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On Entropy and Informative Distance in a Time Interval

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ABSTRACT

Measures of uncertainty in past and residual lifetime distributions have been proposed in the information-theoretic literature. Di Crescenzo and Longobardi [4] and Ebrahimi and Pellerey [7] introduced past and residual entropy functions. Misagh and Yari [11] and Sunoj et al. [15] proposed the interval entropy functions as useful dynamic measures of uncertainty for two sided truncated random variables. In this paper, we aim to study the use of interval entropy in characterization of distribution functions. Furthermore, we presented an informative distance which are based on a time interval and are more general than the well-known Kullback-Leibler divergence measure. The newly proposed measures are consistent in that they are valid in both past and residual lifetimes.
KEYWORDS: Time interval, Entropy, Characterization, Distance.

1. INTRODUCTION

Reliability analysis is a branch of statistics which deals with death in biological organisms and failure in mechanical systems. There are several uncertainty measures that play a central role in understanding and describing reliability. Recently the residual entropy was considered in Ebrahimi and Pellerey [7] which basically measures the expected uncertainty contained in remaining lifetime of a system. The residual entropy has been used to measure the wear and tear of components and to characterize, classify and order distributions of lifetimes Belzunce *et al* [1] and Ebrahimi [5]. The notion of past entropy was introduced in Di Crescenzo and Longobardi [3] which can be viewed as the entropy of the inactivity time of a system.

Sunoj et al. [15] has studied the use of information measures for doubly truncated random variables which plays a significant role in studying the various aspects of a system when it fails between two time points.

Ebrahimi and Kirmani [6] introduced the residual discrimination measure and studied the minimum discrimination principle. Di Crescenzo and Longobardi [4] have considered the past discrepancy measure and presented a characterization of the proportional reversed hazards model. Furthermore, the use of information measures for doubly truncated random variables was explored in Misagh Yari [10, 11]. In this paper, continuing their work, we propose a new measure of discrepancy between two doubly truncated life distributions. The remaining of this paper is organized as follows. In section 2, some results including uniqueness of interval entropy are presented. Section 3 is devoted to definitions of dynamic measures of discrimination, including residual and past lifetimes and also the notion of interval discrimination measure is introduced. In section 4 we present some results and properties on newly presented measures.

2. Uncertainty in a time interval

Let X be a non-negative absolutely continuous random variable describing a system failure time. We denote the probability density function of X as $f(x)$, the failure distribution as $F(x) = P(X \le x)$ and the survival function as $\bar{F}(x) = P(X > x)$. The Shannon [14] information measure of uncertainty is defined as:

$$H(X) = -E(\log f(X)) = \int_0^\infty f(x) \log f(x)\, dx , \tag{2.1}$$

where log denotes the natural logarithm. The entropy (2.1) is not scale invariant because $H(cX) = \log|c| + H(X)$, but it is translation invariant, so that $H(c + X) = H(X)$ for some constant c. The latter property can be interpreted as the shift independence of Shannon information.

We can rewrite the Shannon entropy as:

$$\begin{aligned} H(X) &= 1 - E[r(X)] \\ &= 1 - E[\tau(X)]. \end{aligned}$$

We recall that hazard rate (HR) and reversed hazard rate (RHR) of random lifetime X is defined as $r_F(t) = f(t)/\bar{F}(t)$ and $\tau_F(t) = f(t)/F(t)$ respectively. The HR and RHR have been used in the literature of reliability in both theory and applications of them.

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Ebrahimi and Pellerey [7] considered the residual entropy of the non-negative continuous random variable X at time t as:

$$H_X(t) = \int_t^\infty \frac{f(x)}{\bar{F}(t)} \log \frac{f(x)}{\bar{F}(t)} dx. \quad (2.2)$$

The entropy (2.2), in fact, measures the uncertainty represented by residual lifetime distribution X at time $t > 0$. The residual entropy has been used to measure the wear and tear of systems and to characterize, classify and order distributions of lifetimes. See Belzunce et al. [1] and Ebrahimi and Kirmani [6]. Di Crescenzo and Longobardi [3] considered the past entropy and motivated its use in real-life situations. They also discussed its relationship with the residual entropy. Formally, the past entropy of X at time t is defined as follows:

$$\bar{H}_X(t) = \int_0^t \frac{f(x)}{F(t)} \log \frac{f(x)}{F(t)} dx. \quad (2.3)$$

The entropy (2.3) measures the uncertainty about a system which is observed only at deterministic inspection times, and is found to be down at time t, then the uncertainty relies on which instant in $(0, t)$ it has failed.

Now Recall that the probability density function of $(X|t_1 < X < t_2)$ for all $0 < t_1 < t_2$ is given by $f(x)/\left(F(t_2) - F(t_1)\right)$. Sunoj et al. [15] introduced the notion of interval entropy of X in the interval (t_1, t_2) as the uncertainty contained in $(X|t_1 < X < t_2)$ which is denoted by

$$IH(t_1, t_2) = -\int_{t_1}^{t_2} \frac{f(x)}{F(t_2)-F(t_1)} \log \frac{f(x)}{F(t_2)-F(t_1)} dx. \quad (2.4)$$

We can rewrite the interval entropy as:

$$\begin{aligned} H(t_1, t_2) = {} & 1 - \frac{1}{F(t_2)-F(t_1)} \int_{t_1}^{t_2} f(x) \log r(x)\, dx \\ & + \frac{1}{F(t_2)-F(t_1)} \{\bar{F}(t_2) \log \bar{F}(t_2) - F(t_1) \log F(t_1) \\ & + [F(t_2) - F(t_1)] \log[F(t_2) - F(t_1)]\}. \end{aligned}$$

Note that interval entropy can be negative and also it can be $-\infty$ or $+\infty$. Given that a system has survived up to time t_1 , and has been found to be down at time t_2 , $IH(t_1, t_2)$ measures the uncertainty about its lifetimes between t_1 and t_2 . Misagh and Yari [11] introduced a shift-dependent version of $IH(t_1, t_2)$.The entropy (2.4) has been used to characterize and ordering random lifetime distributions. See Misagh and Yari [10] and Sunoj et al. [15].

Example 2.1. Suppose X and Y be random lifetimes of two systems with joint density function $f(x, y) = \frac{1}{4}, 0 < x < 2, 0 < y < 4 - 2x$. The marginal densities of X and Y are $f(x) = \frac{1}{2}(2 - x)$, $0 < x < 2$, and $g(y) = \frac{1}{8}(4 - y), 0 < y < 4$, respectively. Simply, $H(X) = \frac{1}{2}$ and $H(Y) = \frac{1}{2} + \log 2$, but because X and Y, belong to different domains, the use of differential entropy to compare X and Y informatively is not interpretable. Interval entropy of X and Y in the interval $\left(1, \frac{3}{2}\right)$ are $IH_X\left(1, \frac{3}{2}\right) = -0.712$ and $IH_Y\left(1, \frac{3}{2}\right) = -0.624$ respectively. Therefore, in interval $\left(1, \frac{3}{2}\right)$ the system X is less informative than Y.

Hereafter, we study the connection of interval entropy with uniform distribution. Suppose X denotes random lifetime of a component with uniform distribution in the interval (α, β), then we have $IH_X(t_1, t_2) = \log(t_2 - t_1)$. The expected uncertainty contained in uniform distribution is related to the distance between two points of time considered. For example in $U(1,11)$ we can conclude $IH_X(1,3) = IH_X(7,9)$. Generally, for subsets A and B of $(1,11)$ so that $\|A\| = k\|B\|$, we have $IH_X(B) - IH_X(A) = \log \frac{1}{k}$, where $\|A\|$ denotes the length of interval A and k is a positive real value, but it cannot be concluded that $IH_X(B) = kIH_X(A)$.

Proposition 2.1. Suppose X is a absolutely continues random variable, then for $0 < t_1 < t_2$, $IH_X(t_1, t_2) \le \log(t_2 - t_1)$.
Proof. From (2.4), it is seen that the entropy of a random variable uniformly distributed on (t_1, t_2) equals to $\log(t_2 - t_1)$. Considering the maximum entropy principal (See Cover and Tomas [2]), the uniform distribution maximizes uncertainty under the constraint in which the probability density function is focused on the finite interval (t_1, t_2).

Remark 2.1. For fixed t_1, suppose $IH_X(t_1, t_2) = k$ which k is a positive constant; then for $t_2 < \log(k - t_1)$, we get $IH_X(t_1, t_2) > \log(t_2 - t_1)$ that conflict proposition 2.1. So, in all points of time, the interval entropy of a random variable cannot be constant.

Remark 2.2. Note that $\log(t_2 - t_1)$ is not always reliance; for example, suppose X be an exponential random variable with mean$1/\theta$, then we have $IH_X(t_1,t_2) = \theta E(X|t_1 < X < t_2) - \theta t_1 - \log\frac{f(t_1)}{F(t_2)-F(t_1)}$ whereas $\lim_{t_1\to 0, t_2\to\infty} IH_X(t_1,t_2) = 1 - \log\theta$ but $\lim_{t_1\to 0, t_2\to\infty} \log(t_2 - t_1) = \infty$.

The general characterization problem is to obtain when the interval entropy uniquely determines the distribution function. The following proposition attempts to solve this problem. We first give definition of general failure rate (GFR) functions extracted from Navarro and Ruiz [12].

Definition 2.1. The GFRs of a random variable X having density function $f(x)$ and cumulative distribution function $F(x)$ are given by $h_1^X(t_1,t_2) = \frac{f(t_1)}{F(t_2)-F(t_1)}$ and $h_2^X(t_1,t_2) = \frac{f(t_2)}{F(t_2)-F(t_1)}$.

Remark 2.3. GFR functions determine distribution function uniquely. See Navarro and Ruiz [12].

Proposition 2.2. Let X be a non-negative and continuous random variable, and assume $IH(t_1,t_2)$ be increasing with respect to t_1 and decreasing with respect to t_2, then $IH(t_1,t_2)$ uniquely determines $F(x)$.
Proof. By differentiating $IH(t_1,t_2)$ with respect to t_j , we have

$$\frac{\partial IH_X(t_1,t_2)}{\partial t_1} = h_1(t_1,t_2)[IH_X(t_1,t_2) - 1 + \log h_1(t_1,t_2)],$$

and

$$\frac{\partial IH_X(t_1,t_2)}{\partial t_2} = -h_2(t_1,t_2)[IH_X(t_1,t_2) - 1 + \log h_2(t_1,t_2)],$$

thus, for fixed t_1 and arbitrary t_2, $h_1(t_1,t_2)$ is a positive solution of the following equation

$$g(x_{t_2}) = x_{t_2}[IH_X(t_1,t_2) - 1 + \log x_{t_2}] - \frac{\partial\, IH(t_1,t_2)}{\partial\, t_1} = 0. \quad (2.5)$$

Similarly, for fixed t_2 and arbitrary t_1, we have $h_2(t_1,t_2)$ as a positive solution of the following equation

$$\gamma(y_{t_1}) = y_{t_1}[IH_X(t_1,t_2) - 1 + \log y_{t_1}] + \frac{\partial\, IH(t_1,t_2)}{\partial\, t_2} = 0. \quad (2.6)$$

By differentiating g and γ with respect to x_{t_2} and y_{t_1}, we get $\frac{\partial\, g(x_{t_2})}{\partial\, x_{t_2}} = \log x_{t_2} + IH(t_1,t_2)$, and $\frac{\partial\, \gamma(y_{t_1})}{\partial\, y_{t_1}} = \log y_{t_1} + IH(t_1,t_2)$. Furthermore, second-order derivatives of g and γ with respect to x_{t_2} and $y_{t_1}$are $\frac{1}{x_t} > 0$ and $\frac{1}{y_t} > 0$ respectively. Then the functions g and γ are minimized at points $x_{t_2} = e^{-IH(t_1,t_2)}$ and $y_{t_1} = e^{-IH(t_1,t_2)}$ respectively. In addition, $g(0) = -\frac{\partial\, IH(t_1,t_2)}{\partial\, t_1} < 0$, $g(\infty) = \infty$ and $\gamma(0) = -\frac{\partial\, IH(t_1,t_2)}{\partial\, t_1} < 0$, $\gamma(\infty) = \infty$. So, both functions g and γ first decrease and then increase with respect to x_{t_2} and y_{t_1} respectively. which conclude that equations (2.5) and (2.6) has unique roots $h_1(t_1,t_2)$ and $h_2(t_1,t_2)$ respectively. Now, $IH(t_1,t_2)$ uniquely determines GFRs and by virtue of Remark 2.3, the distribution function.

3. Informative distance in a time interval

In this section, we review some basic definitions and facts for measures of discrimination between two residual and past lifetime distributions. We introduce a measure of discrepancy between two random variables at an interval of time.

Let X and Y be to two non-negative absolutely continues random variable describing times to failure of two systems. We denote the probability density functions of X and Y as $f(x)$ and$g(y)$, failure distributions as $F(x) = P(X \le x)$ and $G(y) = G(Y \le y)$ and the survival functions as $\bar{F}(x) = P(X > x)$ and $\bar{G}(y) = G(Y > y)$ respectively, with$F(0) = G(0) = 1$. Kullback-Leibler [8] informative distance between F and G is defined by

$$I_{X,Y} = \int_0^\infty f(x) \log\frac{f(x)}{g(x)} dx. \quad (3.1)$$

where log denotes natural logarithm. Distance (3.1) is known as relative entropy and it is shift and scale invariant. However it is not metric, since symmetrization and triangle inequality does not hold. The application of $I_{X,Y}$ as a distance in residual and past lifetimes has increasingly studied in recent years. In particular, Ebrahimi and Kirmani [6] considered the residual Kullback-Leibler discrimination information of non-negative lifetimes of the systems X and Y at time t as:

$$I_{X,Y}(t) = \int_t^\infty \frac{f(x)}{\bar{F}(t)} \log \frac{f(x)/\bar{F}(t)}{g(x)/\bar{G}(t)} dx. \quad (3.2)$$

Given that both systems have survived up to time t, $I_{X,Y}(t)$ identifies with the relative entropy of remaining lifetimes $(X|X > t)$ and$(Y|Y > t)$. Furthermore, the Kullback-Leibler distance for two past lifetimes was studied in Di Crescenzo and Longobardi [4] which is dual to (3.2) in the sense that it is an informative distance between past lifetimes $(X|X < t)$ and $(Y|Y < t)$. Formally, the past Kullback-Leibler distance of non-negative random lifetimes of the systems X and Y at time t is defined as:

$$\bar{I}_{X,Y}(t) = \int_0^t \frac{f(x)}{F(t)} \log \frac{f(x)/F(t)}{g(x)/G(t)} dx. \quad (3.3)$$

Given that at time t, both systems have been found to be down, $\bar{I}_{X,Y}(t)$ measures the informative distance between their past lives.

Along a similar line, we define a new discrepancy measure that completes studying information distance between random lifetimes X and Y.

Definition 3.1. The interval distance between random lifetimes X and Y at interval (t_1, t_2) is the Kullback-Leibler discrimination measure between the truncated lives $(X|t_1 < X < t_2)$ and $(Y|t_1 < Y < t_2)$:

$$ID_{X,Y}(t_1, t_2) = \int_{t_1}^{t_2} \frac{f(x)}{F(t_2)-F(t_1)} \log \frac{f(x)/[F(t_2)-F(t_1)]}{g(x)/[G(t_2)-G(t_1)]} dx. \quad (3.4)$$

Remark 3.1. Clearly $ID_{X,Y}(0,t) = \bar{I}_{X,Y}(t)$, $ID_{X,Y}(t,\infty) = I_{X,Y}(t)$ and $ID_{X,Y}(0,\infty) = I_{X,Y}$.

Given that both systems X and Y have survived up to time t_1, and have seen to be down at time t_2, $ID_{X,Y}(t_1,t_2)$ measures the discrepancy between their unknown failure times in the interval (t_1, t_2). $ID_{X,Y}(t_1,t_2)$ satisfies all properties of Kullback-Leibler discrimination measure and can be rewritten as:

$$ID_{X,Y}(t_1, t_2) = -\int_{t_1}^{t_2} \frac{f(x)}{F(t_2)-F(t_1)} \log \frac{g(x)}{G(t_2)-G(t_1)} dx - IH_X(t_1, t_2), \quad (3.5)$$

where $IH_X(t_1, t_2)$ is the interval entropy of X in (2.4).

An alternative way of writing the distance (3.5) is the following:

$$ID_{X,Y}(t_1, t_2) = \log \frac{G(t_2)-G(t_1)}{F(t_2)-F(t_1)} + \frac{1}{F(t_2)-F(t_1)} \int_{t_1}^{t_2} f(x) \log \frac{f(x)}{g(x)} dx. \quad (3.6)$$

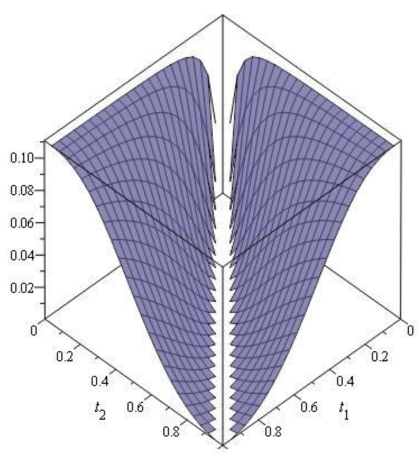

Figure 1. Informative distance between random variables in example 3.1.

The following example clarifies the effectiveness of the interval discrimination measure.

Example 3.1. Suppose X and Y be random lifetimes of two systems with common support $(0,1)$ and the density functions $f(x) = 5x^4$ and $g(y) = 3y^2$ respectively. The relative entropy between X and Y is $ID_{X,Y} = 0.111$. Suppose both systems are survived up to time t_1; if both systems are found to be down at time t_2, then the distance between their unknown failure times must be measured by the interval distance. Three dimensional plot of interval distance between X and Y is given in

Figure 1. In all areas, interval distance between X and Y is smaller than relative entropy, for instance $ID_{X,Y}(0.2,0.4) = 0.053$ and $ID_{X,Y}(0.6,0.8) = 0.012$.

In the following proposition we decompose the Kullback-Leibler discrimination function in terms of residual, past and interval discrepancy measure. The proof is straightforward.

Proposition 3.1. Let X and Y are two non-negative random lifetimes of two systems. For all$t_1 < t_2$, the Kullback-Leibler discrimination measure can be decomposed as follows:

$$I_{X,Y} = [F(t_2) - F(t_1)]ID_{X,Y}(t_1, t_2) + F(t_1)\bar{I}_{X,Y}(t_1) + \bar{F}(t_2)I_{X,Y}(t_2) + I_{U,V}(t_1, t_2), \quad (3.7)$$

where

$$I_{U,V}(t_1, t_2) = F(t_1) \log\frac{F(t_1)}{G(t_1)} + \bar{F}(t_2) \log\frac{\bar{F}(t_2)}{\bar{G}(t_2)} + [F(t_2) - F(t_1)] \log\frac{F(t_2)-F(t_1)}{G(t_2)-G(t_1)}$$

is the Kullback-Leibler distance between two trivalent discrete random variables.

The proposition 3.1 admits the following interpretation. The Kullback-Leibler discrepancy measure between random lifetimes of systems X and Y composed from four parts: (i) The discrepancy between the past lives of two systems at time t_1. (ii) The discrepancy between residual lifetimes of X and Y that have both survived up to time t_2. (iii) The discrepancy between the lifetimes of both systems in the interval (t_1, t_2). (iv) The discrepancy between two random variables which determines if the systems have been found to be failing before t_1, between t_1 and t_2 or after t_2.

4. RESULTS ON INTERVAL ENTROPY AND INFORMATIVE DISTANCE

In this section we study the properties of $ID(t_1, t_2)$ and point out certain similarities with those of $I_{X,Y}(t)$ and $\bar{I}_{X,Y}(t)$. The following proposition gives lower and upper bounds for the interval distance. We first give definition of likelihood ratio ordering.

Definition 4.1. X is said to be larger than Y in likelihood ratio ($X \overset{LR}{\geq} Y$) if $\frac{f(x)}{g(x)}$ is increasing in x over the union of the supports of X and Y.

Several results regarding the ordering in definition 4.1. was provided in Shaked and Shanthikumar [13].

Proposition 4.1. Let X and Y are random variables with common support $(0, \infty)$. Then

(i) $X \overset{LR}{\geq} Y$ implies

$$\log\frac{h_1^X(t_1,t_2)}{h_1^Y(t_1,t_2)} \leq ID_{X,Y}(t_1, t_2) \leq \log\frac{h_2^X(t_1,t_2)}{h_2^Y(t_1,t_2)}, \quad (4.1)$$

when $\frac{f(x)}{g(x)}$ is decreasing in $x > 0$, then the inequalities in (4.1) are reversed.

(ii) Decreasing $g(x)$ in $x > 0$, implies

$$\log\frac{1}{h_1^Y(t_1,t_2)} \leq ID_{X,Y}(t_1, t_2) + IH_X(t_1, t_2) \leq \log\frac{1}{h_2^Y(t_1,t_2)}, \quad (4.2)$$

for increasing $g(x)$ then the inequalities in (4.2) are reversed.

Proof. Because of increasing $\frac{f(x)}{g(x)}$ in > 0 , from (3.4), we have

$$ID_{X,Y}(t_1, t_2) \leq \int_{t_1}^{t_2} \frac{f(x)}{F(t_2)-F(t_1)} \log\frac{f(t_2)/[F(t_2)-F(t_1)]}{g(t_2)/[G(t_2)-G(t_1)]} dx = \log\frac{h_2^X(t_1,t_2)}{h_2^Y(t_1,t_2)},$$

and

$$ID_{X,Y}(t_1, t_2) \geq \int_{t_1}^{t_2} \frac{f(x)}{F(t_2)-F(t_1)} \log\frac{f(t_1)/[F(t_2)-F(t_1)]}{g(t_1)/[G(t_2)-G(t_1)]} dx = \log\frac{h_1^X(t_1,t_2)}{h_1^Y(t_1,t_2)},$$

which gives (4.1). When $\frac{f(x)}{g(x)}$ is decreasing, the proof is similar. Furthermore, for all $t_1 < x < t_2$ decreasing $g(x)$ in $x > 0$ implies$g(t_2) < g(x) < g(t_1)$, then from (3.5) we get

$$ID_{X,Y}(t_1,t_2) \leq -\log h_2^Y(t_1,t_2) - IH_X(t_1,t_2),$$

and

$$ID_{X,Y}(t_1,t_2) \geq -\log h_1^Y(t_1,t_2) - IH_X(t_1,t_2),$$

so that (4.2) holds. When $g(x)$ is increasing the proof is similar.

Remark 4.1. Consider X and Y are two non-negative random variables corresponding to weighted exponential distributions with positive rates λ and μ respectively and with common positive real weight function$\omega(\cdot)$. The densities of X and Y are $f(x) = \frac{\omega(x)e^{-\lambda x}}{h(\lambda)}$, and $g(x) = \frac{\omega(x)e^{-\mu x}}{h(\mu)}$ respectively, where $h(\cdot)$ denotes the Laplace transform of $\omega(\cdot)$ given by $h(\theta) = \int_0^\infty \omega(x)e^{-\theta x}\,dx$, $\theta > 0$, therefore, for $\lambda \neq \mu$ the interval distance between X and Y at interval (t_1,t_2) is the following

$$ID_{X,Y}(t_1,t_2) = \log\frac{G(t_2)-G(t_1)}{F(t_2)-F(t_1)} + \log\frac{h(\mu)}{h(\lambda)} - (\lambda-\mu)E(X|t_1 < X < t_2). \quad (4.3)$$

Remark 4.2. Let X be a non-negative random lifetime with density function $f(x)$ and cumulative distribution function $F(t) = P(X \leq t)$. Then the density function and cumulative distribution function for the weighted random variable X_ω associated to a positive real function $\omega(\cdot)$ are$f_\omega(x) = \frac{\omega(x)}{E(\omega(X))}f(x)$, and $F_\omega(t) = \frac{E(\omega(X)|X\leq t)}{E(\omega(X))}F(t)$, respectively, where $E(\omega(X)) = \int_0^\infty \omega(x)f(x)dx$. Then, from (3.6) we have

$$ID_{X,X_\omega}(t_1,t_2) = \log E(\omega(X)|t_1 < X < t_2) - E\big(\log(\omega(X))\big|t_1 < X < t_2\big). \quad (4.4)$$

A similar expression is available in Maya and Sunoj [8] for past life time. Due to (4.4) and from non-negativity of $ID_{X,X_\omega}(t_1,t_2)$ we have

$$\log E(\omega(X)|t_1 < X < t_2) \geq E\big(\log(\omega(X))\big|t_1 < X < t_2\big)$$

which is a direct result of Markov inequality for concave functions.

Example 4.1. For $\omega(x) = x^{n-1}$ and $h(\theta) = (n-1)!/\theta^n$ the distributions of random variables in Remark 4.1 called Erlang distributions with scale parameters λ and μ and with common shape parameter. The conditional mean of $(X|t_1 < X < t_2)$ is the following

$$E(X|t_1 < X < t_2) = \frac{1}{F(t_2)-F(t_1)}\int_{t_1}^{t_2} x\frac{x^{n-1}\lambda^n e^{-\lambda x}}{(n-1)!}dx = \frac{\gamma(n+1,\lambda t_2)-\gamma(n+1,\lambda t_1)}{\lambda(n-1)!(F(t_2)-F(t_1))}$$

where $\gamma(\alpha,x) = \int_0^x e^{-u}u^{\alpha-1}du$ is the incomplete Gamma function. From (4.3) we obtain

$$ID_{X,Y}(t_1,t_2) = \log\frac{\gamma(n,\mu t_2)-\gamma(n,\mu t_1)}{\gamma(n,\lambda t_2)-\gamma(n,\lambda t_1)} - n\ \log\frac{\lambda}{\mu} + (\lambda-\mu)\frac{\gamma(n+1,\lambda t_2)-\gamma(n+1,\lambda t_1)}{\lambda(n-1)!(F(t_2)-F(t_1))}.$$

In the following proposition, sufficient condition for $ID_{X_1,Y}(t_1,t_2)$ to be smaller than $ID_{X_2,Y}(t_1,t_2)$ is provided.

Proposition 4.2. Consider three non-negative random variables X_1 , X_2 and Y with probability density functions f_1 , f_2 and g respectively. $X_1 \overset{LR}{\geq} Y$ implies $ID_{X_1,Y}(t_1,t_2) \leq ID_{X_2,Y}(t_1,t_2)$.

Proof. From (3.5) we have

$$\begin{aligned} ID_{X_1,Y}(t_1,t_2) - ID_{X_2,Y}(t_1,t_2) &= -ID_{X_2,X_1}(t_1,t_2) + \int_{t_1}^{t_2}\left(\frac{f_1(x)}{F_1(t_2)-F_1(t_1)} - \frac{f_2(x)}{F_2(t_2)-F_2(t_1)}\right)\log\frac{f_1(x)}{g(x)}dx \\ &\leq \int_{t_1}^{t_2}\left(\frac{f_1(x)}{F_1(t_2)-F_1(t_1)} - \frac{f_2(x)}{F_2(t_2)-F_2(t_1)}\right)\log\frac{f_1(x)}{g(x)}dx \\ &\leq \log\frac{f_1(t_2)}{g(t_2)}\int_{t_1}^{t_2}\left(\frac{f_1(x)}{F_1(t_2)-F_1(t_1)} - \frac{f_2(x)}{F_2(t_2)-F_2(t_1)}\right)dx = 0, \end{aligned}$$

where the first inequality comes from the fact that $ID_{X_2,X_1}(t_1,t_2) \geq 0$ and the second one follows from the increasing $\frac{f_1(x)}{g(x)}$ in $x > 0$.

Example 4.2. Let $\{N(t), t \geq 0\}$ be a non-homogeneous Poisson process with a differentiable mean function $M(t) = E(N(t))$ such that $M(t)$ tends to ∞ as t tends to ∞ . Let R_n denote the time of the occurrence of the n-th event in such a process. R_n has the following density function $f_n(x) = \frac{(M(t))^{n-1}}{(n-1)!} f_1(x), x > 0, n = 1,2,3, \ldots$, where $f_1(x) = -\frac{d}{dx}\exp(-M(x)), x > 0$, clearly $f_n(x)/f_1(x)$ is increasing in x. It follows from proposition 3.3 that for all $m \leq n$ $ID_{X_n,X_1}(t_1,t_2) \leq ID_{X_m,X_1}(t_1,t_2)$.

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