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Approximate Solution of a Class of Fredholm Integral Equations of the Second Kind with Hypersingular Kernels

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ABSTRACT

In this paper a method is presented for solving a class of hyper singular integral equations of the second kind. The unknown function is approximated as a truncated series of Legendre polynomials. The integrals are computed in terms of gamma functions. Some examples are given to show the applicability of the method.

Keywords: Hyper singular integral equations, Prandtl's integral equation, Legendre polynomials.

1. INTRODUCTION

Hyper singular integral equations of the second kind have more applications in some branches of mathematical physics and mechanics. These equations arise in viscous fluid problems, water-wave problems [15], Harmonic problems [9], elliptic wing theory [1], 3D potential theory problems [10], radiative equilibrium [7] and radiative heat transfer and electromagnetics [15]. Recently many numerical solution methods are presented to solve these equations [6]. Babolian in [3] introduced a weakly singular integral equation and used Legendre polynomials to solve it. For the treatment of Cauchy type singular integral equations a numerical method is presented in [13]. Taylor-series expansion is used For a class of Fredholm integral equations of the second kind, in [4]. In these methods authors usually use interpolating polynomials to approximate the kernel functions also they use collocation methods and galerkin methods for numerical solution of the system of linear equations [5,8,10,11,12,14].

Following singular integral equation of the second kind

$$u(x)=f(x)+\frac{\alpha\left(1-x^2\right)^{1/2}}{\pi}\int_{-1}^{1}\frac{u(t)}{(t-x)^2}dt \quad , \quad -1<x<1 \tag{1.1}$$

is placed on finite interval in which u(±1) = 0 is generalized state for oval wing of Prandtl's equation, where $\alpha>0$ is a known value, $f(x)$ and $u(x)$ are known and unknown functions respectively. Equation (1.1) is referred to as Hadamard finite element, and is hyper singular [1]. Accurate solution of equation (1.1) was obtained by using a simple approximating polynomial for u(x) in [1] and by reducing it to differential problem of Riemann-Hilbert on (-1, 1) interval in [2]. In this paper, equation (1.1) is solved using Legendre polynomials as kernel function, obtained answer satisfy with solutions offered in [1] and [2].

2. General Method

In equation (1.1) it is assumed that

$$u(x)=\left(1-x^2\right)^{1/2}\psi(x), \tag{2.1}$$

where $\psi(x)$ is a smooth function. Let us approximate $\psi(x)$ with a truncated series as follows

$$\psi(x)=\sum_{j=0}^{n}a_j\,p_j(x), \tag{2.2}$$

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where $p_j(x)$, $j=0,1,2,\ldots$ are Legendre polynomials. After having substitution (2.1) in (1.1) and simplification we get

$$\sum_{j=0}^{n} a_j \left[p_j(x) - \frac{\alpha}{\pi} \int_{-1}^{1} \frac{\left(1-t^2\right)^{1/2} p_j(t)}{(t-x)^2} \right] = \frac{f(x)}{\left(1-x^2\right)^{1/2}}. \tag{2.3}$$

Assuming

$$B_j'(x) = \int_{-1}^{1} \frac{\left(1-t^2\right)^{1/2} p_j(t)}{(t-x)^2} dt, \qquad F(x) = \frac{f(x)}{\left(1-x^2\right)^{1/2}}, \tag{2.4}$$

we have

$$B_j(x) = \int_{-1}^{1} \frac{\left(1-t^2\right)^{1/2} p_j(t)}{(t-x)} dt. \tag{2.5}$$

With (2.4), (2.5) and (2.3) we have

$$\sum_{j=0}^{n} a_j \left[p_j(x) - \frac{\alpha}{\pi} B_j'(x) \right] = F(x), \qquad -1 < x < 1 \tag{2.6}$$

Let us define

$$C_j(x) = p_j(x) - \frac{\alpha}{\pi} B_j'(x) \tag{2.7}$$

By substituting (2.7) in (2.6), we get the following simplified form

$$\sum_{j=0}^{n} a_j \, C_j(x) = F(x), \qquad -1 < x < 1 \tag{2.8}$$

Referring to [1], let us define

$$A_j(x) = \int_{-1}^{1} \frac{\left(1-t^2\right)^{1/2} t^j}{(t-x)} dt, \qquad j = 0,1,\ldots,n \tag{2.9}$$

which was obtained in [1] as follows

$$\begin{aligned} A_0(x) &= -\pi x, \\ A_j(x) &= -\pi x^{j+1} + \sum_{i=0}^{j-1} \frac{1+(-1)^i}{4} \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{i+1}{2}\right)}{\Gamma\left(\frac{i+4}{2}\right)} x^{j-1-i}, \quad j = 1,2,\ldots \end{aligned} \tag{2.10}$$

We can express $B_j(x)$, $j=0,1,2,\ldots$ using $A_j(x)$, $j=0,1,2,\ldots$ as follows

$$B_0(x) = \int_{-1}^{1} \frac{\left(1-t^2\right)^{1/2} p_0(t)}{(t-x)} dt = \int_{-1}^{1} \frac{\left(1-t^2\right)^{1/2}}{(t-x)} dt = A_0(x),$$

$$B_1(x) = \int_{-1}^{1} \frac{\left(1-t^2\right)^{1/2} p_1(t)}{(t-x)} dt = \int_{-1}^{1} \frac{\left(1-t^2\right)^{1/2} t}{(t-x)} dt = A_0(x),$$

$$B_2(x) = \int_{-1}^{1} \frac{\left(1-t^2\right)^{1/2} p_2(t)}{(t-x)} dt = \int_{-1}^{1} \frac{\left(1-t^2\right)^{1/2}\left(\frac{3}{2}t^2 - \frac{1}{2}\right)}{(t-x)} dt = \frac{3}{2} A_2(x) - \frac{1}{2} A_0(x),$$

Then we can conclude that

$$B_j(x) = p_j(u), \quad u^j = A_j(x), \quad j = 0,1,2,\dots \tag{2.11}$$

where $p_j(u)$, $j = 1,2,\dots$ are Legendre polynomials with respect to variable *u*. For instance

$$B_3(x) = P_3(u) = \frac{5}{2}u^3 - \frac{3}{2}u = \frac{5}{2}A_3(x) - \frac{3}{2}A_1(x).$$

After calculating $B_j(x)$, then $B'_j(x)$ and finally $C_j(x)$, $(j = 0,1,2,\dots)$ are calculated.

With substituting $C_j(x)$'*s*, in (2.8), the unknown factors a_j, $j = 0,1,2,\dots$ can be calculated in two ways:

First method: It is an analytical method. Left hand side of (2.8) is simplified and united with right hand side (equaling factors of x's similar powers) and by solving resultant system, a_j, $j = 0,1,2,\dots$ are calculated.

Second method: it is a numerical method (collocation method). $n+1$ arbitrary points x_i, $i = 0,1,2,\dots,n$ are selected from $(-1,\ 1)$ which might be equally spaced or not. Thus the following system of $n+1$ linear equations yields the unknowns parameters a_j, $j = 0,1,2,\dots,n$.

$$\sum_{j=0}^{n} a_j\, C_j(x_i) = F(x_i) \quad , \quad -1 < x_i < 1\,, \quad i = 0,1,2,\dots,n. \tag{2.12}$$

3. Numerical examples

Example 1: In the hyper singular integral equation (1.1) assume that

$$f(x) = \frac{2\pi\, k}{\beta}\left(1 - x^2\right)^{1/2}, \qquad \beta > 0\,,\ \alpha = \frac{\pi}{\beta}, \tag{3.1}$$

where β and k are constants. It gives

$$C_j(x) = p_j(x) - \frac{1}{2\beta}B'_j(x)\ , \qquad j = 0,1,2,\dots,n\ , \quad F(x) = \frac{2\pi\, k}{\beta}. \tag{3.2}$$

Whit substituting (3.2) in (2.8) and simplified left side of it and unifying it with right hand side (equaling factors of x's similar powers in both sides of equation), it gives

$$a_\circ = \frac{4k}{1 + \frac{2\beta}{\pi}}, \qquad a_1 = a_2 = \dots = 0. \tag{3.3}$$

Thus

$$\psi(x) = \frac{4k}{1 + \frac{2\beta}{\pi}}\ , \qquad u(x) = \frac{4k}{1 + \frac{2\beta}{\pi}}\left(1 - x^2\right)^{1/2}, \tag{3.4}$$

which is the exact solution of the equation (1.1) with (3.1).

In numerical method, by putting $\beta = k = 1$, $n = 10$ and selecting 11 points $x_i = \left(\frac{2}{11}\right)i$, $(i = -5,-4,\dots,0,\ \dots,4,5)$ after solving resultant system we get

Table 1: Absolute error for Example 2, Case b.

x	$n=5$	$n=10$	$n=15$	$n=20$
-0.5	$0.51E-7$	$0.12E-11$	$0.22E-19$	$0.23E-25$
-0.2	$0.45E-8$	$0.50E-12$	$0.79E-20$	$0.81E-26$
0	0	0	0	0
0.2	$0.45E-8$	$0.50E-12$	$0.79E-20$	$0.81E-26$
0.5	$0.51E-7$	$0.12E-11$	$0.22E-19$	$0.23E-25$

$$a_{\circ}=2.44406, \quad a_1=a_2=....a_{10}=0. \tag{3.5}$$

Then

$$\psi(x)=2.444406, \quad u(x)=2.444406\left(1-x^2\right)^{1/2}. \tag{3.6}$$

(3.6) is the same answer which was obtained in [1].

Example 2: Consider the following hyper singular integral equation

$$\frac{1}{\pi}\int_{-1}^{1}\frac{u(t)}{(t-x)^2}dt=f(x), \qquad -1<x<1, \quad u(\pm 1)=0. \tag{3.7}$$

Equation (3.7) was solved analytically in [2] as follows

$$u(x)=\frac{1}{\pi}\int_{-1}^{1}f(t)\,Ln\left|\frac{t-x}{1-tx+\left\{(1-t^2)(1-x^2)\right\}^{1/2}}\right|dt \qquad -1<x<1. \tag{3.8}$$

The integral part in (3.8) is difficult to solve analytically, then we prefer to solve (3.7) approximately. After some minor corrections the method proposed in section 2 is used easily to solve (3.7). Here we apply the method for two different equations.

Case a: Let $f(x)=x^2$. In this case we get the exact solution with (3.8) as follows

$$u(x)=\left(-\frac{1}{3}x^2-\frac{1}{6}\right)\left(1-x^2\right), \qquad -1<x<1. \tag{3.9}$$

With $n=5$ our method gives

$$u_{app}(x)=\left(-0.33333333x^2-0.16666666\right)\left(1-x^2\right) \tag{3.10}$$

Case b: Let $f(x)=\sin x$. In this case we cannot solve (3.8), then we apply the approximate method and. Table 1 shows the numerical results obtained for Case b.

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