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Some Properties of a Set-valued Homomorphism on Modules

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ABSTRACT

The main of this paper is to introduce a set-valued homomorphism in which induced by a module homomorphism and another set-valued homomorphism. We discuss the generalized lower and upper rough submodules of a module to obtain some results in T-rough modules.
Keywords: Approximation space; Commutative ring; Module; Rough module; T -rough set; Set-valued homomorphism; T-rough module
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1. INTRODUCTION

After the pioneering work of Zadeh (1965), there has been a great effort to obtain fuzzy analogies of classical theories. Various uncertainties in real world applications can bring difficulties in determining the crisp membership functions of fuzzy sets.

There have been involved not only vagueness (lack of sharp class boundaries), but also ambiguity (lack of information). Hence many extensions have been developed to represent these uncertainties in membership values, such as interval-valued fuzzy sets. In other hand, the notion of rough sets has been introduced by Pawlak (1981, 1982, 1985), and Pawlak and Skowron (2007). Rough set theory is an extension of set theory, in which a subset of a universe is described by a pair of ordinary sets called the lower and upper approximations. The lower approximation of a given set is the union of all the equivalence classes which are subsets of the set, and the upper approximation is the union of all the equivalence classes which have a non-empty intersection with the set. It soon invoked a natural question concerning possible connection between rough sets and algebraic systems and fuzzy sets. The algebraic approach to rough sets has been studied by Iwinski (1987), Bonikowaski (1995), Zhang, Wu (2001). Banerjee and Pal (1996), Nanda (1992), Biswas (1994), Biswas and Nanda (1994) discussed the notion of rough sets and rough subgroups. Kuroki (1997) introduced the notion of rough ideals in semigroups. Davvaz (2004) has given the notion of rough subring with respect to a subring of a ring. Dubois and Prade (1987, 1990) combined fuzzy sets and rough sets in a fruitful way by defining rough fuzzy sets and fuzzy rough sets. Qi-Mei Xiao and Zhen-Liang Zhang (2006) discussed the lower and the upper approximations of prime ideals and fuzzy prime ideals in a semigroup with details. Davvaz (2008) defined a T - rough homomorphism in a group and introduced the T -rough set with respect to a subgroup of a group. Based on the definition, Hosseini et al. (2012) studied some properties of T –rough set in semigroups and commutative rings. It is well known that a partition induces an equivalence relation on a set and vice versa. The properties of rough sets can thus be examined via either partition or equivalence classes. Rough sets are a suitable mathematical model of vague concepts, i.e., concepts without sharp boundaries.

In this paper, we introduce the notion of a set-valued homomorphism on a module and the generalized rough module with respect to a submodule. We prove some more general and fundamental properties of the generalized rough sets. We discuss the relations between the upper and lower T –rough modules and the upper and lower approximations of their homomorphism images and generalize some theorems which have been proved in (2008, 2012). We attempt to conduct a further study along this line.

1. Preliminaries

The following definitions and preliminaries are required in the sequel of our work and hence presented in brief. Some of them were in (1981, 1982, 1985 and 2004). Suppose that U is a non-empty set. A partition or classification of U is a family Θ of non-empty subsets of U such that each element of U is contained in exactly one element of Θ. It is vitally important to recall that an equivalence relation θ on a set U is a reflexive, symmetric and transitive binary relation on U. Each partition Θ induces an equivalence relation θ on U by setting

$x\theta y \Leftrightarrow x$ and y are in the same class of Θ.

Conversely, each equivalence relation θ on U induces a partition Θ of U whoseclasses have the form

$[x]_\theta = \{y \in U \mid x\theta y\}$.

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Definition 2.1. A pair (U, θ) where $U = \emptyset$ and θ is an equivalence relation on U is called an approximation space.

Definition 2.2. For an approximation space (U, θ) by a rough approximation in (U, θ) we mean a mapping $Apr_\theta : P(U) \longrightarrow P(U) \times P(U)$ defined by for every $X \in P(U)$, $Apr_\theta(X) = (\underline{Apr\theta}(X), \overline{Apr\theta}(X))$, where

$\underline{Apr}_\theta(X) = \{x \in U \mid [x]_\theta \subseteq X\}$, $\overline{Apr}_\theta(X) = \{x \in U \mid [x]_\theta \cap X \neq \emptyset\}$.

$\underline{Apr}_\theta(X)$ is called the lower rough approximation of X in (U, θ) whereas $\overline{Apr}_\theta(X)$ is called the upper rough approximation of X in (U, θ).

Definition 2.3. Given an approximation space (U, θ) a pair (A, B) in $P(U) \times P(U)$ is called a rough set in (U, θ) if $(A, B) = (\underline{Apr}_\theta(X), \overline{Apr}_\theta(X))$ for some $X \in P(U)$.

Definition 2.4. A subset X of U is called definable if $\underline{Apr}_\theta(X)$, $= Apr_\theta(X)$. If $X \subseteq U$ is given by a predicate P and $x \in U$, then

1. $x \in \underline{Apr}_\theta(X)$, means that x certainly has property P,
2. $x \in \overline{Apr}_\theta(X)$ means that x possibly has property P ,
3. $x \in U \setminus \overline{Apr}_\theta(X)$ means that x definitely does not have property P.

2. Set-valued Homomorphism

Throughout the paper, R is a commutative ring, M, N are R-modules and if X be a set, the set of all non-empty subsets of X denoted by $P^*(X)$. We define the concept of a set-valued homomorphism and give some important examples of them. We show that every module homomorphism is a set-valued homomorphism. We also investigate some basic properties of the generalized lower and upper submodules induced by a set-valued homomorphism.

Definition 3.1. (2008) Let X and Y be two non-empty sets and $B \in P^*(Y)$. Let $T : X \to P^*(Y)$ be a set-valued mapping. The lower inverse and upper inverse of B under T are defined by

$L_T(B) = \{x \in X \mid T(x) \subseteq B\}$; $U_T(B) = \{x \in X \mid T(x) \cap B \neq \emptyset\}$.

They are called the T-lower approximation or the lower T-rough and the T-upper approximation or the upper T-rough with respect to B, respectively.

Definition 3.2. (2008) Let X and Y be two non-empty sets and $B \in P^*(Y)$. Let $T : X \to P^*(Y)$ be a set-valued mapping, then $(L_T(B), U_T(B)) \in P^*(X) \times P^*(X)$ is called the T-rough set of X with respect to B or the generalized rough set with respect to B.

Example 3.3. (i) Let (U, θ) be an approximation space and $T : U \to P^*(U)$ be a set-valued mapping where $T(x) = [x]_\theta$, then for any $B \in P^*(U)$, $L_T(B) = \underline{Apr}(B)$ and $U_T(B) = \overline{Apr}(B)$. So a rough set is a T-rough set.

Definition 3.4. (i) Suppose that A be a non-empty subset of M. The subset A is called an R-submodule of M or just submodule of M if $a - b$, $ra \in A$ for all a, $b \in A$ and $r \in R$.

(ii) Let M be an R-module. A proper submodule N of M is called prime if given $r \in R$, $m \in M$ then $rm \in N$ implies $m \in N$ or $rM \subseteq N$.

(iii) Let M be an R-module. A proper submodule N of M is called primary if given $r \in R$, $m \in M$ then $rm \in N$ implies $m \in N$ or $r^m M \subseteq N$ for some $m \in N$.

(iv) A mapping $f : M \to N$ is called an R-module homomorphism or just a module homomorphism provided that for all u, $v \in M$ and $r \in R$:

$f(u + v) = f(u) + f(v)$ and $f(ru) = r f(u)$.

Definition 3.5. Let M and N be R-modules and $T : M \to P^*(N)$ be a set-valued mapping. T is called a set-valued homomorphism if

(i) $T(m_1 + m_2) = T(m_1) + T(m_2)$;

(ii) $T(rm) = rT(m)$;

for all $r \in R$ and m, m_1, $m_2 \in M$.

It is clear that $T(0) = \{0\}$ and $T(-m) = -T(m)$ for all $m \in M$.

Example 3.6. (i) The set-valued mapping $T : M \to P^*(N)$ defined by $T(r) = \{0\}$, is a set-valued homomorphism.

(ii) Let $T : M \to P^*(M/A)$ be a set-valued mapping where $T(m) = \{m + A\}$ for all $m \in M$ and A is an R-submodule of M. Then T is a set-valued homomorphism.

(iii) Let θ be a complete congruence relation in M, i.e., $[x]_\theta + [y]_\theta = [x + y]_\theta$ and $r[x]_\theta = [rx]_\theta$ for all x, $y \in M$ and $r \in R$. Define $T : M \to P^*(M)$ by $T(x) = [x]_\theta$, then T is a set-valued homomorphism.

(iv) Let $f : M \to N$ be a module homomorphism and B be a non-empty subset of N. Then the set-valued mapping $T : M \to P^*(N)$ defined by $T(r) = \{f(r)\}$, is a set-valued homomorphism. It is called a set-valued homomorphism induced by a module homomorphism.

(v) Let $T : M \rightarrow P^*(N)$ be a set-valued mapping where $M = Z$ and $N = Z\times Z$ and $T(r) = \{(0, r)\}$, for all $r \in Z$, then T is a set-valued homomorphism.

The following lemmas have been proved in [12].

Lemma 3.7. Let $A\in P^*(N)$ be an R-submodule of N and $T : M \rightarrow P^*(N)$ be a set-valued homomorphism and $L_T(A)$ and $U_T(A)$ be non-empty, then the both of them are R-submodules of M.

Lemma 3.8. Assume M, N, Q be R-modules and $T : M \rightarrow P^*(N)$ be a set-valued homomorphism and $f : Q \rightarrow M$ be a module homomorphism, then T of is a set-valued homomorphism from $Q \rightarrow P^*(N)$ such that $U_{Tof}(B) = f^{-1}(U_T(B))$ and $L_{Tof}(B) = f^{-1}(L_T(B))$ for all $B\in P^*(N)$.

Lemma 3.9. Assume M, N, Q be R-modules and $T : M \rightarrow P^*(N)$ be a set-valued homomorphism and $f : N \rightarrow Q$ be a module homomorphism, then T_f is a set-valued homomorphism from $M \rightarrow P^*(Q)$ defined by $T_f(m) = f(T(m))$ such that $L_{T_f}(A) = L_T(f^{-1}(A))$ and $U_{Tf}(A) = U_T(f^{-1}(A))$ for all $A\in P^*(Q)$.

Theorem 3.10. Let M and N be two R-modules and $f : M \rightarrow N$ be an isomorphism and $T_2 : N \rightarrow P^*(N)$ be a set-valued homomorphism. If $T_1(m) = \{u\in M \mid f(u) \in T_2(f(m))\}$ for all $m\in M$, then T_1 is a set-valued homomorphism from M to $P^*(M)$.

Proof. First, we show that T_1 is a well-defined mapping. Suppose $m_1 = m_2$, we have

$y_1\in T_1(m_1) \Leftrightarrow f(y_1) \in T_2(f(m_1)) = T_2(f(m_2))$

$\Leftrightarrow y_1\in T_1(m_2)$.

Then $T_1(m_1) = T_1(m_2)$. Now we show that $T_1(m_1 + m_2) = T_1(m_1) + T_1(m_2)$.

Suppose $y \in T_1(m_1 + m_2)$, then

$f(y) \in T_2(f(m_1 + m_2)) = T_2(f(m_1) + f(m_2)) = T_2(f(m_1)) + T_2(f(m_2))$.

Hence there exist $a \in T_2(f((m_1))$ and $b \in T_2(f((m_2))$ such that $f(y) = a + b$.

Since f is onto, then there exist $d, c \in M$ such that $f(c) = a, f(d) = b$. On the other hand, we have $f(c) \in T_2(f(m_1))$, then $c \in T_1(m_1)$ and also $f(d) \in T_2(f(m_2))$. Therefore $d \in T_1(m_2)$ and $f(y) = a + b = f(c) + f(d) = f(c + d)$.

Since f is one to one, it implies $y = c + d$. So $y \in T_1(m_1) + T_1(m_2)$. It follows

$T_1(m_1 + m_2) \subseteq T_1(m_1) + T_1(m_2)$.

Conversely, assume that $y \in T_1(m_1) + T_1(m_2)$, then there are $a \in T_1(m_1)$, $b\in T_1(m_2)$ such that $y = a + b$. Hence

$f(y) = f(a) + f(b) = f(a + b) \in T_2(f(m_1)) + T_2(f(m_2)) = T_2(f(m_1 + m_2))$

$\Rightarrow y\in T_1(m_1 + m_2)$.

So $T_1(m_1) + T_1(m_2) \subseteq T_1(m_1 + m_2)$. Also suppose $r \in R$ and $m \in M$, then

$T_1(rm) = \{u \in M \mid f(u) \in T_2(f(rm))\}$

$= \{u \in M \mid f(u) \in T_2(rf(m))\}$

$= \{u \in M \mid f(u) \in rT_2(f(m))\}$.

Suppose $u \in T_1(m)$ and $r \in R$. By definition, $f(u) \in T_2(f(m))$, therefore $f(ru) = rf(u)\in T_2(f(rm))$. It implies that $ru\in T_1(rm)$ which shows $rT_1(m) \subseteq T_1(rm)$. Now suppose that $f(u) \in T_2(f(rm)) = rT_2(f(m))$. So there is a $t \in T_1(m)$ such that $u = rt\in rT_1(m)$. It yields that $T_1(rm) \subseteq rT_1(m)$. Hence $T_1(rm) = rT_1(m)$.

Theorem 3.11. Let M and N be two R-modules and $f : M \rightarrow N$ be an isomorphism and let $T_2 : N \rightarrow P^*(N)$ be a set-valued homomorphism. If $T_1(m) = \{u \in M \mid f(u) \in T_2(f(m))\}$ for all $m \in M$, and A is a non-empty subset of N, then

(1) $f(L_{T_1}(A)) = L_{T_2}(f(A))$;

(2) $f(U_{T_1}(A)) = U_{T_2}(f(A))$.

Proof. (1). If $y \in f(L_{T_1}(A))$, then there exists $m \in L_{T_1}(A)$ such that $y = f(m)$. But if $m \in L_{T_1}(A)(A)$, then $T_1(m) \subseteq A$. Now, let $w \in T_2(f(m))$, since f is onto, there exists $z \in M$ such that $w = f(z)$. So,

$w = f(z) \in T_2(f(m)) \Rightarrow z \in T_1(m) \subseteq A$

$\Rightarrow w = f(z) \in f(A)$

$\Rightarrow T_2(f(m)) \subseteq f(A)$

$\Rightarrow y\in L_{T_2}(f(A))$.

Therefore $f(L_{T_1}(A)) \subseteq L_{T_2}(f(A))$.

Conversely, if $y\in L_{T_2}(f(A))$; then $T_2(y) \subseteq f(A)$. On the other hand, f is onto, then there is $m \in M$ such that $y = f(m)$. Hence, we have $T_2(f(m)) \subseteq f(A)$.

Let $u \in T_1(m)$, then $f(u) \in f(A)$, therefore there exists $a \in A$ such that $f(u) = f(a)$. But f is one to one, so $u = a$. Hence we have

$u\in A \Rightarrow T_1(m) \subseteq A \Rightarrow m \in L_{T_1}(A) \Rightarrow y = f(m) \in f(L_{T_1}(A))$.

Hence $L_{T_2}(f(A)) \subseteq f(L_{T_1}(A))$.

(2). If $y \in f(U_{T_1}(A))$, then there exists $m \in U_{T_1}(A)$ such that $y = f(m)$.

But if $y \in U_{T_1}(A)$, then $T_1(m) \cap A \neq \emptyset$. Let $a \in T_1(m) \cap A$. Therefore

$$f(a) \in T_2(f(m)) \cap f(A) \Rightarrow T_2(f(m)) \cap f(A) \neq \emptyset$$
$$\Rightarrow f(m) \in U_{T_2}(f(A))$$
$$\Rightarrow y \in U_{T_2}(f(A))$$

It means that $f\left(U_{T_1}(A)\right) \subseteq U_{T_2}(f(A))$.Conversely, if $y \in U_{T_2}(f(A))$. Since f is onto, then there exists $m \in M$ such that $y = f(m)$ and $T_2(y) \cap f(A) \neq \emptyset$.

So, we have $T_2(f(m)) \cap f(A) \neq \emptyset$. Hence there is a $z \in T_2(f(m)) \cap f(A)$. It means that there exists $a \in A$ such that $z = f(a) \in T_2(f(m))$. Therefore $a \in T_1(m) \cap A$. It obtains that m ∈ UT1 (A). Hence $y = f(m) \in f\left(U_{T_1}(A)\right)$. It follows that $U_{T_2}(f(A)) \subseteq f\left(U_{T_1}(A)\right)$.

Finally, the following corollaries which have been proved [14, 24] are a special case of Theorem 3.10, Theorem 3.11 and Lemma 3.7.

Corollary 3.12. Let M and N be two R-modules and $f : M \rightarrow N$ be an isomorphism and let $T_2 : N \rightarrow P^*(N)$ be a set-valued homomorphism. If $T_1(m) = \{u \in M \mid f(u) \in T_2(f(m))\}$ for all $m \in M$, and A is a non-empty subset of N, then the following hold:

(1) $L_{T_1}(A)$ is an R-submodule of M (prime, primary) if and only if $L_{T_2}(A)$ $(f(A))$ is an R-submodule(prime, Primary) of N;

(2) $U_{T_1}(A)$ is an R-submodule (prime, primary) of M if and only if U_{T_2} $(f(A))$ is an R-submodule(prime, primary) of N.

Corollary 3.13. Let M and$N$ be two R-modules and $f : M \rightarrow N$ be an isomorphism. Let θ_2 be a congruence relation on N and Aa subset of M.

Let

$\theta_1 = \{(m_1, m_2) \in M \times M \mid (f(m_1), f(m_2)) \in \theta_2\}$

if θ_2 is complete congruence relation, then

(1)$f(\underline{Apr}_{\theta_1}(A)) = \underline{Apr}_{\theta_2}(f(A))$;

(2) $f\left(\overline{Apr}_{\theta_1}(A)\right) = \overline{Apr}_{\theta_2}(f(A))$.

Corollary 3.14. Let M and N be two R-modules and $f : M \rightarrow N$ be an isomorphism. Let θ_2 be a complete congruence relation on N and Aa subset of M. Let

$\theta_1 = \{(m_1, m_2) \in M \times M \mid (f(m_1), f(m_2)) \in \theta_2\}$

then $\underline{Apr}_{\theta_1}(A)$is an R-submodule of Miff $\underline{Apr}_{\theta_2}(f(A))$ is an R-submodule12 of N .

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