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A Note on Modified Control Limits for Autoregressive Process

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ABSTRACT

In this his paper we aim to study the application of six-sigma control limits in autoregressive data. We investigate auto-correlated data in traditional control limits and present modified control limits which guarantee the certain inferences and reliable consequences. We have evaluated the behavior of traditional and modified control limits for independent and serially correlated data. We use average run length criterion to discover a shift in the mean of process and to compare the efficiency of traditional and newly presented control limits.
Keywords: SPC, Uncertain Process, Autoregressive.

1. INTRODUCTION

In traditional statistical process control (SPC) it is assumed that the observations are independently and normally distributed, but independence of data may not be realized practically. We aim to use Average Run Length (ARL) to determine the impression of dependency on the efficiency of traditional control limits for the mean. ARL criterion has been of interest in the literature of SPC. [3].

Suppose X_t denotes independent observations of a continuous quality characteristics following normal distribution with mean μ_t and variance σ_X^2. We assume that a harmony causes a shift in the mean at an unknown time T,

$$\mu_t = \begin{cases} \mu & ,t < T, \\ \mu + \delta\sigma_X & ,t \geq T. \end{cases}$$

The six-sigma control limits are the form of $\mu \pm 3\sigma_X$, [5]. After a shift in the mean, an observation may not fall between lower and upper control limits (LCL and UCL). We denote the probability of falling an observation within control limits by $\mathrm{P}(\delta) = \Phi(\delta + 3) - \Phi(\delta - 3)$, where Φ is the cumulative distribution function (CDF) of the standard normal distribution.

Suppose $\mathcal{N}$ be the number of observations until the first out-of-control one; $\mathcal{N}$ follows from geometric distribution with parameter $\mathrm{P}(\delta)$ and The ARL is the average of $\mathcal{N}$, i.e.: $\mathrm{ARL}(\delta) = 1/(1 - \mathrm{P}(\delta))$.

The remaining of this paper is organized as follows. In section 2 we provide some basic notions of auto-regressive process, complemented by some examples. In section 3, modified control limits are presented and some conclusions are given in section 4.

2. Autoregressive Data

Suppose Y_t denotes the AR(1) time series, i.e.:

$$Y_t - \mu = \theta(Y_{t-1} - \mu) + \varepsilon_t,$$

where ε_ts are pair-wise independently and identically random variables, with mean 0 and variance σ_ε^2 ; it is easy to prove that $\sigma_Y^2 = \sigma_\varepsilon^2 / (1 - \theta)$. Let $Y_1, \cdots, Y_n$ are observations of AR(1) process. In traditional SPC, central line of control chart is determined by sample mean $\bar{Y} = (1/n)\sum_{i=1}^{n} Y_i$ and $\overline{MR}/1.13$ is used as standard deviation, where $MR_i = |Y_i - Y_{i-1}|$ and $\overline{MR} = (1/n)\sum_{i=1}^{n} MR_i$, [5]. To evaluate the behavior of AR(1) process in traditional control chart, we simulate a sequence of size 1000 from AR(1) with $\mu = 0$ and $\sigma_\varepsilon = 1$ and five values of $\theta = -0.8, -0.4, 0, 0.5, 0.9$. This simulation is repeated 10,000 times and the results is given in TABLE I. The values of $E[\overline{MR}/1.13]$ and $ARL(\delta)$ are the average of the values generated in every step of simulation process.

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TABLE I
EVALUATING AUTOREGRESSIVE DATA IN TRADITIONAL SIX-SIGMA CONTROL LIMITS

θ	σ_Y	$E[\overline{MR}/1.13]$	LCL	UCL	ARL(0)	ARL(1)
-0.8	2.7778	2.6448	-7.9344	7.9344	32582.98	1186.43
-0.4	1.1905	1.2241	-3.6723	3.6723	3451.85	219.18
0	1.0000	1.0012	-3.0036	3.0036	353.93	61.22
0.5	1.3334	0.7985	-2.3955	2.3955	41.99	13.52
0.9	2.2942	0.6998	-2.0994	2.0994	9.23	3.46

For negative θs, the width of control limits is so larger than of it for negative θs. The values of two last columns show puzzling and unexpected number of observations until first out-of-control case. Vice versa, for positive θs, nearly all of observations are recognized as out-of-control. Therefore, using traditional control limits to evaluating serially correlated data may cause serious problems in the interpretation of SPC design. A suggestion to overpowering these problems is to construct LCL and UCL based on σ_Y instead of $\overline{MR}/1.13$.

3. Modified Control Limits

In this section, we present modified control limits based on longevous variation of process. Subsequently, we evaluate the new limits using ARL criterion.

Suppose $\mathrm{Y_t}$ is an AR(1) process with parameter θ, i.e:

$$(Y_t - \mu_t) = \theta(Y_{t-1} - \mu_{t-1}) + \varepsilon_t, \tag{1}$$

where $|\theta| < 1$ and ε_t is an i.i.d normally distributed sequence with mean 0 and variance σ_ε^2. The value of Y_t is exactly determined by Y_{t-1} and so, the AR(1) model is known as Morkov process, [2]. Let $Y_1, \cdots, Y_n$ are the realizations of AR(1) process; similar to i.i.d data, We estimate the central control with $\bar{Y} = (1/n)\sum_{i=1}^{n} Y_i$ and LCL and UCL with $\bar{Y} \pm 3\hat{\sigma}_Y$. In practice, the values of θ, σ_Y and σ_ε can be estimated by the following equalities:

$$\begin{aligned} r &= \frac{\sum_{t=1}^{n-1}(Y_t - \bar{Y})(Y_{t-1} - \bar{Y})}{\sum_{t=1}^{n}(Y_t - \bar{Y})^2} \\ &= -\frac{\hat{\theta}}{1+\hat{\theta}^2} \end{aligned}$$

and

$$\begin{aligned} \hat{\sigma}_Y &= \frac{1}{n}\sum_{t=1}^{n}(Y_t - \bar{Y})^2 \\ &= \left(1 + \hat{\theta}^2\right)\hat{\sigma}_\varepsilon \end{aligned}$$

where r denotes the autocorrelation estimator [2]. In the remaining of this section, we assume values of θ and σ_Y are known; otherwise we can estimate the unknown parameters from a realization of process which is in access in practical situations. Now, suppose a phenomenon shift the mean from μ to $\mu + \delta\sigma_Y$ at an unknown instant T. Therefore, we rewrite the model (1) as:

$$(Y_t - \mu) - \delta\sigma_Y = \theta(Y_{t-1} - \mu) - \theta\delta\sigma_Y + \varepsilon_t.$$

For $Y_{t-1} = s$, the value of Y_t is equal to

$$v = \theta s + (1-\theta)\mu + (1-\theta)\delta\sigma_Y + \varepsilon_t.$$

The run length equals one if the value of v falls out of control limits. Otherwise, ARL comes from one plus run length of AR(1) process with starting point v. We denote the ARL of AR(1) process by $R_\theta(\delta, s)$ that s is the beginning state of process.

$$R_\theta(\delta, s) = 1 + \int_{\mu-3\sigma_Y}^{\mu+3\sigma_Y} R_\theta(\delta, v)\, \varphi\big(v - \theta s - (1-\theta)(\mu + \delta\sigma_Y)\big) dv,$$

where φ is the probability density function of normal random variable ε. Concerning Fredholm Integral, [1], there is a recurrent sequence of functions $\{R_0, R_1, R_2, \cdots\}$ so that

$$\lim_{k\to\infty} R_k(\delta, s) = R_\theta(\delta, s)$$

and

$$R_k(\delta, s) = 1 + E\left[R_{k-1}(\delta, V) I_{(\mu - 3\sigma_Y, \mu + 3\sigma_Y)}(V)\right]$$

where V is a random variable normally distributed with mean $\eta = \theta s + (1 - \theta)(\mu + \delta\sigma_Y)$ and variance 1. E and I denotes statistical expectation and identifier function respectively. R_0 is an arbitrary continuous function on $(\mu - 3\sigma_Y, \mu + 3\sigma_Y)$; for example, $R_0 = 1$ can be a suitable choice. Solving or simulating Fredholm integral to reach the value of $R_\theta(\delta, s)$ is complicated. To get to this goal, we choose an abridged way by simulating AR(1) process until one of v falls outside of control limits. According to definition of ARL criterion, The number of v's up to the first uncertain one will set as the ARL value. For $s = 0$, we repeat the simulation 10,000 times per every θ; relevant values of $R_\theta(\delta, 0)$ are the mean of ARL values generated in every step of simulation that is presented in TABLE II. Compatible with i.i.d case, we denote the quantities of $R_\theta(\delta, 0)$ by $ARL(\delta)$. Despite of TABLE I, the ARL values in TABLE II show logical and desired values for autocorrelated data. So, using longevous variation instead of pseudo variation, improves traditional control limits to take advantage in AR(1) data.

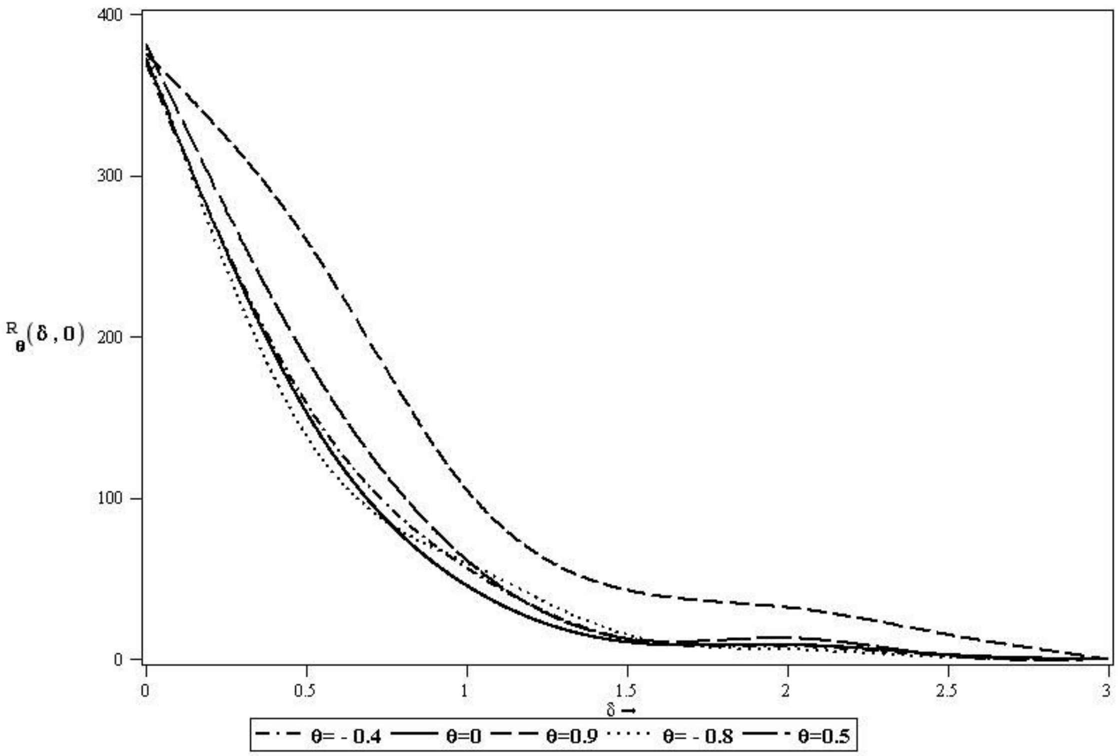

Fig. 1. Curve of $R_\theta(\delta, 0)$ for modified control limits.

For $\theta = 0$, the values of $R_\theta(\delta, 0)$ is the same as $ARL(\delta)$ values depicted in TABLE I; We know that for $\theta = 0$, AR(1) process generates i.i.d normally distributed random variables [2]. Comparing ARL values of different θ shows that for positive values of θ, the modified control limits has low sensitivity to explore the shift in the mean or AR(1) process. ARL values for $\theta = 0.9$ are too large comparative to others. For $\theta = -0.8$ and 0, the inequality $R_{\theta=-0.8}(\delta = 0.5, 0) < R_{\theta=0}(\delta = 0.5, 0)$ indicates that for δ close to 0.5, modified control limits has high sensitivity to detecting changes in the mean of process. For other values of θ, except for $\theta = 0.9$, modified control limits is powerful as well as Traditional control limits in discovering the behavior of uncertain process.

TABLE II

EVALUATING AUTOREGRESSIVE DATA IN MODIFIED SIX-SIGMA CONTROL LIMITS

θ	ARL(0)	ARL(0.5)	ARL(1)	ARL(1.5)	ARL(2)	ARL(2.5)
-0.8	378.64	139.03	59.66	15.14	6.09	1.13
-0.4	368.95	159.29	56.82	12.59	8.05	2.01
0	372.64	153.61	45.76	10.54	8.98	2.76
0.5	381.38	187.08	61.77	12.68	13.09	1.99
0.9	375.32	260.45	105.49	42.76	32.03	15.22

We plot the interpolated curves of $R_\theta(\delta, 0)$ in Fig. 1; all curves have decreasing trend and tend to zero. Except the similar origin with other curves, $R_{\theta=-0.9}(\delta, 0)$ is very elevated and more different from independent case curve. In the region close to $\delta = 0.5$ ARL for $\theta = -0.8$ is less than the others. Therefore, we conclude that in detecting small changes of mean; the more negative θ, the more efficient the modified control limits will be.

4. Conclusion

In this paper, we have examined the behavior of independent and serially correlated data in traditional and modified control limits. Using Traditional control limits may cause miss understandings of production process. We have adopted traditional control limits using longevous variation instead of pseudo variation. In conclusion, if one is interested in checking auto-correlated process for changes in the mean, for negative and not too large positive θ modified control limits are more efficient than traditional control limits.

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