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Numerical Solution of Fractional Differential Equations by Using the Jacobi Polynomials

Seyedahmad Beheshti[1], Hassan Khosravian-Arab[2], Iman Zare[3]

[1,2,3] Khomein Branch, Islamic Azad University, Khomein, Iran

ABSTRACT

In this paper, we present a new accurate and simple approach for the numerical solution of initial value problems with Caputo type fractional derivative of order $\alpha > 0$. The method is based upon expanding fractional derivative of unknown solution in term of Jacobi polynomials. The properties of Jacobi polynomial are then utilized to reduce the fractional initial value problem to the solution of algebraic equations. Through several numerical examples, the accuracy and validity of the method are reported.

Keywords: Fractional Calculus, Caputo fractional derivatives, Fractional initial value problem, Jacobi polynomials, Jacobi-Gauss quadrature.

1. INTRODUCTION

In this paper, we consider the fractional differential equation

$$y^{(\alpha)}(t) = f(t, y(t)),\ \ \alpha > 0,$$

subject to the following initial conditions

$$y^{(p)}(0) = y_0^{(p)},\ p = 0,\cdots,[\alpha],$$

where f is an arbitrary function, $y^{(p)},\ p = 0,\cdots,[\alpha]$ is the p -th derivative of y and $y_0^{(p)},\ p = 0,\cdots,[\alpha]$ are the special initial conditions. Furthermore $y^{(\alpha)}(t)$ is the Caputo type fractional derivative of order α , defined by

$$y^{(\alpha)}(t) = \frac{d^\alpha}{dt^\alpha}\{y(t)\} = \frac{1}{\Gamma([\alpha]+1-\alpha)}\int_0^t \frac{f^{([\alpha]+1)}(\tau)}{(t-\tau)^{\alpha-[\alpha]}}d\tau.$$

For more details on fractional derivative concepts and another definitions for the fractional derivative, see [17, 23].

Fractional initial value problems (FIVPs) arise in some application areas. See for instance the fractional oscillation equation [19, 23], the linear and nonlinear fractional Bagley-Torvik equation [12], the Basset equation [11, 20], the fractional Lorenz system [3, 29], the fractional dynamical systems [13, 31], and etc.

Several methods have recently been proposed to solve the FIVPs. In [22], Oturanc et al. gave an analytical method for fractional differential equation, Galeone [15] used the Adams multistep methods for FIVPs, In [4], Arikoglu et al. used differential transform method for FIVPs. As the other numerical approach to solve the fractional ordinary and partial differential equations, we refer to [10, 24, 26, 28, 7, 30, 21].

In this paper, by establishing a relationship between Jacobi polynomials and fractional derivatives, a unified method for solution of FIVPs is presented. The Jacobi polynomials have been used extensively in mathematical analysis and numerical solution of differential equations (cf. [6]). Our proposed methods are based on the approximation of unknown solution by using a n -term Jacobi polynomials expansion. Then properties of Jacobi polynomial and Jacobi-Gauss quadrature are utilized to reduce the FIVP to a system of algebraic equations. The main advantage of the presented method is that it gains efficient results even with using a small number of discretization parameter n .

2. SOME PRELIMINARIES

In this section, we briefly review the Jacobi polynomial and Jacobi-Gauss quadrature rule [6, 14, 16].

***Corresponding Author:** Seyedahmad Beheshti, Khomein Branch, Islamic Azad University, Khomein, Iran.
Email: sabeheshti@iaukhomein.ac.ir ,Tel: + 989188665412

2.1 Jacobi Polynomials

The well-known Jacobi polynomials $P_n^{(a,b)}$, $n = 0,1,\cdots$ are given explicitly by

$$P_i^{(a,b)}(x) = \sum_{m=0}^{i} \frac{(-1)^{i-m}(1+b)_i(1+a+b)_{i+m}}{m!(i-m)!(1+b)_m(1+b+a)_i}\left(\frac{x+1}{2}\right)^m, \quad (1)$$

where

$$(a)_k = a(a+1)\cdots(a+k-1), (a)_0 = 1. \quad (2)$$

These polynomials can be expressed equivalently by other explicit formula or by the Rodrigues formula [6, 14, 16]. In practice, we can use the recurrence Bonnet's relation to generate Jacobi polynomials,

$$P_0^{(a,b)}(x) = 1, P_1^{(a,b)}(x) = \frac{1}{2}[(a-b)+(a+b+2)x],$$

$$P_{n+1}^{(a,b)}(x) = (a_n x + b_n)P_n^{(a,b)}(x) - \gamma_n P_{n-1}^{(a,b)}(x), n = 1,2,\cdots,$$

with

$$a_n = \frac{(2n+a+b+1)(2n+a+b+2)}{2(n+1)(n+a+b+1)},$$

$$b_n = \frac{(2n+a+b+1)(a^2-b^2)}{2(n+1)(n+a+b+1)(2n+a+b)},$$

$$\gamma_n = \frac{(n+a)(n+b)(2n+a+b+2)}{(n+1)(n+a+b+1)(2n+a+b)}.$$

These expressions show that $P_n^{(a,b)}(x)$ are analytic functions of parameters a and b. The classical Jacobi polynomials correspond to the parameters $a, b > -1$. For these parameters, the Jacobi polynomials are orthogonal on the canonical interval $[-1,1]$ with respect to the weight function $(1-x)^a(1+x)^b$. i.e.,

$$\int_{-1}^{1} P_n^{(a,b)} P_m^{(a,b)}(1-x)^a(1+x)^b\,dx = \begin{cases} 0, & if m \neq n, \\ \dfrac{2^{a+b+1}}{n!(2n+a+b+1)} \dfrac{\Gamma(n+a+1)\Gamma(n+b+1)}{\Gamma(n+a+b+1)}, & if m = n. \end{cases} \quad (3)$$

As a result, all the zeros of $P_n^{(a,b)}(x)$ are simple and belong to the interval $(-1,1)$.

2.2 Jacobi-Gauss Points and Quadratures

For a given positive integer n, we denote the Jacobi-Gauss points, with parameters a and b, by $\{\xi_i^{(a,b)}\}_{i=0}^n$, which is the set of $n+1$ roots of $P_{n+1}^{(a,b)}(x)$.

The Jacobi-Gauss quadrature rule, with parameters a and b, is based on Jacobi-Gauss points $\{\xi_i^{(a,b)}\}_{i=0}^n$, and can be used to approximate the integral of a function over the range $[-1,1]$ with weight $(1-x)^a(1+x)^b$ as

$$\int_{-1}^{+1} f(x)(1-x)^a(1+x)^b\,dx; \ \sum_{i=0}^{n} \omega_i^{(a,b)} f(\xi_i^{(a,b)}), \quad (4)$$

where $\omega_i^{(a,b)}, i = 0,\cdots,n$ are Jacobi-Gauss quadrature weights. The Jacobi-Gauss points and weights may be determined by accurate and stable methods [16]. Also it is well known that Jacobi-Gauss quadrature has degree of exactness $2n+1$, i.e. it is exact whenever $f(x)$ is a polynomial of degree equal or less than $2n+1$.

3. The Present Method

At first, we present the following theorem that has a key rule in our method.

Theorem 3.1 *For* $\alpha > 0$ *and* $0 < x < \ell$*, we have*

$$\frac{d^\alpha}{dx^\alpha}\left\{x^\alpha P_i^{(0,\alpha)}(\frac{2x}{\ell}-1)\right\}=g_i P_i^{(\alpha,0)}(\frac{2x}{\ell}-1),$$

where $g_i=\dfrac{\Gamma(i+\alpha+1)}{\Gamma(i+1)}, i=0,\cdots,n$.

Proof 3.1 *By substituting* $t=\frac{2x}{\ell}-1$ *in Eq.(1), we have*

$$x^\alpha P_i^{(0,\alpha)}(\frac{2x}{\ell}-1)=\sum_{m=0}^{i}\frac{(-1)^{i-m}(1+\alpha)_{i+m}}{m!(i-m)!(1+\alpha)_m}\frac{x^{m+\alpha}}{\ell^m},$$

now by taking the Riemann-Liouville fractional derivative of order α of both side, we get

$$\frac{d^\alpha}{dx^\alpha}\left\{x^\alpha P_i^{(0,\alpha)}(\frac{2x}{\ell}-1)\right\}=\sum_{m=0}^{i}\frac{(-1)^{i-m}(1+\alpha)_{i+m}}{m!(i-m)!(1+\alpha)_m}\frac{1}{\ell^m}\frac{d^\alpha}{dx^\alpha}\left\{x^{m+\alpha}\right\}.$$

By noting that $\frac{d^\alpha}{dx^\alpha}\left\{x^{m+\alpha}\right\}=\frac{\Gamma(m+\alpha+1)}{\Gamma(m+1)}x^m, \alpha>0, m=0,1,\cdots$, we conclude

$$\frac{d^\alpha}{dx^\alpha}\left\{x^\alpha P_i^{(0,\alpha)}(\frac{2x}{\ell}-1)\right\}=\sum_{m=0}^{i}\frac{(-1)^{i-m}(1+\alpha)_{i+m}\Gamma(m+\alpha+1)}{m!(i-m)!(1+\alpha)_m\Gamma(m+1)}\left(\frac{x}{\ell}\right)^m$$
$$=\frac{\Gamma(i+\alpha+1)}{\Gamma(i+1)}P_i^{(\alpha,0)}(\frac{2x}{\ell}-1).$$

For sake of simplicity, we consider the case $0<\alpha<2$. Furthermore, the case $\alpha\geq 2$ does not seem to be of major practical interest. However the presented method is simply extended for the case $\alpha\geq 2$. Indeed, we consider the following fractional initial value problem

$$y^{(\alpha)}(x)=f(x,y(x)),\ 0<x<\ell, \tag{5}$$
$$y(0)=a, \tag{6}$$
$$y'(0)=b, \tag{7}$$

where the second initial condition (7) is for $\alpha>1$ only. For $\alpha\leq 1$, we approximate $y(x)$ as

$$y(x);\ \tilde{y}_n(x)=a+\sum_{i=0}^{n}c_i x^\alpha P_i^{(0,\alpha)}(\frac{2x}{\ell}-1), \tag{8}$$

and for $1<\alpha\leq 2$, we approximate $y(x)$ by

$$y(x);\ \tilde{y}_n(x)=a+bx+\sum_{i=0}^{n}c_i x^\alpha P_i^{(0,\alpha)}(\frac{2x}{\ell}-1). \tag{9}$$

Note that the approximation (8) satisfies initial condition (6) and the approximation (9) satisfies initial conditions (6) and (7). For summarizing the both cases, we set

$$\tilde{y}_n(x)=a+b^*x+\sum_{i=0}^{n}c_i x^\alpha P_i^{(0,\alpha)}(\frac{2x}{\ell}-1), \tag{10}$$

where

$$b^*=\begin{cases}0, & \alpha\leq 1,\\ b, & 1<\alpha\leq 2.\end{cases}$$

In view of Theorem 3.1, we can approximate $y^{(\alpha)}(x)$ as

$$y^{(\alpha)}(x);\ \ \tilde{y}_n^{(\alpha)}(x)=\sum_{i=0}^{n}c_i g_i P_i^{(\alpha,0)}(\frac{2x}{\ell}-1). \tag{11}$$

By substituting (10) and (11) in fractional ODE (5), we get

$$\sum_{i=0}^{n}c_i g_i P_i^{(\alpha,0)}(\frac{2x}{\ell}-1)=f\left(x,a+b^*x+\sum_{i=0}^{n}c_i x^{\alpha}P_i^{(0,\alpha)}(\frac{2x}{\ell}-1)\right).$$

Now by multiplying both side of above equations by $P_k^{(\alpha,0)}(\frac{2x}{\ell}-1)\,(1-\frac{x}{\ell})^{\alpha}$ and integrating them in the interval $[0,\ell]$, we get

$$\sum_{i=0}^{n}c_i g_i\int_0^{\ell}P_i^{(\alpha,0)}(\frac{2x}{\ell}-1)P_k^{(\alpha,0)}(\frac{2x}{\ell}-1)\,(1-\frac{x}{\ell})^{\alpha}\,dx=$$

$$\int_0^{\ell}f\left(x,a+b^*x+\sum_{i=0}^{n}c_i x^{\alpha}P_i^{(0,\alpha)}(\frac{2x}{\ell}-1)\right)P_k^{(\alpha,0)}(\frac{2x}{\ell}-1)\,(1-\frac{x}{\ell})^{\alpha}\,dx.$$

By applying transformation $x=\frac{\ell}{2}(\xi+1)$, and by using orthogonality property (3), finally we get

$$\frac{2^{2\alpha+2}}{\ell(2k+\alpha+1)}g_k c_k=$$

$$\int_{-1}^{1}f\left(\ell\frac{\xi+1}{2},a+\ell b^*\frac{\xi+1}{2}+\sum_{i=0}^{n}c_i\left(\ell\frac{\xi+1}{2}\right)^{\alpha}P_i^{(0,\alpha)}(\xi)\right)P_k^{(\alpha,0)}(\xi)(1-\xi)^{\alpha}\,d\xi.$$

Now, in order to obtain high order accuracy, the integral term in above equation is approximated by using Jacobi Gauss quadrature (4) and we get

$$\frac{2^{2\alpha+2}}{\ell(2k+\alpha+1)}g_k c_k=$$

$$\sum_{j=0}^{n}\omega_{n,j}^{(\alpha,0)}f\left(\hat{\tau}_{n,j}^{(\alpha,0)},a+b^*\hat{\tau}_{n,j}^{(\alpha,0)}+\sum_{i=0}^{n}c_i\left(\hat{\tau}_{n,j}^{(\alpha,0)}\right)^{\alpha}P_i^{(0,\alpha)}(\tau_{n,j}^{(\alpha,0)})\right)P_k^{(\alpha,0)}(\tau_{n,j}^{(\alpha,0)}),$$

$$k=0,\cdots,n,$$

where $\hat{\tau}_{n,j}^{(\alpha,0)}=\ell\frac{\tau_{n,j}^{(\alpha,0)}+1}{2}$, $j=0,\cdots,n$. The above equations form a (nonlinear) system of algebraic equations. By solving it, we can find unknown coefficients c_i, $i=0,\cdots,n$ and the approximation (10) is followed.

4. NUMERICAL ILLUSTRATION

This section is devoted to the numerical experiments. We implemented the proposed method for numerical solution of FIVPs with Matlab in a personal computer. Meanwhile, we use Matlab routines provided by Gautschi [16] in our implementation to generate Jacobi-Gauss nodes and weights.

Table. 1 lists some FIVPs, which are used as test examples in our simulations and reports. In this table, $E_{a,b}$ is Mittag-Leffler function[23],

$$E_{a,b}(t):=\sum_{k=0}^{\infty}\frac{t^k}{\Gamma(ak+b)},\quad a,b>0.$$

This function plays the same role in differential equations of fractional order which the exponential function e^t plays in ordinary differential equations; in fact, $E_{1,1}(t)=e^t$.

At first, we apply our method to Ex. 1, with $\alpha=1.3$, $\ell=20$ and $n=2,4,6$ and $n=30$. The resulting

solutions and exact solution are plotted in Fig. 4.a. We see that, the presented method is provides accurate results even with using $n = 6$. Also the error of obtained solution with $n = 30$ is plotted in Fig. 4.b.
To explore the dependence of errors on the parameter n , we use the following notation

$$e_n = \underset{1 \le j \le 10000}{Max} \left| y(t_j) - \tilde{y}_n(t_j) \right|, \qquad t_j = j\frac{\ell}{10000},\ j = 1, \cdots, 10000,$$

and we plot e_n for Ex. 1 ($\alpha = 1.8$ and $\ell = 6$), Ex. 2 ($\alpha = 1.2$ and $\ell = 1$) and Ex. 3 ($\alpha = 0.3$, $\ell = 1$, $c = 100$ and $k = 10$) in Fig. 5. This figure shows that the presented method converges quickly.
To make a comparison, we consider the four numerical methods [1, 10, 18, 25] which are investigated in [2]. In Tables. 2 and 3, the obtained results of these methods and our method, for Ex. 1 with $\alpha = 0.25, \alpha = 1.5$ and $\ell = 6.4$ are reported.
In final, we apply our method to Ex. 4, with $n = 30$ and $\alpha = 0.2, 0.4, \cdots, 1.8, 2$. The results are plotted in Fig. 6. We mention that when $\alpha = 1$ and $\alpha = 2$ the exact solutions are $y(x) = \frac{1}{2}\left((1+x)e^{-x} - cos(x)\right)$ and $y(x) = \frac{1}{2}x^2 e^{-x}$ respectively. These exact solutions are highlighted in this figure.

Also in Table. 4, the obtained values of $\tilde{y}_n(x)$ in $x = 1,2,3,4,5$ with $n = 10,50,100$ are presented. Through this Table, the accuracy and convergence rate of presented method for Ex. 4 are shown.

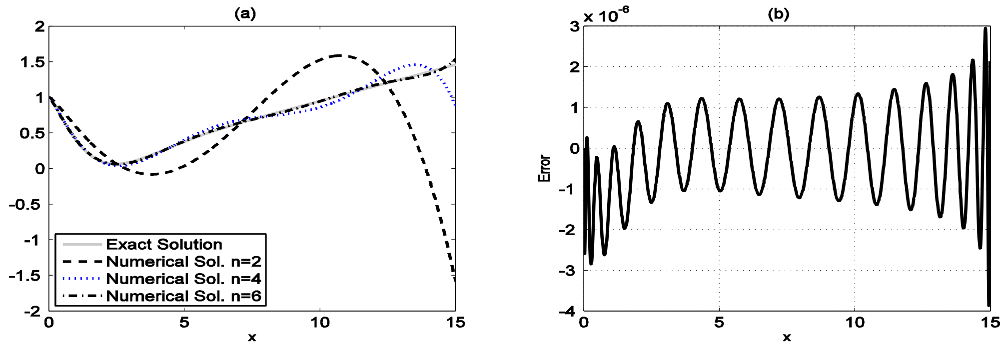

Figure 1: (left) Comparison between exact solution and obtained solutions by the present method for Ex. 1 with $n = 2,4,6$. (right) Plot of error $y(x) - \tilde{y}_n(x)$ for $n = 30$.

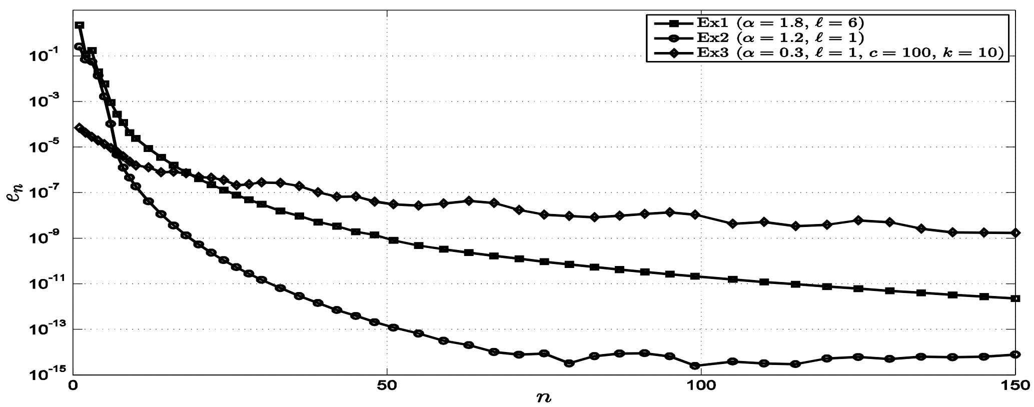

Figure 2: Error e_n as a function of discretization parameter n for Examples 1-3.

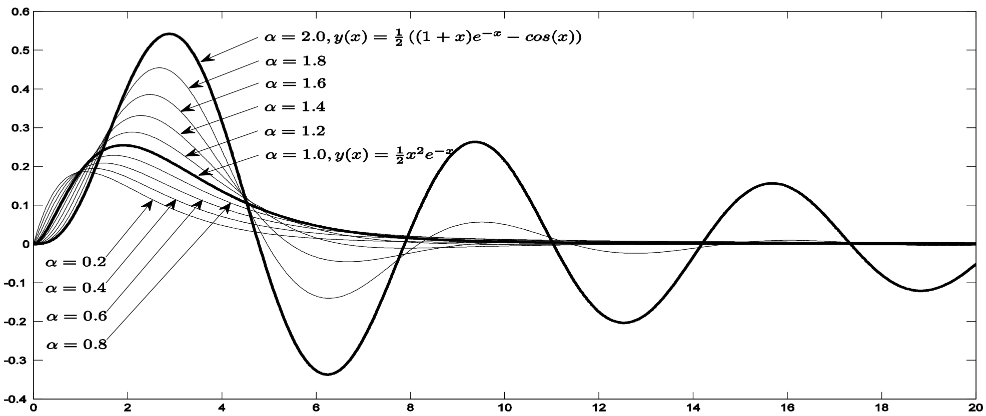

Figure 3: Comparison of obtained solution of Ex. 4 by the present method with $n = 3$ for various values of α

Table 1: Some Fractional Initial Values Problems

Ex.1 [2, 27]	ODE	$y^{(\alpha)}(x) = 0.1x - y(x), 0 < \alpha < 2$
	Initial Co.	$y(0) = 1, y'(0) = 0^{\dagger}$
	Exa Sol.	$y(x) = 0.1x\left(1 - E_{\alpha,2}(-x^{\alpha})\right) + y(0)E_{\alpha,1}(-x^{\alpha})$
Ex. 2 [2, 9]	ODE	$y^{(\alpha)} = \frac{40320}{\Gamma(9-\alpha)}x^{8-\alpha} - 3\frac{\Gamma(5+\frac{\alpha}{2})}{\Gamma(5-\frac{\alpha}{2})}x^{4-\frac{\alpha}{2}} + \frac{9}{4}\Gamma(1+\alpha) + (\frac{3}{2}x^{\frac{\alpha}{2}} - x^4)^3 - y^{\frac{3}{2}}, 0 < \alpha < 2$
	Initial Co.	$y(0) = 0, y'(0) = 0^{\dagger}$
	Exa Sol.	$y(x) = x^8 - 3x^{4+\frac{\alpha}{2}} + \frac{9}{4}x^{\alpha}$
Ex. 3 [8, 5]	ODE	$cy^{(\alpha)}(x) + ky(x) = \begin{cases} 0 & x = 0 \\ 1 & x \neq 0, 0 < \alpha < 1 \end{cases}$
	Initial Co.	$y(0) = 0,$
	Exa Sol.	$\frac{1}{k}\left(1 - E_{\alpha}(-kx^{\alpha}/c)\right)$
Ex. 4 [19, 23]	ODE	$y^{(\alpha)}(x) = -y + xe^{-x}, 0 < \alpha < 2$
	Initial Co.	$y(0) = 0, y'(0) = 0^{\dagger}$
	Exa Sol.	No Exact Solution

Table 2: Comparison of errors between some numerical methods and our method at different x for Ex. 1 with $\alpha = 0.25$

	Methods (with $h = 0.00625(n = 1024)$)				Presented Method	
x	**Ref.[10]**	**Ref.[1]**	**ef.[18]**	**Ref.[25]**	$n = 10$	$n = 100$
0.8	-5.01e-5	-2.76e-5	-3.29e-5	1.06e-3	-2.77e-4	-1.83e-6
1.6	2.46e-5	-1.36e-5	-1.62e-5	5.34e-4	8.89e-4	2.36e-6
2.4	-1.61e-5	-8.91e-6	-1.06e-5	3.58e-4	-6.48e-4	1.12e-7
3.2	-1.19e-5	-6.58e-6	-7.83e-6	2.70e-4	4.35e-4	-7.88e-7
4.0	-9.39e-6	-5.20e-6	-6.17e-6	2.17e-4	-1.71e-4	-1.37e-6
4.8	-7.73e-6	-4.28e-6	-4.91e-6	1.82e-4	1.33e-4	3.35e-7
5.6	-6.69e-6	-3.76e-6	-4.72e-6	1.57e-4	-5.58e-4	2.10e-6
6.4	-9.55e-6	-7.02e-6	-7.25e-6	1.35e-4	-3.50e-3	-2.36e-5

Table 3: Comparison of errors between some numerical methods and our method at different x for Ex. 1 with $\alpha = 1.5$

	Methods (with $h = 0.1(n = 64)$)				Presented Method	
x	**Ref.[10]**	**Ref.[1]**	**Ref.[18]**	**Ref.[23]**	$n = 10$	$n = 50$
0.8	3.53e-4	2.63e-5	3.57e-5	1.26e-6	-2.46e-5	-8.1011e-9
1.6	1.77e-4	1.60e-5	2.84e-5	-2.53e-6	8.24e-6	-5.63e-9
2.4	-1.96e-4	3.65e-6	1.19e-5	-5.67e-6	-1.19e-5	-7.88e-10
3.2	-4.07e-4	-3.73e-6	-1.06e-6	-4.60e-6	1.50e-5	1.59e-9
4.0	-3.58e-4	-5.52e-6	-6.89e-6	-2.75e-6	-1.81e-5	1.80e-9
4.8	-1.55e-4	-3.98e-6	-6.92e-6	-8.23e-7	2.08e-5	-4.26e-10
5.6	4.62e-4	-1.62e-6	-4.24e-6	1.86e-6	3.39e-6	2.10e-9
6.4	1.50e-4	7.68e-8	-1.37e-6	3.47e-6	1.43e-4	1.83e-8

Table 4: The resulting value of the presented method on Ex. 4 for $\tilde{y}_n(x)$ in $x = 1,2,3,4,5$ with $n = 10,50,100$.

x	$n = 10$	$n = 50$	$n = 100$
$x = 1$	**1.5785**33757460$e-1$	**1.5785166255**28$e-1$	**1.5785166255**69$e-1$
$x = 2$	**3.41473**0076501$e-1$	**3.41472659**2927$e-1$	**3.41472659**3070$e-1$
$x = 3$	**3.2216**11163682$e-1$	**3.221638352**932$e-1$	**3.221638352876**$e-1$
$x = 4$	**1.8793**46621946$e-1$	**1.879371242**407$e-1$	**1.879371242520**$e-1$
$x = 5$	**6.2654**64972031$e-2$	**6.264072**201256$e-2$	**6.26407219**0534$e-2$

5. CONCLUSION

In the present work, a numerical procedure has been developed for obtaining the solution of fractional initial value problems. In this method the Jacobi polynomials were used to reduce the fractional initial value problem to a system of algebraic equations. The method is characterized by simplicity, efficiency and it is also readily implemented. By some numerical examples, we analyzed the accuracy and validity of the presented method through simulations. We believe that the presented method in this work can be extended to solve the multiterm fractional differential equations.

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