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Some Notes on Homotopy Analysis Method for Solving the Fornberg-Whitham Equation

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ABSTRACT

In this paper, the homotopy analysis method is applied to obtain an approximate analytical solution of the Fornberg-Whitham equation. The homotopy analysis method can give very good approximations by means of a few terms, if initial guess and auxiliary linear operator are good enough. Comparison of the results with those of homotopy analysis method and exact solutions shows the accuracy of the HAM method. The homotopy analysis method contains the auxiliary parameter $\hbar$, which provides us with a simple way to adjust and control the convergence region of rate series solution. In this paper, it is shown that some parts of the solutions given by the outhors of the article "*F. Abidi, K. Omranib, The homotopy analysis method for solving the Fornberg-Whitham equation and comparison with Adomian's decomposition method, Computers and Mathematics with Applications 59 (2010) 2743-2750.*" are incorrect. By correcting the solution, more discussions are done.

KEYWORDS: Fornberg-Whitham equation, Homotopy analysis method.

1- INTRODUCTION

Most of engineering problems are nonlinear, and in most cases it is difficult to solve them, especially analytically.

In 1992, Liao employed the basic ideas of the homotopy in topology to propose a general analytic method for nonlinear problems, namely homotopy analysis method (HAM) [11,12], and then modified it, step by step (see [6] and its references, [13]). This method has been successfully applied to solve many types of nonlinear problems by others [3,8,9,15]. In this article, we follow the Homotopy analysis method (HAM) which was presented in [2] to find the approximate analytical solution of the Fornberg-Whitham equation.

The study of traveling wave solutions [7] has become one of the important issues in many areas of physics [17]. The Fornberg-Whitham equation [16] given as

$$u_t + u_{xxt} + u_x = uu_{xxx} - uu_x + 3u_x u_{xx}, \tag{1}$$

has a type of traveling wave solution called a kink-like wave solution and anti kink-like wave solutions. Such kinds of traveling wave solutions have never been found for the Fornberg-Whitham equation. Eq. (1) was used to study the qualitative behaviour of wave-breaking [7,17]. Some methods such as the Adomian decomposition method and homotopy perturbation method are special cases of HAM [1,4,5,6,14]. With the present method, numerical results can be obtained by using a few iterations. The HAM contains the auxiliary parameter $\hbar$, which provides us with a simple way to adjust and control the convergence region of solution series for any values of x and t [13].

2 Basic idea of homotopy analysis method

In this paper, we apply the homotopy analysis method to the discussed problem. To show the basic ideas of the HAM, let us consider a differential equation

$$N[u(x,t)] = 0 \tag{2}$$

where N is a nonlinear operator, $u(x,t)$ is an unknown function and x and t denote spatial and temporal independent variables, respectively.

By means of generalizing the traditional concept of homotopy [12] constructs the so-called zero-order deformation equation

$$(1-p)L[\phi(x,t;p) - u_0(x,t)] = p\hbar N[\phi(x,t;p)] \tag{3}$$

where $p \in [0,1]$ is an embedding parameter, $\hbar$ is a nonzero auxiliary parameter, L is an auxiliary linear operator, $u_0(x,t)$ is an initial guess of $u(x,t)$ and $\phi(x,t;p)$ is an unknown function. It should be emphasized that one has great freedom to choose the initial guess, the auxiliary linear operator, the auxiliary parameter $\hbar$ in HAM. Obviously, when $p=0$ and $p=1$, it holds

$$\phi(x,t;0) = u_0(x,t), \quad \phi(x,t;1) = u(x,t), \tag{4}$$

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respectively. Thus, as p increases from 0 to 1, the solution $\phi(x,t;p)$ varies from the initial guess $u_0(x,t)$ to the solution $u(x,t)$. Expanding $\phi(x,t;p)$ in Taylor series with respect to p, one has

$$\phi(x,t;p) = u_0(x,t) + \sum_{m=1}^{\infty} u_m(x,t)p^m, \tag{5}$$

where

$$u_m(x,t) = \frac{1}{m!}\frac{\partial^m \phi(x,t;p)}{\partial p^m}|_{p=0}. \tag{6}$$

If the auxiliary linear operator, the initial guess, the auxiliary parameter $\hbar$ and the auxiliary function are so properly chosen, then, as proved by Liao [12], the series (5) converges at $p = 1$ and one has

$$u(x,t) = u_0(x,t) + \sum_{m=1}^{\infty} u_m(x,t), \tag{7}$$

which most be one of solutions of the original nonlinear equation, as proved by Liao [12]. As $\hbar = -1$, Eq. (3) becomes

$$(1-p)L[\phi(x,t;p) - u_0(x,t)] + pN[\phi(x,t;p)] = 0, \tag{8}$$

which is used in the homotopy perturbation method [10].

According to the definition (6), the governing equation of can be deduced from the zero-order deformation equation (3). Let us define the vector

$$\vec{u}_n = \{u_0(x,t), u_1(x,t), \ldots, u_n(x,t)\}. \tag{9}$$

Differentiating Eq. (3) m times with respect to the embedding parameter p and then setting $p = 0$ and finally dividing them by $m!$, we have the so-called mth-order deformation equation,

$$L[u_m(x,t) - \chi_m u_{m-1}(x,t)] = \hbar \Re_m[\vec{u}_{m-1}(x,t)], \tag{10}$$

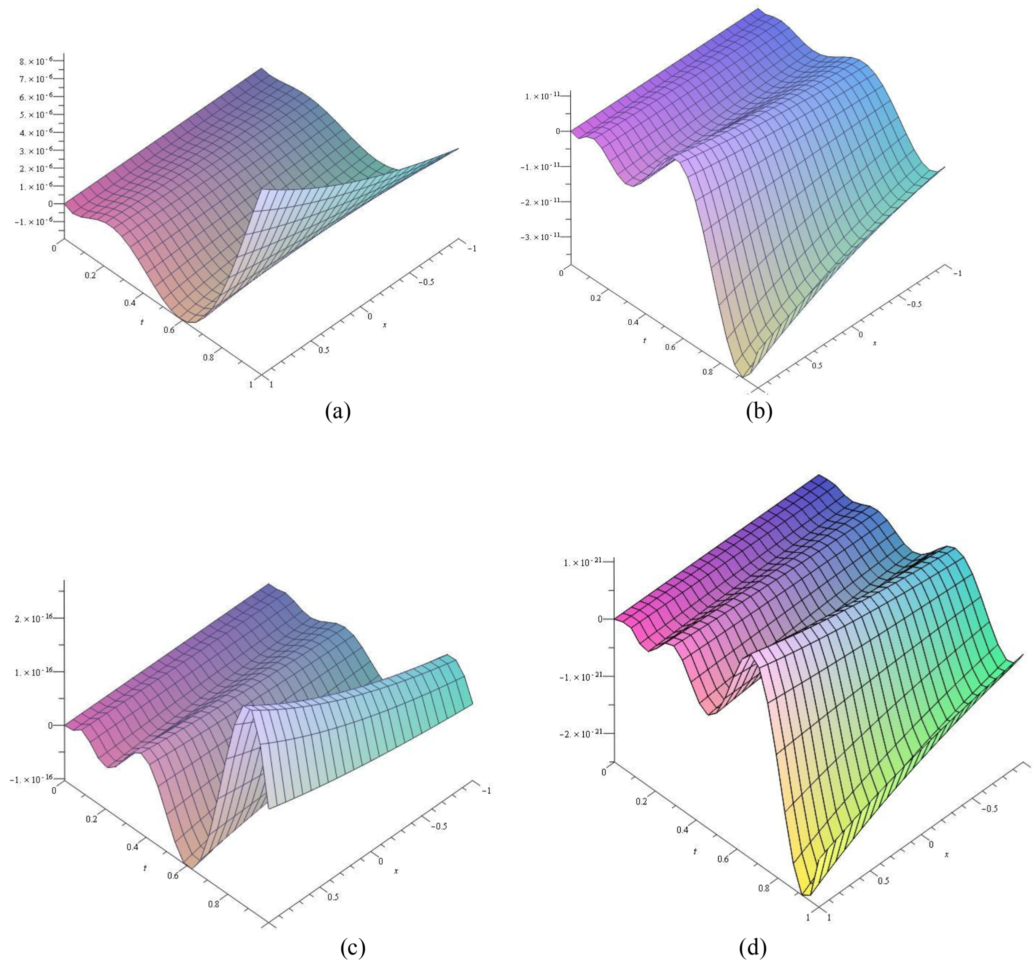

Figure 1: The behavior of the error $(u_{exact} - u_{app})$ obtained by: (a) n=5, (b) n=10, (c) n=15, (d) n=20.

where

$$\Re(\vec{u}_{m-1}) = \frac{1}{(m-1)!}\frac{\partial^{m-1}N[\phi(x,t;p)]}{\partial p^{m-1}}\bigg|_{p=0}, \tag{11}$$

and

$$\chi_m = \begin{cases} 0, & m \le 1, \\ 1, & m \ge 2. \end{cases} \tag{12}$$

It should be emphasized that $u_m(x,t)$ for $m \ge 1$ is governed by the linear equation (10) with the linear boundary conditions that come from the original problem, which can be easily solved by symbolic computation software such as Maple and Mathematica.

3 Result analysis

In this section, we apply HAM to solve the Fornberg-Whitham equation. In our work, we use the Maple Package to calculate the numerical solutions obtained by this method. Consider the Fornberg-Whitham equation

$$u_t + u_{xxt} + u_x = uu_{xxx} - uu_x + 3u_x u_{xx}, \tag{13}$$

subject to the initial condition of

$$u(x,0) = \exp(\tfrac{1}{2}x). \tag{14}$$

Then, the exact solution is given by

$$u(x,t) = \exp(\tfrac{1}{2}x - \tfrac{2}{3}t). \tag{15}$$

To solve Eq. (13) by means HAM, we choose the initial approximation

$$u_0(x,t) = u(x,0) = \exp(\tfrac{1}{2}x). \tag{16}$$

Eq. (13) suggests the nonlinear operator as

$$\begin{aligned} N[\phi(x,t;p)] = {} & \frac{\partial\phi(x,t;p)}{\partial t} - \frac{\partial^3\phi(x,t;p)}{\partial x^2\,\partial t} + \frac{\partial\phi(x,t;p)}{\partial x} - \phi(x,t;p)\frac{\partial^3\phi(x,t;p)}{\partial x^3} \\ & + \phi(x,t;p)\frac{\partial\phi(x,t;p)}{\partial x} - 3\frac{\partial\phi(x,t;p)}{\partial x}\frac{\partial^2\phi(x,t;p)}{\partial x^2}, \end{aligned} \tag{17}$$

and the linear operator

$$L[\phi(x,t;p)] = \frac{\partial\phi(x,t;p)}{\partial t}, \tag{18}$$

with the property

$$L(c_1) = 0,$$

where c_1 is the integration constant.
Using the above definition, we construct the zeroth-order deformation equation

$$(1-p)L[\phi(x,t;p) - u_0(x,t)] = p\hbar N[\phi(x,t;p)]. \tag{19}$$

Obviously, when $p = 0$ and $p = 1$,

$$\phi(x,t;0) = u_0(x,t), \quad \phi(x,t;1) = u(x,t).$$

Therefore, as the embedding parameter p increases from 0 to 1, $\phi(x,t;p)$ varies from the initial guess $u_0(x,t)$ to the solution $u(x,t)$. Then, we obtain the mth-order deformation equation

$$L[u_m(x,t) - \chi_m u_{m-1}(x,t)] = h\Re_m[\vec{u}_{m-1}(x,t)], \tag{20}$$

subject to initial condition

$$u_m(x,0) = 0,$$

where

$$\begin{aligned} \Re(\vec{u}_{m-1}) = {} & \frac{\partial u_{m-1}(x,t)}{\partial t} - \frac{\partial^3 u_{m-1}(x,t)}{\partial x^2\,\partial t} + \frac{\partial u_{m-1}(x,t)}{\partial x} \\ & + \sum_{k=1}^{m-1}\left[-u_k(x,t)\frac{\partial^3 u_{m-1-k}(x,t)}{\partial x^3} + u_k(x,t)\frac{\partial u_{m-1-k}(x,t)}{\partial x} - 3\frac{\partial u_k(x,t)}{\partial x}\frac{\partial^2 u_{m-1-k}(x,t)}{\partial x^2}\right]. \end{aligned} \tag{21}$$

Now, the solution of the mth-order deformation equation (20) for $m \ge 1$ becomes

$$u_m(x,t) = \chi_m u_{m-1}(x,t) + \hbar L^{-1}[\Re_m(\vec{u}_{m-1})], \tag{22}$$

From (16) and (22) we now successively obtain

Table 1: $\|u_{exact} - u_{app}\|_2^2$ for different values of $\hbar$ and n.

$\hbar$	$n = 5$	$n = 10$	$n = 15$
-2	$1.8e-01$	$3.7e-02$	$5.4e-03$
-1.9	$9.8e-02$	$1.0e-02$	$7.5e-04$
-1.8	$4.8e-02$	$1.0e-02$	$7.3e-05$
-1.7	$2.1e-02$	$3.8e-04$	$4.6e-06$
-1.6	$7.9e-03$	$4.2e-05$	$1.4e-07$
-1.5	$2.1e-03$	$2.3e-06$	$1.3e-09$
-1.4	$3.4e-04$	$2.7e-08$	$7.9e-13$
-1.3	$8.3e-06$	$5.7e-14$	$8.7e-16$
-1.2	$2.3e-06$	$1.5e-11$	$8.8e-16$
-1.1	$1.9e-05$	$1.9e-09$	$1.9e-13$
-1	$8.8e-05$	$6.7e-08$	$4.1e-11$
-0.9	$4.2e-04$	$7.3e-07$	$2.7e-09$
-0.8	$1.0e-03$	$7.8e-06$	$5.3e-08$
-0.7	$2.3e-03$	$5.4e-05$	$8.6e-07$
-0.6	$6.6e-03$	$2.2e-04$	$1.1e-05$
-0.5	$1.8e-02$	$8.1e-04$	$4.1e-04$
-0.4	$4.2e-02$	$3.6e-03$	$4.1e-04$
-0.3	$8.7e-02$	$1.6e-02$	$2.8e-03$
-0.2	$1.6e-01$	$5.6e-02$	$1.9e-02$
-0.1	$2.8e-01$	$1.7e-01$	$1.0e-01$
0	$4.7e-01$	$4.7e-01$	$4.7e-01$

$$
\begin{aligned}
u_0(x,t) &= \exp(\tfrac{1}{2}x),\\
u_1(x,t) &= \exp(\tfrac{1}{2}x)\left[\frac{\hbar t}{2}\right],\\
u_2(x,t) &= \exp(\tfrac{1}{2}x)\left[\frac{4\hbar+3\hbar^2}{8}t+\frac{\hbar^2t^2}{8}\right],\\
u_3(x,t) &= \exp(\tfrac{1}{2}x)\left[t(\frac{48}{96}\hbar+\frac{72}{96}\hbar^2+\frac{27}{96}\hbar^3)+t^2(\frac{24}{96}\hbar^2+\frac{18}{96}\hbar^3)+t^3(\frac{1}{48}\hbar^3)\right],\\
u_4(x,t) &= \exp(\tfrac{1}{2}x)\left[t(\frac{192\hbar+432\hbar^2+324\hbar^3+81\hbar^4}{384})+t^2(\frac{144\hbar^2+216\hbar^3+81\hbar^4}{384})\right.\\
&\qquad \left.+t^3(\frac{24\hbar^3+18\hbar^4}{384})+t^4(\frac{\hbar^4}{384})\right],
\end{aligned}
$$

$$
\begin{aligned}
u_5(x,t) &= \exp(\tfrac{1}{2}x)\left[t(\frac{3840\hbar+11520\hbar^2+12960\hbar^3+6480\hbar^4+1215\hbar^5}{7680})\right.\\
&\qquad +t^2(\frac{3840\hbar^2+8640\hbar^3+6480\hbar^4+1620\hbar^5}{7680})+t^3(\frac{960\hbar^3+1440\hbar^4+540\hbar^5}{7680})\\
&\qquad \left.+t^4(\frac{80\hbar^4+60\hbar^5}{7680})+t^5(\frac{\hbar^5}{3840})\right],\\
u_6(x,t) &= \exp(\tfrac{1}{2}x)\left[t(\frac{46080\hbar+172800\hbar^2+259200\hbar^3+194400\hbar^4+72900\hbar^5+10935\hbar^6}{92160})\right.\\
&\qquad +t^2(\frac{57600\hbar^2+172800\hbar^3+194400\hbar^4+97200\hbar^5+18225\hbar^6}{92160})\\
&\qquad +t^3(\frac{19200\hbar^3+43200\hbar^4+32400\hbar^5+8100\hbar^6}{92160})\\
&\qquad \left.+t^4(\frac{2400\hbar^4+3600\hbar^5+1350\hbar^6}{92160})+t^5(\frac{120\hbar^5+90\hbar^6}{92160})+t^6(\frac{\hbar^6}{46080})\right],
\end{aligned}
$$

and so on where as u_2, u_3 and u_4 were incorrect in reference [2]. At last, we use seven terms in evaluating the approximate solution as

$$u_{app} = \sum_{i=0}^{6} u_i. \tag{23}$$

Then,

$$
\begin{aligned}
u_{app} &= \exp(\tfrac{1}{2}x)\Big[t(\tfrac{276480\hbar+518400\hbar^2+518400\hbar^3+291600\hbar^4+87480\hbar^5+10935\hbar^6}{92160}) \\
&+t^2(\tfrac{172800\hbar^2+345600\hbar^3+291600\hbar^4+116640\hbar^5+18225\hbar^6}{92160}) \\
&+t^3(\tfrac{38400\hbar^3+64800\hbar^4+38880\hbar^5+8100\hbar^6}{92160}) \\
&+t^4(\tfrac{3600\hbar^4+4320\hbar^5+1350\hbar^6}{92160})+t^5(\tfrac{144\hbar^5+90\hbar^6}{92160})+t^6(\tfrac{\hbar^6}{46080})\Big].
\end{aligned} \tag{24}
$$

4 Conclusions and discussion

In this paper, HAM has successfully developed for solving Fornberg-Whitham equation. It is obvious to see that the HAM is very powerful and efficient technique in finding analytical solutions for wide classes of nonlinear problems. It is worth pointing out that this method presents a rapid convergence for the solutions. HAM provides accurate numerical solution for nonlinear problems in comparison with other methods. The auxiliary parameter $\hbar$ can be employed to adjust the convergence region of the homotopy analysis solution. To investigate the influence of $\hbar$ on the solution series, we computed Table 1 which shows $\|u_{exact}-u_{app}\|_2^2$ on the interval $x\in[-1,1]$ and $t\in[0,1]$. According to this table which was computed for $n=5,10$ and 15, it is easy to discover that the valid region of $\hbar$ which corresponds to the low values of $\|u_{exact}-u_{app}\|_2^2$ was in the interval $[-1.5,-0.9]$.

Figure 1, shows the behavior of the error for different values of n. In Figure 2, we study the diagrams of the results obtain by HAM for $n=5,10,15$, and the $\hbar$-curve of $u_t(0,0)$ given by the

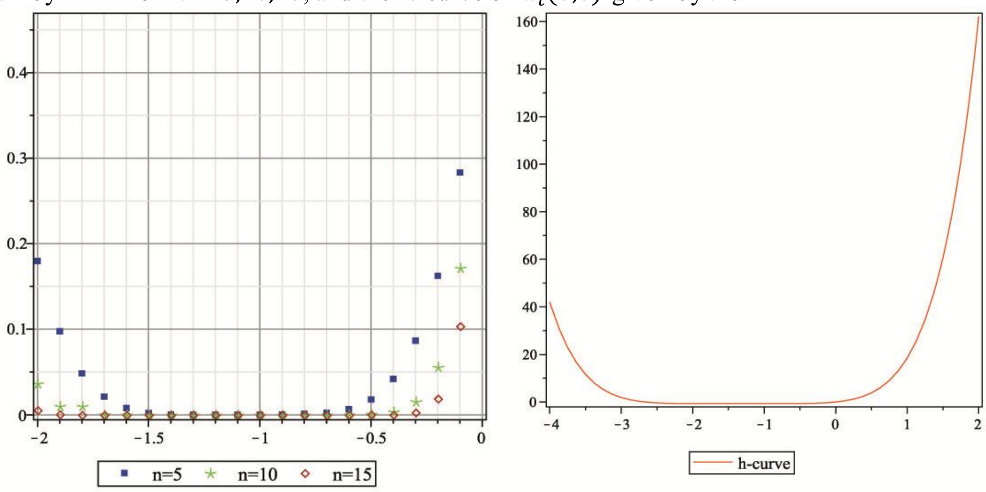

Figure 2: Left: The results obtained by HAM for various $\hbar$ by $n=5,10,15$, **Right:** The $\hbar$-curve of $u_t(0,0)$ given by the 7th-order HAM approximate solution.

7th-order HAM approximate solution. These diagrams shows that the valid region of $\hbar$ was in $[-1.5,-0.9]$, as we have obtained from Table 1 early. In Table 2 and 3, we compute the absolute errors for differences between the exact solution and the approximate solution for $\hbar=-1$ and $\hbar=-1.3$ at some points.

The results show that HAM is powerful mathematical tool for solving nonlinear partial differential equations and systems of nonlinear partial differential equations having wide applications in engineering.

Table 2: Absolute errors for differences between the exact solution and 7th-order HAM approximate given by HAM for $\hbar=-1$.

x_i/t_i	0.2	0.4	0.6	0.8	1
−4	5.55e − 07	1.60e − 06	2.40e − 06	8.78e − 07	2.17e − 06
−2	1.51e − 06	4.36e − 06	6.52e − 06	2.39e − 06	5.90e − 06
0	4.10e − 06	1.18e − 05	1.77e − 05	6.49e − 06	1.60e − 05
2	1.11e − 05	3.22e − 05	4.82e − 05	1.76e − 05	4.36e − 05
4	3.03e − 05	8.76e − 05	1.31e − 04	4.79e − 05	1.18e − 04

Table 3: Absolute errors for differences between the exact solution and 7th-order HAM approximate given by HAM for $\hbar = -1.3$.

x_i/t_i	0.2	0.4	0.6	0.8	1
-4	$2.35e-12$	$1.93e-11$	$4.74e-10$	$9.33e-09$	$1.33e-07$
-2	$6.39e-12$	$5.24e-11$	$1.29e-09$	$2.53e-08$	$6.62e-07$
0	$1.74e-11$	$1.42e-10$	$3.50e-09$	$6.89e-08$	$9.84e-07$
2	$4.72e-11$	$3.87e-10$	$9.51e-09$	$1.87e-07$	$2.67e-06$
4	$1.28e-10$	$1.05e-09$	$2.59e-08$	$5.09e-07$	$7.27e-06$

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