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Structure Theorems For O-Minimal Expansions Of Groups

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ABSTRACT

In this study the structure for O-minimal expansions of groups is considered Let $\Re$ be an o-minimal expansion of an ordered group (R,0,1,+,<) with distinguished positive element 1. We first prove that the following are equivalent: (1) $\Re$ is semibounded, (2) $\Re$ has no poles, (3) $\Re$ cannot define a real closed field with domain R and order <, (4) $\Re$ is eventually linear and (5) every $\Re$ –definable set is a finit union of cones. As a corollary we get that Th($\Re$) has quantifier elimination and universal axiomatization in the language with symbols for the ordered group operations, bounded $\Re$–definable sets and symbols for each definable endomorphis of the group (R,0,+).
KEY WORDS: o-minimal, Structure Theorems, Groups.

1- INTRODUCTION

In all of the paper $\Re$ is an expansion of O-minimal from an ordered group $(R, 0, 1, +, <)$ with distinguished element 0<1. And what is meant by being definable is being definable by parameter in $\Re$. Also the function $f : R \longrightarrow R$ is definable whenever graph f is a definable set in $\Re$. A O-minimal structure, is a structure in the form of $\mathcal{N} = (N, <, (c)_{c\in C}, (f)_{f\in F}, (R)_{R\in U})$ in which $(N, <)$ is a dense ordered set with no initial and terminal points, C is a collection of constant elements N, F is a collection of functions of N^n (for different n s) in N and U is a collection of relations on N^n (for different n s) so that any definable subset of N is in the form of a finite union of points and intervals with terminal points at $N \cup \{-\infty, +\infty\}$.

2- Concepts

Definition 1. We call the definable subset $C \subseteq R^n$ as a K-cone whenever it is in the following form:

$$C = \left\{b + \Sigma_{i=1}^{k} v_i(t_i) : b \in B, t_1, ..., t_k \in R^{>0}\right\}$$

Where $B \subseteq R^n$ is a definable bounded set and $v_1, \dots, v_k$ are independent elements of Λ^n which are linearly independent, i.e., for every $t_1, ..., t_k \in R$, if $\Sigma_{i=1}^{k} v_i(t_i) = 0$ then $t_1 = t_2 = \cdots = t_k = 0$.

Definition 2. The definable function $f : R \longrightarrow R$ is said to be linearly bounded whenever there is an endomorphism $\lambda \in \Lambda$ so that for every large enough value of x we have: $|f(x)| \leq \lambda(x)$.

Also the structure of $\Re$ is said to be linearly bounded whenever every definable function $f : R \longrightarrow R$ is linearly bounded.

Definition3. We say $\Re$ is semi-bounded whenever every definable subset $\Re$ in the reduced structure of $(R, 0, 1, +, <, (B_i)_{i\in I}, (\lambda)_{\lambda\in\Lambda})$ is also definable, in which $(B_i)_{i\in I}$ is the collection of all definable bounded subsets $\Re$.

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We say $\Re$ is "eventually linear" whenever for every definable function $f: R \longrightarrow R$, there exists endomorphism $\lambda \in \Lambda$ and element $c \in R$ so that we can eventually have $f(x) = \lambda(x) + c$.
The main result of this research which is stated below is the generalization of a theorem in (Peterzil, 1992) which was proved by Peterzil for O-minimal expansions from collective ordered group of real numbers $(\mathbb{R}, 0, 1, +, <)$.

Theorem 1. For a O-minimal expansion of $\Re$ from an ordered Abelian group the following conditions are equivalent:

(1) $\Re$ is semi – bounded

(2) $\Re$ has no poles

(3) We can not define a real closed field whose universe is an unbounded sub – interval from $\Re$ and whose order is compatible with order $<$.

(4) $\Re$ is eventually linear

(5) [structure theorem for semi – bounded $\Re$] : Every definable set $X \subseteq R^n$ can be partitioned into a finite number of definable normal cones in addition if $X \subseteq R^n$ is definable and if $\mathcal{F}$ is a finite collection of definable function from X to R, then a partition like $\mathcal{C}$ from X exists as a finite number of definable normal cones so that every function $f \in \mathcal{F}$ maintains every cone $C \in \mathcal{C}$. In other words if $C \in \mathcal{C}$ is a K-cone and $C := B + \Sigma_{i=1}^{k} v_i(t_i)$ then there exists $\mu_1, \dots, \mu_k \in \Lambda$ so that for every $b \in B$ and $t_1, \dots, t_k \in R^{>0}$ we have :

$$f\left(b + \Sigma_{i=1}^{k} v_i(t_i)\right) = f\mid_B (b) + \Sigma_{i=1}^{k} \mu_i(t_i).$$

If we combine the above result with the following theorem of (Loyeys, Peterzil, 1993) we will reach to a conclusion of eliminating the relative quantifier for semi – bounded $\Re$.
Theorem 2. [Loveys, Peterzil]. Assume that

$$\mathcal{V} := (V, +, <, a, (d)_{d \in D}, (P)_{P \in \mathcal{P}})$$

Is an expansion of an ordered vector space $(V, +, <, (d)_{d \in D})$ on an ordered division ring D with predicate symbols of first order $P \in \mathcal{P}$ so that $\mathcal{P}$ includes predicate symbols of first order for all of the definable subsets – a of $[-a, a]^n$ in the vector structure. So $Th(\mathcal{V})$ in its own language has the property of quantifier elimination.

3- Propositions

Theorem 3- Every theory of complete model, is existential universal axiomatization $(\forall\exists)$.
Proof refer to (Marker, 2002, EX, 3.4.12)

Theorem 4 – The property of relative quantifier elimination for semi bounded $\Re$.

Assume that $\Re$ is semi – bounded. In this case $Th(\Re)$ has the property of quantifier elimination and in the language $\mathcal{L}_{sb}(\Re)$ constituted of symbols 0, 1, $+$, $-$, $<$, a symbol for every member Λ and a symbol for every definable bounded set from $\Re$ is universal axiomatization.

Proof: To prove that $Th(\Re)$ has the property of quantifier elimination, according to theorem no.1 it is, sufficient to show that m-cones in definable $\Re$ have no quantifiers. Since $\Re$ is semi bounded so every definable subset $X \subseteq R^m$ is partitioned into a finite number of definable normal cones. But this subject, regarding to the fact that $\Re$ is a vector space over the division ring Λ, is concluded from theorem no.2.

To prove the universal axiomatization of $Th(\Re)$ since $Th(\Re)$ has the quantifier elimination property so it is a complete model; therefore according to theorem no.3 it is an existential universal axiomatization. On the other hand since $\Re$ is O – minimal then it has definable Skolem functions. But according to structure theorem no.1 every definable function in $\Re$ is discretely defined by $\mathcal{L}_{sb}(\Re)$ terms. As a result of this the existential universal axioms have universal equivalents in $\mathcal{L}_{sb}(\Re)$.

In theorem no.1, the proof fro part (5) ⇒(1) is easily obtained from the definitions since if $X \subseteq R^n$ is definable in $\Re$ then according to part (5), it is partitioned into a finite number of definable normal cones $B + \Sigma_{i=1}^{k} v_i(t_i)$. Since the bounded sets B and endomorphisms v_i in the reduced structure $(R, 0, 1, +, <, (B_i)_{i\in I}, (\lambda)_{\lambda\in\Lambda})$ are definable, therefore the set X in the reduced structure is definable, so $\Re$ is semi – bounded. Part (1)⇒(2) is established in general condition and its proof is given in (Prillay, et.al, 1989). Here to complete the proof, we will present an improved form of the proof (Peterzil, 1992) which is also a summarized form of all the proof (Prillay, et.al, 1989). The proof of part (2) ⇒(3) is easily obtained.

Because if you assume that in $\Re$, the real closed field F is definable, then the bijective function $x \mapsto \frac{1}{x}$ maps the definable bounded interval (0, 1) to an unbounded interval in R. therefore $\Re$ will have a pole. In (Peterzil, 1992), part (3) ⇒(4) has been proved for the special condition $R = \mathbb{R}$ using the following two theorems:

Theorem no.5 [Peterzil]

A O-minimal expansion from an ordered group $(\mathbb{R}, 0, 1, +, <)$ is eventually linear if and only if has no pole.

Theorem no.6 [Marker, Peterzil, Pillay]

If a O-minimal expansion from an ordered group $(\mathbb{R}, 0, 1, +, <)$ is not linear (i.e. there is a definable function $f : \mathbb{R} \longrightarrow \mathbb{R}$ which is not discretely linear) then it defines a field over a subinterval (probably bounded) whose order conforms to the order $<$.

The proof of part (4) ⇒(5) is difficult. Peterzil in (Peterzil, 1992) gives a proof for the special case $R = \mathbb{R}$ using the condition known as "Partitioning condition" then in (Van den dries, 1998) a definable nonempty set $X \subseteq R^m$ is defined as:

$\dim X = \max\{i_1 + \cdots + i_m : X \text{ includes one } (i_1, \ldots, i_m) - \text{cell}\}$

Proposition no.1 If $\Re$ is semi – bounded, then $\Re$ has no poles.

Proof: Assume that $\Re$ is semi – bounded and $\Re$ - definable bijective function $\sigma : (a, b) \longrightarrow (c, d)$ exists, where $a, b \in R$ and $c = -\infty$ or $d = +\infty$.

Since $\Re$ is semi – bounded, then $\Re$-definable bijective σ in a reduced structure from $\Re$ in the form $\Re_\sigma := (R, 0, 1, <, (B_i)_{i\in I}, (\lambda)_{\lambda\in\Lambda})$ is also definable, where $(B_i)_{i\in I}$ is a finite collection of bounded definable subsets $B_i \subseteq R^{m_i}$ (m_i' s are positive integers).

Assume that $\Re'$ is a $|\Re|^+$ - saturated elementary extension from $\Re_\sigma$ (so R' has infinite members relative to R, because $P = \{x > r \mid r \in R\}$ is a 1-type). Therefore $\Re' = (R', 0, +, (\lambda)_{\lambda\in\Lambda})$ is a vector space over division ring $\Lambda(\Re_\sigma)$. Because $\Re'$ is an elementary extension of $\Re_\sigma$.

So definable endomorphisms of $\Re_\sigma$ in $\Re'$ are also definable. Thus $\Re'$ closed with respect to the scalar multiplication $\lambda \cdot r' = \lambda(r')$ and is a vector space over the division ring $\Lambda(\Re_\sigma)$.

We consider subspace $S = \{r \in R$ exist so that, $S \in R' : |S| < r\}$ of $\Re'$. Suppose T is complementary of S in $\Re'$: $\Re' = S \oplus T$. According to the definition S consists of all the R-finite members of R′. Therefore the order in the structure $\Re' = S \oplus T$, is dictionary order $(x_1 + y_1 < x_2 + y_2 \Longleftrightarrow x_1 < x_2 \vee (x_1 = x_2 \wedge y_1 < y_2))$. Because $B_i \subseteq R^{m_i}$ is bounded and definable so the explanation of B_i in $\Re'$ includes S^{m_i}. On the other hand every automorphism $\tau : T \rightarrow T$ for the ordered vector space $(T, 0, +, (\lambda)_{\lambda\in\Lambda})$ naturally includes an automorphism $\tau' : R' \rightarrow R'$ with the definition $t \oplus s \mapsto \tau(t) \oplus s$ for the ordered vector space $(R', 0, +, (\lambda)_{\lambda\in\Lambda})$.

Proposition no.2 Assume that x indicates the mass of the function id_R and $f \in \mathcal{G}$ then we have:

(1) $v(f) < v(\mathrm{x})$ If and only If $v(\Delta f) < 0$.

(2) $v(f) > v(\mathrm{x})$ If and only If $v(\Delta f) > 0$.

(3) $v(f) = v(\mathrm{x})$ If and only If $v(\Delta f) = 0$ if and only if there exist $r \in \Lambda^*$ and $u \in \mathcal{G}$ under the condition $v(u) > v(\mathrm{x})$ so that $f = r(\mathrm{x}) + u$.

(4) If $g \in \mathcal{G}$ and $v(f) = v(g) < 0$ then there exists $r \in \Lambda^*$ and $u \in \mathcal{G}$ under the condition $v(u) > v(g)$ so that $f = r(g) + u$.

Proof: (1) assume that $v(f) < v(\mathrm{x})$ thus for every $r \in \Lambda^{>0}$, $|f| > r|\mathrm{x}|$ so by choosing $g = r(\mathrm{x})$ in the previous lemma, we will have:
$\lim_{x\longrightarrow+\infty} |\Delta f(x)| > \lim_{x\longrightarrow+\infty} \Delta g(x) = r$ So according to the note 2.2.4 $\lim_{x\longrightarrow+\infty} |\Delta f(x)| = +\infty$, so $v(\Delta f) < 0$. Inverse) assume $v(\Delta f) < 0$ in this case $.\lim_{x\longrightarrow+\infty} |\Delta f(x)| = +\infty$. So fro every $r \in \Lambda^{>0}$, we have $|f| > r(\mathrm{x})$.

(2) Is done similar to (1)

(3) According to parts (1) and (2) we have $v(f) = v(\mathrm{x})$ if and only if $v(\Delta f) = 0$

To show the next claim in part (3), assume $v(\Delta f) = 0$, then there exist $r \in R^*$ that $\lim_{x \longrightarrow +\infty} |\Delta f(x)| = r$, thus if we put $u := f - r(\mathrm{x})$ then $f = r(\mathrm{x}) + u$ ، $u \in \mathcal{G}$ ، $r \in \Lambda^*$ and $v(u) > v(\mathrm{x})$, Because $\lim_{x \longrightarrow +\infty} \Delta u(x) = 0$ and therefore $v(\Delta u) > 0$ and according to part (2) $v(u) > v(\mathrm{x})$.

Inverse) assume there exist $r \in \Lambda^*$ and $u \in \mathcal{G}$ under the condition $v(u) > v(\mathrm{x})$, therefore we have $v(\Delta u) > 0$ and thus $\lim_{x \longrightarrow +\infty} \Delta u(x) = 0$. Since $f = r(\mathrm{x}) + u$ then $\lim_{x \longrightarrow +\infty} |\Delta f(x)| = r$ where $r \in R^*$ as a result $v(\Delta f) = 0$.

(4) Since $v(f) = v(g) < 0$ then $v(f \circ g^{-1}) = v(\mathrm{x})$. According to previous part, $v(\Delta(f \circ g^{-1})) = 0$ and $u_1 \in \mathcal{G}$ ، $r \in \Lambda^*$ exist that $f \circ g^{-1} = r(\mathrm{x}) + u_1$ and $v(u_1) > v(f \circ g^{-1})$. Therefore $f = r(g) + u_1 \circ g$ by setting $u = u_1 \circ g$ we have: $f = r(g) + u$ and $v(u) > v(f) = v(g)$. Because $v(u_1) > v(f \circ g^{-1})$ then for every $|u_1| < r|f \circ g^{-1}|$ ، $r \in \Lambda^{>0}$ as a result for big enough values of x we have $|u_1(x)| < r|f \circ g^{-1}(x)|$ since $\lim_{x \longrightarrow +\infty} |g(x)| = +\infty$, so for big enough values of x we have: $|u_1(g(x))| < r|f \circ g^{-1}(g(x))| = r|f(x)|$ so far big enough values of x: $|u_1 \circ g(x)| < r|f(x)|$ as a result $|u_1 \circ g| < r|f|$.

Proportion no.3 $\Re$ has no poles if and only if $\Re$ is eventually linear.

Proof: Assume that V has no poles, we show that $\Re$ is not eventually linear and definable function $f : R \longrightarrow R$ is eventually nonlinear. We can assume that for big enough values of x, function f is eventually nonlinear and positive, (since $\Re$ is O– minimal then according to monotonic theory, definable function f from one point on is either strictly positive or strictly negative. We have the following three conditions:

(1) $v(f) > v(\mathrm{x})$, In this case $v(\Delta f) > 0$, so $\lim_{x \longrightarrow +\infty} \Delta f(x) = 0$, therefore Δf is a bijective between an unbounded definable interval and a bounded definable set. As a result Δf defines a pole in $\Re$ which is contrary to the assumption, so $\Re$ is eventually linear.

(2) $v(f) < v(\mathrm{x})$, In this case $v(\Delta f) < 0$. In other words, $\lim_{x \longrightarrow +\infty} |\Delta f(x)| = +\infty$. In this case according to (Miller, Sarchenko, 1998) the inverse function f^{-1}, is definable for big enough values of x and $v(f^{-1}) > v(\mathrm{x})$. As a result $v(\Delta f^{-1}) > 0$. So like the previous case Δf^{-1} defines a pole which contradicts the assumption.

(3) $v(f) = v(\mathrm{x})$, according to proposition no.2, there exists definable function under the condition that u of the function is not zero (because otherwise, since $f = r(\mathrm{x}) + u$, function f becomes eventually linear which is contrary to the assumption) and therefore we have: $v(u) > v(\mathrm{x})$, then $v(\Delta u) > 0$ and consequently Δu defines a pole which contradicts the assumption.

Inverse) assume that $\Re$ is eventually linear. We show that $\Re$ has no poles.

If $\Re$ is eventually linear, then there exist $k \in R$, in which $f\mid_{(k,+\infty)}$ is linear. If x goes to infinity then from linearity of function f, the value of this function becomes infinity too. So $\Re$ has no pole.

Lemma.no1. assume that $D = B + \Sigma_{i=1}^{m} v_i(t_i) \subseteq R^n$ is a normal m-cone and function $f : D \longrightarrow R$ is definable and endomorphisms of $\lambda_1, ..., \lambda_m \in \Lambda$ exist so that $\forall b \in B, \forall t_1, ..., t_m > 0,\ f\left(b + \Sigma_{i=1}^{m} v_i(t_i)\right) = f\mid_B (b) + \Sigma_{i=1}^{m} \lambda_i(t_i)$.

In this case, for every cone $D' = B' + \Sigma_{i=1}^{m'} v_i'(t_i') \subseteq D$, there exist endomorphisms of $\mu_1, ..., \mu_{m'} \in \Lambda$ so that

$\forall b \in B, \forall t_1, ..., t_m > 0,\ f\left(b + \Sigma_{i=1}^{m} v_i(t_i)\right) = f\mid_B (b) + \Sigma_{i=1}^{m} \lambda_i(t_i)$

Besides, if $\lambda_1 = \cdots = \lambda_m = 0$ then $\mu_1 = \cdots = \mu_{m'} = 0$.

Proof: According to the lemma on the sub cones, for every $1 \leq j \leq m$ we have: $v_j' \in \langle v_1, ..., v_m \rangle^{\geq 0} \subseteq \langle v_1, ..., v_m \rangle$; but according to the assumption, function f is linear along any endomorphisms of v_j' s and therefore the endomorphisms of $\mu_1, ..., \mu_{m'}$ which each one of them is a proper linear combination of λ_i s, exist so that:

$\forall b' \in B',\ \forall t_1', ..., t_m' > 0,\ f\left(b' + \Sigma_{i=1}^{m'} v_i'(t_i')\right) = f\mid_{B'} (b') + \Sigma_{i=1}^{m'} \mu_i(t_i')$

Besides, since every endomorphism of μ_j is written in the form of a linear combination of endomorphisms of $\lambda_1, ..., \lambda_m$, so if $\lambda_1 = \cdots = \lambda_m = 0$ then $\mu_1 = \cdots = \mu_{m'} = 0$.

Theorem no.7 [structural theorem] If $\Re$ is an eventually linear O-minimal structure then:

$(1)_n$ Every definable set $X \subseteq R^n$ can be partitioned into a finite number of definable normal cones.

$(2)_n$ If the set $X \subseteq R^n$ is definable and $f_1, ..., f_k : X \longrightarrow R$ are definable functions, then set X can be partitioned into a finite number of normal cones so that fro every cone like $B + \Sigma_{i=1}^{m} v_i(t_i)$ and every $1 \leq j \leq k$, there exists endomorphisms of $\lambda_{1j}, ..., \lambda_{mj} \in \Lambda(\Re)$ that:

$$f_j\left(b + \Sigma_{i=1}^{m} v_i(t_i)\right) = f_j\mid_B (b) + \Sigma_{i=1}^{m} \lambda_{ij}(t_i).$$

The proof for structure theorem, we prove rules $(1)_n, (2)_n$ using parallel inductive reasoning over n. Establishment of cases $(1)_0, (2)_0, (1)_1, (2)_1$, since $\Re$ is O-minimal, are obvious.

One. $(n \geq 1)(2)_{n-1}, (1)_n \Rightarrow (2)_n$

If we proof $(2)_n$ only for a function like f, then using lemma no.1, it can be obtained for k function using induction on k.

Now we proof $(2)_n$ for $f : X \to R$ using induction on $l\dim(X)$. If $l\dim(X) = 0$ then x is a bounded set, therefore the rule is established.

Assume that $l\dim(X) = m + 1$ and the rule is established fro every definable set Y in which $l\dim(Y) < l\dim(X)$. Using $(1)_n$ and the induction assumption, without loosing universalization, we can consider X as a normal $m + 1$ - cone like $B' + \Sigma_{i=1}^{m+1} v_i(t_i)$.

Assume that $\{v_1, ..., v_{m+1}, v_{m+2}, ..., v_n\}$ is a definable base from R^n which includes $\{v_1, ..., v_{m+1}\}$. Consider the linear monomorphism L for R^n which is defined as $L(v_i) := e_{n-i}$ for

$i \in \{1, ..., n-1\}$, where e_i s are standard base vectors. It is obvious that if we prove the rule for $m+1$-cone $L(X)$ then the result is also established for X. Therefore we can assume that X is in the form $B + \Sigma_{i=n-m}^{n} e_i(t_i)$.

Assume that $(x_1, ..., x_n)$ is the coordinate member of X and $\pi : R^n \longrightarrow R^{n-1}$ is the image function over n-1 first coordinate. Also assume that:
$X_0 := \{(\bar{x}, x) \in X : \exists \delta > 0 \forall y \in (-\delta, \delta) \forall z \in (x-\delta, x+\delta) \Delta_y f(\bar{x}, z) = \Delta_y f(\bar{x}, x)\}$ is a definable set on which the function f with respect to the last variable X_n is locally linear. Assume that $K, K' : \pi(X) \longrightarrow R$ are definable functions given in the form: $K(\bar{x}) := \inf\{x \geq 0 : \forall z \geq x, (\bar{x}, z) \in X_0\}$ and $K'(\bar{x}) := 0$. These functions according to the definition of $X_0$and eventual according to the definition of $X_0$and eventual linearity of function f are well – defined.

Two. Proof over $(K, +\infty)_{\pi(X)} = \{(\bar{x}, x) : \bar{x} \in \pi(X), K(\bar{x}) < x < +\infty\}$

For every $\bar{x} \in \pi(X)$, there exists definable function $c : \pi(X) \longrightarrow R$ with the definition $\bar{x} \mapsto c_{\bar{x}}$ and endomorphism of $\lambda_{\bar{x}} \in \Lambda$ so that for every $x > K(\bar{x})$ we have: $f(\bar{x}, x) = c_{\bar{x}} + \lambda_{\bar{x}}(x)$. But will have:

$$\begin{aligned}
\lim_{t \longrightarrow +\infty} \Delta f(x, t) &= \lim_{t \longrightarrow +\infty}(f(\bar{x}, t+1) - f(x, t)) \\
&= \lim_{t \longrightarrow +\infty}(\lambda_{\bar{x}}(t+1) - \lambda_{\bar{x}}(t)) \\
&= \lim_{t \longrightarrow +\infty} \lambda_{\bar{x}}(1) \\
&= \lambda_{\bar{x}}(1) \\
&= \lambda_{\bar{x}}
\end{aligned}$$

we can partition $\pi(X)$ into a finite number of definable subsets so that on each and every one of them, the function $\bar{x} \mapsto \lambda_{\bar{x}}$ is constant. Assume $A \subseteq \pi(X)$ is one of these subsets and the function $\bar{x} \mapsto \lambda_{\bar{x}}$ on it, is equal to the constant valueλ.

Now we apply $(2)_{n-1}$ on $X = A, K = 1$ and $f_1 = c\bar{x}$. For convenience we show the function $\bar{x} \longmapsto c_{\bar{x}}$ by c.

Thus A is partitioned into a finite number of definable normal cones $\tilde{A} = \tilde{B} + \Sigma_{i=1}^{l} v_i(t_i)$ (l is dependent on $\tilde{A}$) so that corresponding to each one of them there exist endomorphisms of $\tilde{\lambda}_1, ..., \tilde{\lambda}_l \in \Lambda$ that $c(b + \Sigma_{i=1}^{l} v_i(t_i)) = c\,|_{\tilde{B}}\,(b) + \Sigma_{i=1}^{l} \tilde{\lambda}_i(t_i)$ so on $(K\,|_{\tilde{A}}, +\infty)_{\tilde{A}}$ we have:

$$f\left(b + \Sigma_{i=1}^{l} v_i(t_i), x\right) = c\,|_{\tilde{B}}\,(b) + \Sigma_{i=1}^{l} \tilde{\lambda}_i(t_i) + \lambda(x)$$

Again using $(2)_{n-1}$ we can partition $\tilde{A}$ into a finite number of normal cones so that if $\bar{A} := \bar{B} + \Sigma_{i=1}^{k} w_i(t_i)$ is one of them then there exist endomorphisms of $\mu_1, ..., \mu_k \in \Lambda$ that:

$$K\left(b + \Sigma_{i=1}^{k} w_i(t_i)\right) = K\,|_{\bar{B}}\,(b) + \Sigma_{i=1}^{k} \mu_i(t_i)$$

Now according to lemma no.1 there exists endomorphisms $\zeta_1, \ldots, \zeta_k \in \Lambda$ that on $(K \mid_{\bar{A}}, +\infty)_{\bar{A}}$ we have:

$$f\left(b + \Sigma_{i=1}^{k} w_i(t_i), x\right) = c \mid_B (b) + \Sigma_{i=1}^{k} \zeta_i(t_i) + \lambda(x).$$

But

$$\begin{aligned}(K \mid_{\bar{A}}, +\infty)_{\bar{A}} &= \{(\bar{x}, x) : \bar{x} \in \bar{A}, K \mid_{\bar{A}} (\bar{x}) < x < +\infty\} \\ &= \{(b + \Sigma_{i=1}^{k} w_i(t_i), x) : b \in B, t_i > 0, x > K \mid_{\bar{B}} (b) + \Sigma_{i=1}^{k} \mu_i(t_i)\} \\ &= \Gamma(K \mid_{\bar{B}}) + \Sigma_{i=1}^{k+1} u_i(t_i)\end{aligned}$$

Where $\Gamma(K \mid_{\bar{B}})$ is the graph of the function $K \mid_{\bar{B}}$ and for every $1 \leq i \leq k, u_i := (w_i, \mu_i)$, and $u_{k+1} = e_n$. So $(K \mid_{\bar{A}}, +\infty)_{\bar{A}}$ is K+1-cone $\Gamma(K \mid_{\bar{B}}) + \Sigma_{i=1}^{k+1} u_i(t_i)$. Thus we have:

$$\begin{aligned}f\left((b, K \mid_B (b)) + \Sigma_{i=1}^{k+1} u_i(t_i)\right) &= f\left(b + \Sigma_{i=1}^{k} w_i(t_i), K \mid_B (b) + \Sigma_{i=1}^{k} \mu_i(t_i) + t_{k+1}\right) \\ &= c \mid_{\bar{B}} (b) + \Sigma_{i=1}^{k} \zeta_i(t_i) + \\ &\quad \lambda(K \mid_{\bar{B}} (b) + \Sigma_{i=1}^{k} \mu_i(t_i) + t_{k+1}) \\ &= c \mid_{\bar{B}} (b) + \Sigma_{i=1}^{k} \zeta_i(t_i) + \lambda(K \mid_{\bar{B}} (b)) \\ &\quad + \Sigma_{i=1}^{k} \lambda(\mu_i(t_i)) + \lambda(t_{k+1}) \\ &= c \mid_B (b) + \lambda(K \mid_B (b)) + \Sigma_{i=1}^{k} (\zeta_i, \lambda\mu_i)(t_i) + \\ &\quad \lambda(t_{k+1}) \\ &= f(b, K \mid_{\bar{B}} (b)) + \Sigma_{i=1}^{k} (\zeta_i, \lambda\mu_i)(t_i) + \lambda(t_{k+1}) \\ &= f \mid_{\Gamma(K|_B)} (b, K \mid_B (b)) + \Sigma_{i=1}^{k+1} \lambda_i(t_i)\end{aligned}$$

Where for every $1 \leq i \leq k, \lambda_i = \zeta_i + \lambda\mu_i$ and $\lambda_{k+1} = \lambda$.

So the rule on $(K, +\infty)_{\pi(X)}$ is established.

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