that our approach produce better numerical solutions in the sense that max $|e_i|$is a minimum.

**KEYWORDS:** Eighth-order boundary-value problem; boundary value formulae; Numerical results.

## 1. INTRODUCTION

Eighth-order differential equations govern the physics of some hydrodynamic stability problems. When an infinite horizontal layer of fluid is heated from below and is subjected to the action of rotation, instability sets in. When this instability sets in as over stability, it is modeled by an eighth-order ordinary differential equation [1-3].

Inc and Evans [1] presented the solutions of eighth-order boundary-value problems using Adomian decomposition method. Siddiqi and Twizell [3] presented the solution of eighth-order boundary value problem using polynomial spline. Boutayeb and Twizell [4] developed the finite difference methods for the solution of eighth-order boundary-value problems. Twizell et al. [5] developed numerical methods for eighth, tenth and twelfth-order eigenvalue problems arising in thermal instability.

The eighth-order boundary-value problem using Nonic spline have been solved by Ghazala Akram, Shahid S. Siddiqi [6]. In this paper we use non polynomial spline approximation to develop a family of new numerical methods to obtain smooth approximations to the solution of eighth-order differential equation. The spline functions proposed in this paper have the form $T_9 = \mathrm{span}\{1, \mathrm{x}, \mathrm{x}^2, \mathrm{x}^3, \mathrm{x}^4, \mathrm{x}^5, \mathrm{x}^6, \mathrm{x}^7, \cos(\mathrm{kx}), \sin(\mathrm{kx})\}$ where $k$ is the frequency of the trigonometric part of the spline functions which can be real or pure imaginary and which will be used to raise the accuracy of the method. Thus in each subinterval $x_i \le x \le x_{i+1}$ we have

$$T_9 = \mathrm{span}\{1, \mathrm{x}, \mathrm{x}^2, \mathrm{x}^3, \mathrm{x}^4, \mathrm{x}^5, \mathrm{x}^6, \mathrm{x}^7, \cos(|\mathrm{k}|\,\mathrm{x}), \sin(|\mathrm{k}|\,\mathrm{x})\}$$

Or

$$T_9 = \mathrm{span}\{1, \mathrm{x}, \mathrm{x}^2, \mathrm{x}^3, \mathrm{x}^4, \mathrm{x}^5, \mathrm{x}^6, \mathrm{x}^7, \cosh(|\mathrm{k}|\,\mathrm{x}), \sinh(|\mathrm{k}|\,\mathrm{x})\}$$

In this manuscript the following eighth-order boundary value problem is consider:

$$y^{(8)}(x) + f(x)y(x) = g(x) \qquad x \in [a,b] \qquad (1)$$

with boundary conditions

$$y(a) = A_0\ ,\ y^{(2)}(a) = A_1\ ,\ y^{(4)}(a) = A_2\ ,\ y^{(6)}(a) = A_3$$
$$y(b) = B_0\ ,\ y^{(2)}(b) = B_1\ ,\ y^{(4)}(b) = B_2\ ,\ y^{(6)}(b) = B_3 \qquad (2)$$

where $A_i, B_i$ for $i = 0,1,2,3$ are finite real constants and the functions $f(x)$ and $g(x)$ are continuous on $[a,b]$ . In this paper, in Section 2, the new Non-polynomial spline methods are developed for solving equation (1) along with boundary condition (2). The boundary formulas are develop in Section 3. In Section 4, non-polynomial spline solution of (1)and(2) is determined and in Section 5 numerical experiment, discussion and comparison with other known methods, are given.

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## 2 NUMERICAL METHODS

To develop the spline approximation to the eighth-order boundary-value problem (1)-(2), the interval [a,b] is divided in to $n$ equal subintervals using the grid $x_{i-\frac{1}{2}} = a + (i - \frac{1}{2})h$ , $i = 1,2,3,..,n$ where $h = \frac{b-a}{n}$ Consider the following Non-polynomial Nonic spline $S_i(x)$ is each Subinterval $[x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}]$, $i = 0,1,2,..,n-1$, $x_0 = a$, $x_n = b$,

$$S_i(x) = a_i \cos k(x - x_i) + b_i \sin k(x - x_i) + c_i (x - x_i)^7 + d_i (x - x_i)^6 + e_i (x - x_i)^5 + f_i (x - x_i)^4 + g^*{}_i (x - x_i)^3 + r_i (x - x_i)^2 + q^*{}_i (x - x_i) + p_i \tag{3}$$

where $a_i, b_i, c_i, d_i, e_i, f_i, g^*{}_i, r_i, q^*{}_i$ and $p_i$ are real finite constants and $k$ is free parameter. The spline $S$ is defined in terms of its 2td, 4th, 6th and 8th derivatives and we denote these values at knots as:

$$S_i(x_{i-\frac{1}{2}}) = y_{i-\frac{1}{2}},\ S_i^{(2)}(x_{i-\frac{1}{2}}) = m_{i-\frac{1}{2}},\ S_i^{(4)}(x_{i-\frac{1}{2}}) = M_{i-\frac{1}{2}},\ S_i^{(6)}(x_{i-\frac{1}{2}}) = Z_{i-\frac{1}{2}},$$
$$S_i(x_{i+\frac{1}{2}}) = y_{i+\frac{1}{2}},\ S_i^{(2)}(x_{i+\frac{1}{2}}) = m_{i+\frac{1}{2}},\ S_i^{(4)}(x_{i+\frac{1}{2}}) = M_{i+\frac{1}{2}},\ S_i^{(6)}(x_{i+\frac{1}{2}}) = Z_{i+\frac{1}{2}},$$
$$S_i^{(8)}(x_{i-\frac{1}{2}}) = L_{i-\frac{1}{2}},\ S_i^{(8)}(x_{i+\frac{1}{2}}) = L_{i+\frac{1}{2}}$$
$$\text{For } i = 0,1,2,...,n-1. \tag{4}$$

Assuming $y(x)$ to be the exact solution of the boundary value problem (1) and $y_i$ be an approximation to $y(x_i)$ ,obtained by the spline $S(x_i)$ ,we can obtained the coefficients in (3) in the following form

$$a_i = \frac{\sec(\frac{\theta}{2})(L_{i-\frac{1}{2}} + L_{i+\frac{1}{2}})}{2k^8},\quad b_i = \frac{cec(\frac{\theta}{2})(-L_{i-\frac{1}{2}} + L_{i+\frac{1}{2}})}{2k^8},$$

$$c_i = \frac{-L_{i-\frac{1}{2}} + L_{i+\frac{1}{2}} + k^2(-z_{i-\frac{1}{2}} + z_{i+\frac{1}{2}})}{5040hk^2}, d_i = \frac{L_{i-\frac{1}{2}} + L_{i+\frac{1}{2}} + k^2(z_{i-\frac{1}{2}} + z_{i+\frac{1}{2}})}{1440k^2},$$

$$e_i = \frac{(24+\theta^2)(L_{i-\frac{1}{2}} - L_{i+\frac{1}{2}}) + k^4(-24M_{i-\frac{1}{2}} + 24M_{i+\frac{1}{2}} + h^2(z_{i-\frac{1}{2}} - z_{i+\frac{1}{2}}))}{2880hk^4},$$

$$f_i = -\frac{(8+\theta^2)(L_{i-\frac{1}{2}} + L_{i+\frac{1}{2}}) + k^4(-8M_{i-\frac{1}{2}} - 8M_{i+\frac{1}{2}} + h^2(z_{i-\frac{1}{2}} - z_{i+\frac{1}{2}}))}{38k^4},$$

$$g_i^* = -\frac{1}{34560hk^6}[-(5760 + 240\theta^2 + 7\theta^4)(-L_{i-\frac{1}{2}} + L_{i+\frac{1}{2}}) + k^6(57860(-m_{i-\frac{1}{2}} + m_{i+\frac{1}{2}}) + h^2(240(M_{i-\frac{1}{2}} - M_{i+\frac{1}{2}}) + 7h^2(-z_{i-\frac{1}{2}} + z_{i+\frac{1}{2}})))],$$

$$r_i = \frac{1}{1536k^6}[(384 + 48\theta^2 + 5\theta^4)(L_{i-\frac{1}{2}} + L_{i+\frac{1}{2}}) + k^6(384(m_{i-\frac{1}{2}} + m_{i+\frac{1}{2}}) + h^2(-48(M_{i-\frac{1}{2}} +$$

$$M_{i+\frac{1}{2}})+5h^2(z_{i-\frac{1}{2}}+z_{i+\frac{1}{2}})))],$$

$$q_i^* = \frac{1}{967680hk^8}[(967680+40320\theta^2+1176\theta^4+31\theta^6)(L_{i-\frac{1}{2}}-L_{i+\frac{1}{2}})+k^8(40320h^2(m_{i-\frac{1}{2}}-$$
$$m_{i+\frac{1}{2}})-1176h^4(M_{i-\frac{1}{2}}-M_{i+\frac{1}{2}})-967680(y_{i-\frac{1}{2}}-y_{i+\frac{1}{2}})+31h^6(z_{i-\frac{1}{2}}-z_{i+\frac{1}{2}}),$$

$$p_i = -\frac{1}{92160k^8}[(46080+5760\theta^2+600\theta^4+61\theta^6)(L_{i-\frac{1}{2}}+L_{i+\frac{1}{2}})+k^8(5760h^2(m_{i-\frac{1}{2}}+m_{i+\frac{1}{2}})-$$
$$600h^4(M_{i-\frac{1}{2}}+M_{i+\frac{1}{2}})-46080(y_{i-\frac{1}{2}}-y_{i+\frac{1}{2}})+61h^6(z_{i-\frac{1}{2}}+z_{i+\frac{1}{2}}),$$

where $\theta = kh$ and $i = 0,1,2,..,n-1$. Applying the continuity conditions of the first, third, fifth and seventh derivative, at knots, i.e. $S_{i-1}^{(\lambda)}(x_i) = S_i^{(\lambda)}(x_i)$ where $\lambda = 1,3,5$ and $7$ after simplifying, we get the following spline relation in terms of eighth derivative of spline $L_i$ and $y_i$:

$$y_{i-\frac{9}{2}}-8y_{i-\frac{7}{2}}+28y_{i-\frac{5}{2}}-56y_{i-\frac{3}{2}}+70y_{i-\frac{1}{2}}-56y_{i+\frac{1}{2}}+28y_{i+\frac{3}{2}}-8y_{i+\frac{5}{2}}+y_{i+\frac{7}{2}} =$$
$$h^8[\alpha L_{i-\frac{9}{2}}+\beta L_{i-\frac{7}{2}}+\gamma L_{i-\frac{5}{2}}+\delta L_{i-\frac{3}{2}}+\eta L_{i-\frac{1}{2}}+\delta L_{i+\frac{1}{2}}+\gamma L_{i+\frac{3}{2}}+\beta L_{i+\frac{5}{2}}+\alpha L_{i+\frac{7}{2}}]$$
$$i = 5,6,...,n-5. \qquad (5)$$

Where

$$\alpha = \frac{1}{5040\theta^8\sin(\theta)}(-5040\theta+840\theta^3-42\theta^5+\theta^7+5040\sin(\theta)),$$

$$\beta = \frac{1}{5040\theta^8\sin(\theta)}(-2(\theta(-5040+840\theta^2-42\theta^4+\theta^6)\cos(\theta)+12(-1260\theta+42\theta^5-$$
$$5\theta^7+1680\sin(\theta)))),$$

$$\gamma = \frac{1}{5040\theta^8\sin(\theta)}(-806640\theta-6720\theta^3-672\theta^5+1192\theta^7-60480\cos(\theta)+$$
$$2160\theta^5\cos(\theta)-240\theta^7\cos(\theta)+141120\sin(\theta),$$

$$\delta = \frac{1}{5040\theta^8\sin(\theta)}(1310400\theta+13440\theta^3+2352\theta^5+2536\theta^7+1512000\theta\cos(\theta)+$$
$$15120\theta^3\cos(\theta)+1260\theta^5\cos(\theta)-2382\theta^7\cos(\theta)-282240\sin(\theta),$$

$$\eta = \frac{1}{5040\theta^8\sin(\theta)}(-1512000\theta+15120\theta^3-1260\theta^5+2382\theta^7-2016000\theta\cos(\theta)-$$
$$26880\theta^3\cos(\theta)-6720\theta^5\cos(\theta)-4832\theta^7\cos(\theta)+352800\sin(\theta).$$

If $\theta \to 0$ then $(\alpha,\beta,\gamma,\delta,\eta) \to (\frac{1}{362880},\frac{251}{181440},\frac{913}{22680},\frac{44117}{181440},\frac{15619}{36288})$, and we get polynomial nonic spline functions.

**3 Development of the boundary formulas**

To obtain unique solution we need eight more equations to be associate with (5) so that we use the following boundary conditions. In order to obtain the eighth-order boundary formula we define the following identity

$$u_0^{'}y_0+\sum_{j=0}^{5}a_j^{'}y_{j+\frac{1}{2}}+c^{'}h^2y_0^{(2)}+d^{'}h^4y_0^{(4)}+e^{'}h^6y_0^{(6)}=h^8\sum_{j=0}^{7}b_j^{'}y_{j+\frac{1}{2}}^{(8)} \quad (6)$$

$$u_0^{''}y_0+\sum_{j=0}^{6}a_j^{''}y_{j+\frac{1}{2}}+c^{''}h^2y_0^{(2)}+d^{''}h^4y_0^{(4)}+e^{''}h^6y_0^{(6)}=h^8\sum_{j=0}^{8}b_j^{''}y_{j+\frac{1}{2}}^{(8)} \quad (7)$$

$$u_0^{'''}y_0+\sum_{j=0}^{7}a_j^{'''}y_{j+\frac{1}{2}}+c^{'''}h^2y_0^{(2)}+d^{'''}h^4y_0^{(4)}+e^{'''}h^6y_0^{(6)}=h^8\sum_{j=0}^{9}b_j^{'''}y_{j+\frac{1}{2}}^{(8)} \quad (8)$$

$$u_0^{\circ}y_0+\sum_{j=0}^{8}a_j^{\circ}y_{j+\frac{1}{2}}+c^{\circ}h^2y_0^{(2)}+d^{\circ}h^4y_0^{(4)}+e^{\circ}h^6y_0^{(6)}=h^8\sum_{j=0}^{10}b_j^{\circ}y_{j+\frac{1}{2}}^{(8)} \quad (9)$$

$$u_0^{\circ\circ}y_n+\sum_{j=0}^{8}a_j^{\circ\circ}y_{j+n-\frac{17}{2}}+c^{\circ\circ}h^2y_n^{(2)}+d^{\circ\circ}h^4y_n^{(4)}+e^{\circ\circ}h^6y_n^{(6)}=h^8\sum_{j=0}^{10}b_j^{\circ\circ}y_{j+n-\frac{21}{2}}^{(8)} \quad (10)$$

$$u_0^{*}y_n+\sum_{j=0}^{7}a_j^{*}y_{j+n-\frac{15}{2}}+c^{*}h^2y_n^{(2)}+d^{*}h^4y_n^{(4)}+e^{*}h^6y_n^{(6)}=h^8\sum_{j=0}^{9}b_j^{*}y_{j+n-\frac{19}{2}}^{(8)} \quad (11)$$

$$u_0^{**}y_n+\sum_{j=0}^{6}a_j^{**}y_{j+n-\frac{13}{2}}+c^{**}h^2y_n^{(2)}+d^{**}h^4y_n^{(4)}+e^{**}h^6y_n^{(6)}=h^8\sum_{j=0}^{8}b_j^{**}y_{j+n-\frac{17}{2}}^{(8)} \quad (12)$$

$$u_0^{\bullet}y_n+\sum_{j=0}^{5}a_j^{\bullet}y_{j+n-\frac{11}{2}}+c^{\bullet}h^2y_n^{(2)}+d^{\bullet}h^4y_n^{(4)}+e^{\bullet}h^6y_n^{(6)}=h^8\sum_{j=0}^{7}b_j^{\bullet}y_{j+n-\frac{15}{2}}^{(8)} \quad (13)$$

Where all of the coefficients are arbitrary parameters to be determined.
The local truncation error corresponding to the method (5) is given by

$$\begin{aligned}t_i &= (2\alpha+2\beta+2\gamma+2\delta+\eta-1)h^8y_i^{(8)}+(-\alpha-\beta-\gamma-\delta+\frac{\eta}{2}-\frac{1}{2})h^9y_i^{(9)}+(\frac{1}{4}(65\alpha+\\
&37\beta+17\gamma+5\delta+\frac{\eta}{2})-\frac{11}{24})h^{10}y_i^{(10)}+(\frac{-1}{24}(193\alpha+109\beta+49\gamma+13\delta+\frac{\eta}{2})+\frac{13}{16})h^{11}y_i^{(11)}\\
&+(\frac{1}{192}(4481\alpha+1513\beta+353\gamma+41\delta+\frac{\eta}{2})-\frac{559}{5760})h^{12}y_i^{(12)}\\
&+(\frac{-1}{1920}(21121\alpha+6841\beta+1441\gamma+121\delta+\frac{\eta}{2})+\frac{43}{1280})h^{13}y_i^{(13)}\\
&+(\frac{1}{23040}(324545\alpha+66637\beta+8177\gamma+365\delta+\frac{\eta}{2})-\frac{2473}{193536})h^{14}y_i^{(14)}\\
&+(\frac{-1}{322560}(1979713\alpha+372709\beta+37969\gamma+1093\delta+\frac{\eta}{2})+\frac{2473}{645120})h^{15}y_i^{(15)}\\
&+(\frac{1}{5160960}(24405761\alpha+3077713\beta+198593\gamma+3281\delta+\frac{\eta}{2})-\frac{183311}{154828800})h^{16}y_i^{(16)}\\
&+(\frac{-1}{92897280}(173533441\alpha+19200241\beta+966721\gamma+9841\delta+\frac{\eta}{2})+\frac{10783}{34406400})h^{17}y_i^{(17)}\\
&+(\frac{1}{1857945600}(1884629825\alpha+146120437\beta+4912337\gamma+29525\delta+\frac{\eta}{2})\end{aligned}$$

$$-\frac{2027813}{24524881920})h^{18}y_i^{(18)}+O(h^{19}),\ i=4,5,...,n-4 \tag{14}$$

By using the above truncation error to eliminate the coefficients of various powers $h$ we can obtain classes of the methods. For any choice of $\alpha,\beta,\gamma,\delta$ and $\eta$ whose $2\alpha+2\beta+2\gamma+2\delta+\eta=1$ and with boundary formulas (6)-(13). We obtain the following methods.

**Remark(i): Second-order method**

For $(\alpha,\beta,\gamma,\delta)=(\frac{1}{362880},\frac{502}{362880},\frac{14608}{362880},\frac{88234}{362880})$ and $\eta=\frac{156190}{362880}$ we obtain the second-order method with truncation error $t_i=\frac{1}{12}h^{10}y_i^{(10)}+O(h^{11})$.

**Remark(ii):Fourth-order method**

For $(\alpha,\beta,\gamma,\delta)=(0,0,0,\frac{1}{3})$ and $\eta=\frac{1}{3}$ we obtain the second-order method with truncation error $t_i=-\frac{1}{40}h^{12}y_i^{(12)}+O(h^{13})$.

**Remark(iii):Sixth-order method**

For $(\alpha,\beta,\gamma,\delta)=(0,0,\frac{1}{40},\frac{7}{30})$ and $\eta=\frac{29}{60}$ we obtain the second-order method with truncation error $t_i=-\frac{1}{5040}h^{14}y_i^{(14)}+O(h^{15})$.

**Remark(iv): Eight-order method**

For $(\alpha,\beta,\gamma,\delta)=(0,\frac{1}{5040},\frac{1}{42},\frac{397}{1680})$ and $\eta=\frac{151}{315}$ we obtain the second-order method with truncation error $t_i=\frac{1}{1209600}h^{16}y_i^{(16)}+O(h^{17})$.

**Remark(v): Tenth-order method**

For $(\alpha,\beta,\gamma,\delta)=(-\frac{1}{1209600},\frac{31}{151200},\frac{7193}{302400},\frac{35737}{151200})$ and $\eta=\frac{57977}{120960}$ we obtain the second-order method with truncation error $t_i=\frac{1}{1330560}h^{18}y_i^{(18)}+O(h^{19})$.

**4 Non-Polynomial Spline Solution**

The spline solution of boundary value problem (1) is based on‘ using (6)-(13) and (5) when we ignore the truncation errors in (14) we obtain a system of linear equations. Considering $Y=[y_{\frac{1}{2}},y_{\frac{3}{2}},...,y_{n-\frac{1}{2}}]^T$ and $C=[c_{\frac{1}{2}},c_{\frac{3}{2}},...,c_{n-\frac{1}{2}}]^T$, This system can be written the following matrix equation:

$$(A+h^8BF)Y=C$$

Where

$$A=\begin{bmatrix}
a_0^{'} & a_1^{'} & a_2^{'} & a_3^{'} & a_4^{'} & a_5^{'} & & & & & & & & & \\
a_0^{''} & a_1^{''} & a_2^{''} & a_3^{''} & a_4^{''} & a_5^{''} & a_6^{''} & & & & & & & & \\
a_0^{'''} & a_1^{'''} & a_2^{'''} & a_3^{'''} & a_4^{'''} & a_5^{'''} & a_6^{'''} & a_7^{'''} & & & & & & & \\
a_0^{\circ} & a_1^{\circ} & a_2^{\circ} & a_3^{\circ} & a_4^{\circ} & a_5^{\circ} & a_6^{\circ} & a_7^{\circ} & a_8^{\circ} & & & & & & \\
1 & -8 & 28 & -56 & 70 & -56 & 28 & -8 & 1 & & & & & & \\
 & 1 & -8 & 28 & -56 & 70 & -56 & 28 & -8 & 1 & & & & & \\
 & & \ddots & & & \ddots & & & \ddots & & & & & & \\
 & & & & & & 1 & -8 & 28 & -56 & 70 & -56 & 28 & -8 & 1 \\
 & & & & & & a_0^{\circ\circ} & a_1^{\circ\circ} & a_2^{\circ\circ} & a_3^{\circ\circ} & a_4^{\circ\circ} & a_5^{\circ\circ} & a_6^{\circ\circ} & a_7^{\circ\circ} & a_8^{\circ\circ} \\
 & & & & & & & a_0^{*} & a_1^{*} & a_2^{*} & a_3^{*} & a_4^{*} & a_5^{*} & a_6^{*} & a_7^{*} \\
 & & & & & & & & a_0^{**} & a_1^{**} & a_2^{**} & a_3^{**} & a_4^{**} & a_5^{**} & a_6^{**} \\
 & & & & & & & & & a_0^{\bullet} & a_1^{\bullet} & a_2^{\bullet} & a_3^{\bullet} & a_4^{\bullet} & a_5^{\bullet}
\end{bmatrix}$$

$$B=\begin{bmatrix}
b_0^{'} & b_1^{'} & b_2^{'} & b_3^{'} & b_4^{'} & b_5^{'} & b_6^{'} & b_7^{'} & b_8^{'} & & & & & & & \\
b_0^{''} & b_1^{''} & b_2^{''} & b_3^{''} & b_4^{''} & b_5^{''} & b_6^{''} & b_7^{''} & b_8^{''} & b_9^{''} & & & & & & \\
b_0^{'''} & b_1^{'''} & b_2^{'''} & b_3^{'''} & b_4^{'''} & b_5^{'''} & b_6^{'''} & b_7^{'''} & b_8^{'''} & b_9^{'''} & b_{10}^{'''} & & & & & \\
b_0^{\circ} & b_1^{\circ} & b_2^{\circ} & b_3^{\circ} & b_4^{\circ} & b_5^{\circ} & b_6^{\circ} & b_7^{\circ} & b_8^{\circ} & b_9^{\circ} & b_{10}^{\circ} & b_{11}^{\circ} & & & & \\
\alpha & \beta & \gamma & \delta & \eta & \delta & \gamma & \beta & \alpha & & & & & & & \\
 & \alpha & \beta & \gamma & \delta & \eta & \delta & \gamma & \beta & \alpha & & & & & & \\
 & & \ddots & & & \ddots & & & \ddots & & & & & & & \\
 & & & & & & & \alpha & \beta & \gamma & \delta & \eta & \delta & \gamma & \beta & \alpha \\
 & & & & b_0^{\circ\circ} & b_1^{\circ\circ} & b_2^{\circ\circ} & b_3^{\circ\circ} & b_4^{\circ\circ} & b_5^{\circ\circ} & b_6^{\circ\circ} & b_7^{\circ\circ} & b_8^{\circ\circ} & b_9^{\circ\circ} & b_{10}^{\circ\circ} & b_{11}^{\circ\circ} \\
 & & & & & b_0^{*} & b_1^{*} & b_2^{*} & b_3^{*} & b_4^{*} & b_5^{*} & b_6^{*} & b_7^{*} & b_8^{*} & b_9^{*} & b_{10}^{*} \\
 & & & & & & b_0^{**} & b_1^{**} & b_2^{**} & b_3^{**} & b_4^{**} & b_5^{**} & b_6^{**} & b_7^{**} & b_8^{**} & b_9^{**} \\
 & & & & & & & b_0^{\bullet} & b_1^{\bullet} & b_2^{\bullet} & b_3^{\bullet} & b_4^{\bullet} & b_5^{\bullet} & b_6^{\bullet} & b_7^{\bullet} & b_8^{\bullet}
\end{bmatrix}$$

$F = diag(f_i)$ , $i = 1,2,3,\ldots,n-1.$

The vector C is defined by

$$c_{\frac{1}{2}} = -u_0^{'} y_0 - c^{'} h^2 y_0^{(2)} - d^{'} h^4 y_0^{(4)} - e^{'} h^6 y_0^{(6)} + h^8 (b_0^{'} g_{\frac{1}{2}} + b_1^{'} g_{\frac{3}{2}} + b_2^{'} g_{\frac{5}{2}} + b_3^{'} g_{\frac{7}{2}} +$$

$b_4^{'}g_{\frac{9}{2}} + b_5^{'}g_{\frac{11}{2}} + b_6^{'}g_{\frac{13}{2}} + b_7^{'}g_{\frac{15}{2}}),$

$c_{\frac{3}{2}} = -u_0^{''}y_0 - c^{''}h^2 y_0^{(2)} - d^{''}h^4 y_0^{(4)} - e^{''}h^6 y_0^{(6)} + h^8(b_0^{''}g_{\frac{1}{2}} + b_1^{''}g_{\frac{3}{2}} + b_2^{''}g_{\frac{5}{2}} + b_3^{''}g_{\frac{7}{2}} +$

$b_4^{''}g_{\frac{9}{2}} + b_5^{''}g_{\frac{11}{2}} + b_6^{''}g_{\frac{13}{2}} + b_7^{''}g_{\frac{15}{2}} + b_8^{''}g_{\frac{17}{2}}),$

$c_{\frac{5}{2}} = -u_0^{'''}y_0 - c^{'''}h^2 y_0^{(2)} - d^{'''}h^4 y_0^{(4)} - e^{'''}h^6 y_0^{(6)} + h^8(b_0^{'''}g_{\frac{1}{2}} + b_1^{'''}g_{\frac{3}{2}} + b_2^{'''}g_{\frac{5}{2}} + b_3^{'''}g_{\frac{7}{2}} +$

$b_4^{'''}g_{\frac{9}{2}} + b_5^{'''}g_{\frac{11}{2}} + b_6^{'''}g_{\frac{13}{2}} + b_7^{'''}g_{\frac{15}{2}} + b_8^{'''}g_{\frac{17}{2}} + b_9^{'''}g_{\frac{19}{2}}),$

$c_{\frac{7}{2}} = -u_0^{\circ}y_0 - c^{\circ}h^2 y_0^{(2)} - d^{\circ}h^4 y_0^{(4)} - e^{\circ}h^6 y_0^{(6)} + h^8(b_0^{\circ}g_{\frac{1}{2}} + b_1^{\circ}g_{\frac{3}{2}} + b_2^{\circ}g_{\frac{5}{2}} + b_3^{\circ}g_{\frac{7}{2}} +$

$b_4^{\circ}g_{\frac{9}{2}} + b_5^{\circ}g_{\frac{11}{2}} + b_6^{\circ}g_{\frac{13}{2}} + b_7^{\circ}g_{\frac{15}{2}} + b_8^{\circ}g_{\frac{17}{2}} + b_9^{\circ}g_{\frac{19}{2}} + b_{10}^{\circ}g_{\frac{21}{2}}),$

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.
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$c_{i-\frac{1}{2}} = h^8(\alpha g_{i-\frac{9}{2}} + \beta g_{i-\frac{7}{2}} + \gamma g_{i-\frac{5}{2}} + \delta g_{i-\frac{3}{2}} + \eta g_{i-\frac{1}{2}} + \delta g_{i+\frac{1}{2}} + \gamma g_{i+\frac{3}{2}} + \beta g_{i+\frac{5}{2}} +$

$\alpha g_{i+\frac{7}{2}})\ , \quad i = 5,6,\ldots,(n-5)$

.
.
.

$c_{n-\frac{7}{2}} = -u_0^{\circ\circ}y_n - c^{\circ\circ}h^2 y_n^{(2)} - d^{\circ\circ}h^4 y_n^{(4)} - e^{\circ\circ}h^6 y_n^{(6)} + h^8(b_0^{\circ\circ}g_{n-\frac{21}{2}} + b_1^{\circ\circ}g_{n-\frac{19}{2}} +$

$b_2^{\circ\circ}g_{n-\frac{17}{2}} + b_3^{\circ\circ}g_{n-\frac{15}{2}} + b_4^{\circ\circ}g_{n-\frac{13}{2}} + b_5^{\circ\circ}g_{n-\frac{11}{2}} + b_6^{\circ\circ}g_{n-\frac{9}{2}} + b_7^{\circ\circ}g_{n-\frac{7}{2}} + b_8^{\circ\circ}g_{n-\frac{5}{2}} +$

$b_9^{\circ\circ}g_{n-\frac{3}{2}} + b_{10}^{\circ\circ}g_{\frac{1}{2}}),$

$c_{n-\frac{5}{2}} = -u_0^{*}y_n - c^{*}h^2 y_n^{(2)} - d^{*}h^4 y_n^{(4)} - e^{*}h^6 y_n^{(6)} + h^8(b_0^{*}g_{n-\frac{19}{2}} + b_1^{*}g_{n-\frac{17}{2}} + b_2^{*}g_{n-\frac{15}{2}} +$

$b_3^{*}g_{n-\frac{13}{2}} + b_4^{*}g_{n-\frac{11}{2}} + b_5^{*}g_{n-\frac{9}{2}} + b_6^{*}g_{n-\frac{7}{2}} + b_7^{*}g_{n-\frac{5}{2}} + b_8^{*}g_{n-\frac{3}{2}} + b_9^{*}g_{n-\frac{1}{2}}),$

$c_{n-\frac{3}{2}} = -u_0^{**}y_n - c^{**}h^2 y_n^{(2)} - d^{**}h^4 y_n^{(4)} - e^{**}h^6 y_n^{(6)} + h^8(b_0^{**}g_{n-\frac{17}{2}} + b_1^{**}g_{n-\frac{15}{2}} + b_2^{**}g_{n-\frac{13}{2}} +$

$b_3^{**}g_{n-\frac{11}{2}} + b_4^{**}g_{n-\frac{9}{2}} + b_5^{**}g_{n-\frac{7}{2}} + b_6^{**}g_{n-\frac{5}{2}} + b_7^{**}g_{n-\frac{3}{2}} + b_8^{**}g_{n-\frac{1}{2}}),$

$c_{n-\frac{1}{2}} = -u_0^{\bullet}y_n - c^{\bullet}h^2 y_n^{(2)} - d^{\bullet}h^4 y_n^{(4)} - e^{\bullet}h^6 y_n^{(6)} + h^8(b_0^{\bullet}g_{n-\frac{15}{2}} + b_1^{\bullet}g_{n-\frac{13}{2}} + b_2^{\bullet}g_{n-\frac{11}{2}} +$

$b_3^{\bullet}g_{n-\frac{9}{2}} + b_4^{\bullet}g_{n-\frac{7}{2}} + b_5^{\bullet}g_{n-\frac{5}{2}} + b_6^{\bullet}g_{n-\frac{3}{2}} + b_7^{\bullet}g_{n-\frac{1}{2}}),$

## 5 NUMERICAL RESULTS

**Example 1.** We Consider the following boundary-value problem

$$y^{(8)}(x) + xy = -(48 + 15x + x^3)e^x \ , \ 0 \le x \le 1 \ ,$$
$$y(0) = 0 \ , \ y^{(2)}(0) = 0 \ , \ y^{(4)}(0) = -8 \ , \ y^{(6)}(0) = -24 \ ,$$
$$y(1) = 0 \ , \ y^{(2)}(1) = -4e \ , \ y^{(4)}(1) = -16e \ , \ y^{(6)}(1) = -36e \ .$$

The exact solution for this problem is $y(x) = x(1-x)e^x$ .

This problem has been solved by many authors [2,3]. We applied our methods described in section 2,3 to solve this problem with different value of $h$ and parameters $\alpha, \beta, \gamma, \delta$ and $\eta$ , The computed solutions are compared with the exact solution at grid points. The observed maximum absolute errors are tabulated in Table 1 and compared with the methods in [2,3]. It has been observed that our methods are more efficient.

Table 1 : Maximum absolute errors of Example 1

| x | Our method | Method in[2] | Method in[3] |
|---|---|---|---|
| 0.1 | $2.472\times10^{-13}$ | $2.92\times10^{-10}$ | $8.11\times10^{-8}$ |
| 0.2 | $1.161\times10^{-12}$ | $9.90\times10^{-10}$ | $2.46\times10^{-7}$ |
| 0.3 | $2.399\times10^{-12}$ | $2.80\times10^{-9}$ | $8.45\times10^{-7}$ |
| 0.4 | $4.142\times10^{-12}$ | $1.20\times10^{-8}$ | $2.73\times10^{-6}$ |
| 0.5 | $8.826\times10^{-12}$ | $5.04\times10^{-8}$ | $1.15\times10^{-5}$ |
| 0.6 | $2.471\times10^{-11}$ | $1.18\times10^{-6}$ | $4.87\times10^{-5}$ |
| 0.7 | $2.748\times10^{-11}$ | $4.47\times10^{-6}$ | $5.30\times10^{-4}$ |
| 0.8 | $6.085\times10^{-11}$ | $2.00\times10^{-4}$ | $1.14\times10^{-2}$ |
| 0.9 | $7.329\times10^{-11}$ | - | - |

**Conclusion**

We approximate solution of the eighth-order linear boundary-value problems by using non-polynomial spline. The new methods enable us to approximate the solution at every point of the range of integration. Table 1 shows that our methods produced better in the sense that max $|e_i|$ is minimum in comparison with the methods in [2,3].

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