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Green's Function and its Applications

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ABSTRACT

Green's function, a mathematical function that was introduced by George Green in 1793 to 1841.
Green's functions used for solving Ordinary and Partial Differential Equations in different dimensions and for time-dependent and time-independent problem, and also in physics and mechanics, specifically in quantum field theory, electrodynamics and statistical field theory, to refer to various types of correlation functions.
In this paper, we describe some of the applications of Green's function in sciences, to determine the importance of this function. i.e. Boundary and Initial Value problem, Wave Equation, Kirchhoff Diffusion Equation, Diffraction Theory, Helmholtz Equation and etc.
KEYWORDS: Green's Function, Boundary Value problem, Wave Equation, Kirchhoff Diffusion Equation, Diffraction Theory, Helmholtz Equation, Diffusion Equation, Laplace Equations, Poisson Equations, Bessel Equations, Sturm-Liouville Differential Equation.
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1. INTERODUCTION

George Green (14 July 1793 – 31 May 1841) was largely self-taught British mathematical physicist who wrote "*An Essay on the Application of Mathematical Analysis to the Theories of Electricity and Magnetism (Green, 1828)*". The essay introduced several important concepts, among them a theorem similar to the modern Green's theorem, the idea of potential functions as currently used in physics, and the concept of what are now called Green's functions. George Green was the first person to create a mathematical theory of electricity and magnetism and his theory formed the foundation for the work of other scientists such as James Clerk Maxwell, William Thomson, and others. His work ran parallel to that of the great mathematician Gauss (potential theory).

1.1. Definition

We assume the following ordinary differential equation in the interval (a,b), is given [3, 4, 5, 6, 9, 25, and 26]:

$$Ly = y'' + P_1(x)y' + P_2(x)y = r(x) \qquad a < x < b \tag{1}$$

where $P_1(x), P_2(x), r(x)$ are continue and differentiable on (a,b) (analytic on (a,b)).

If two independent solution $y_1(x)$ and $y_2(x)$ are available for the homogeneous equation $Ly = 0$, then $y^{(h)} = C_1y_1 + C_2y_2$ a general solution of $Ly = 0$ on (a,b).

Now, we suing the method of variation of parameters to get particular solution of $Ly = r(x)$ on (a,b).

If we substitute $y = v_1y_1 + v_2y_2$ into the equation $Ly = r(x)$ and we assume $v_1'y_1 + v_2'y_2 = 0$, we obtain

$$v_1 = \int \frac{-y_2.r(x)}{W(y_1,y_2)}dx \qquad , \qquad v_2 = \int \frac{y_1.r(x)}{W(y_1,y_2)}dx$$

Or

$$y^{(p)} = y_1\int \frac{-y_2.r(x)}{W(y_1,y_2)}dx + y_2\int \frac{y_1.r(x)}{W(y_1,y_2)}dx \tag{2}$$

Or, let x_0 be any point on the interval (a,b) and let the indefinite integrals be replaced by definite integrals with respect to a dummy variable ξ from x_0 to x:

$$y^{(p)} = \int_{x_0}^{x} \frac{y_1(\xi).y_2(x) - y_2(\xi).y_1(x)}{y_1(\xi).y_2'(\xi) - y_2(\xi).y_1'(\xi)} r(\xi)d\xi \tag{3}$$

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This given the solution of $Ly = r(x)$ that satisfies $y(x_0) = y'(x_0) = 0$.

Another special case of (2) is useful in the solution of boundary value problems (BVPs). Let the first integral in (2) be replaced by a definite integral from b to x, and the second by an integral from a to x. The limits of integration (b, x) can be changed to (x, b) if we change the sign, hence

$$y^{(p)} = \int_a^x \frac{y_1(\xi) y_2(x)}{W(\xi)} r(\xi) d\xi + \int_x^b \frac{y_1(x) y_2(\xi)}{W(\xi)} r(\xi) d\xi \tag{4}$$

If we define

$$g(x,\xi) = \begin{cases} \dfrac{y_1(\xi) y_2(x)}{W(\xi)} & a < \xi < x \\ \dfrac{y_1(x) y_2(\xi)}{W(\xi)} & x < \xi < b \end{cases} \tag{5}$$

Equation (4) is equivalent to

$$y^{(p)}(x) = \int_a^b g(x,\xi) r(\xi) d\xi \tag{6}$$

Or, general solution of $Ly = r(x)$ on (a, b) is

$$y = y^{(p)} + y^{(h)} = \int_a^b g(x,\xi) r(\xi) d\xi + C_1 y_1 + C_2 y_2 \tag{7}$$

The formula (6) gives the solution of $Ly = r(x)$ that satisfies

$$\begin{cases} \alpha_1 y(a) + \beta_1 y'(a) = 0 \\ \alpha_2 y(b) + \beta_2 y'(b) = 0 \end{cases} \tag{8}$$

The function $g(x,\xi)$ determined by this analysis is called Green's Function for the boundary value problem associated with (1) and (8).

1.2 Properties of Green Function

We notice that $g(x,\xi)$ as defined in (5) has the following properties:

1. It satisfies the homogeneous form of the given differential equation, that is $g'' + k^2 g = 0$ in each of the intervals $0 \le \xi < x$, $x < \xi \le l$. The behavior of g at $x = \xi$ is, at moment, uncertain.
2. The function g is continuous at $x = \xi$ since
$$\lim_{\xi \to x^-} g(x,\xi) = \frac{\sin(kx).\sin(k(l-x))}{k.\sin(kl)} = \lim_{\xi \to x^+} g(x,\xi)$$
3. The derivative of g with respect to ξ is discontinuous at $x = \xi$. This can be seen as follows:
$$\begin{aligned} g'(x,\xi^-) &= \lim_{\xi \to x^-} g'(x,\xi) = \frac{\cos(kx).\sin(k(l-x))}{\sin(kl)} \\ g'(x,\xi^+) &= \lim_{\xi \to x^+} g'(x,\xi) = -\frac{\sin(kx).\cos(k(l-x))}{\sin(kl)} \end{aligned} \tag{9}$$
where primes denote differentiation with respect to ξ, hence $g'(x,x^+) - g'(x,x^-) = -1$
4. The function $g(x,\xi)$ satisfies the relations
$$g(x,0) = g(x,l) = 0$$
and thus satisfies the boundary conditions of the problem.
5. The function $g(x,\xi)$ is symmetric in its arguments, hence $g(x,\xi) = g(\xi, x)$

2. Applications of green's function

2.1. Green's Functions for the Time-independent Wave Equation

In this section, we shall concentrate on the computation of Green's functions for the time-independent wave equation in one, two and three dimensions (see e.g. [3, 6, 7, 8]). The solution is over all space and the Green's function is not constrained to any particular boundary conditions (except those at $\pm\infty$). It is therefore referred to as a free space Green's function.

2.1.1 The One-dimensional Green's Function

We consider the inhomogeneous wave equation

$$\left(\frac{\partial^2}{\partial x^2} + k^2\right) u(x,k) = f(x) \tag{10}$$

for constant k and $x \in (-\infty, +\infty)$ subject to the boundary conditions

$$u(x,k)\Big|_{\pm\infty} = 0 \quad , \quad \frac{\partial}{\partial x}u(x,k)\Big|_{\pm\infty} = 0$$

Define the Green's function as being the solution to the equation obtained by replacing the source term with a delta function which represents a point source at x_0 say, giving the equation

$$(\frac{\partial^2}{\partial x^2} + k^2)g(x \mid x_0, k) = \delta(x - x_0) \tag{11}$$

Multiplying equation (10) by g and multiplying equation (11) by u and now subtract the two results and integrate to obtain

$$u(x_0, k) = \int_{-\infty}^{+\infty} f(x)\, g(x \mid x_0, k)\, dx$$

Provided u and $\partial u/\partial x$ are zero at $x = \pm\infty$.

This solution is of course worthless without an expression for the Green's function which is given by the solution to the equation

$$(\frac{\partial^2}{\partial x^2} + k^2)g(x \mid x_0, k) = -\delta(x - x_0)$$

$$g(x \mid x_0, k)\big|_{\pm\infty} = 0 \quad , \quad \frac{\partial}{\theta x} g(x \mid x_0, k)\big|_{\pm\infty} = 0 \tag{12}$$

The solution to this equation is based on employing the properties of the Fourier transform.
Writing $X = |x - x_0|$, we express g and δ as Fourier transforms, that is

$$\begin{aligned} g(X,k) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} G(u,k)\exp(iuX)\,du \\ \delta(X) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \exp(iuX)\,du \end{aligned} \tag{13}$$

Substituting these expressions into equation (12) and differentiating givens

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} \left(-u^2 + k^2\right)G(u,k)\exp(iuX)\,du = -\frac{1}{2\pi}\int_{-\infty}^{\infty}\exp(iuX)\,du$$

From which it follows that

$$G(u,k) = \frac{1}{u^2 - k^2}\,.$$

Substituting this result back into equation (13), we obtain

$$g(X,k) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{\exp(iuX)}{u^2 - k^2}\,du = \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{\exp(iuX)}{(u-k)(u+k)}\,du$$

The problem is therefore reduced to that of evaluating the above integral. This can be done using Cauchy's integral formula. Hence the Green's function is given by

$$g(X,k) = 2\pi i\left(\frac{e^{ikX}}{4\pi k} - \frac{e^{-ikX}}{4\pi k}\right) = \frac{-Sin(kX)}{k}\,.$$

2.1.2 The two-dimensional Green's Function

In two dimensions, the same method can be used to obtain the (*free space*) Green's function, that is to solve the equation

$$\left(\nabla^2 + k^2\right)g(\vec{r} \mid \vec{r}_0, k) = -\delta^2(\vec{r} - \vec{r}_0)$$

Subject to some boundary conditions at $|\vec{r}| = \infty$, where $\vec{r} \in \Re^2$, $\vec{r}_0 \in \Re^2$.

Thus, the (outgoing) Green's function can be written in the form

$$g(R,k) = \frac{i}{4\pi}\int_0^{\pi}\exp(ikRCos(\theta))\,d\theta$$

writing the Green's function in this form allows us to employ the result

$$H_0^{(1)}(kR) = \frac{1}{\pi}\int_0^{\pi}\exp(ikRCos(\theta))\,d\theta$$

where $H_0^{(1)}$ is the Hankel function (of the first kind and of order zero).

2.1.3 The Three-dimensional Green's Function

In three dimensions, the free space Green's function is given by the solution to the equation

$$\left(\nabla^2 + k^2\right) g(\vec{r}\mid\vec{r}_0,k) = -\delta^3(\vec{r}-\vec{r}_0)$$

Subject to some boundary conditions at $|\vec{r}| = \infty$, where $\vec{r} \in \Re^3$, $\vec{r}_0 \in \Re^3$.

Thus, the (outgoing) Green's function can be written in the form

$$g(\vec{r}\mid\vec{r}_0,k) = \frac{1}{4\pi|\vec{r}-\vec{r}_0|}\exp(ik|\vec{r}-\vec{r}_0|).$$

2.1.4 Asymptotic Forms

Although the Green's functions for the inhomogeneous wave equation can be computed in the manner given above, their algebra form is not always easy or useful to work with. For this reason, it is worth considering their asymptotic forms, which relate to the case when the field generated by a point source is a long distance away from that source. Thus, asymptotic approximations for these Green's functions are based on considering the case where the source at $\vec{r}_0$ is moved further and further away from the observer at $\vec{r}$. There are two approximations that are important in these respects, which are often referred to as the Fraunhofer and Fresnel approximations.

In one dimension, we do not have asymptotic approximation as such.

In two and three dimensions, we expand the path length between the source and observer in terms of their respective coordinates. First, let us look at the result in two dimensions. In this case,

$$|\vec{r}-\vec{r}_0| = \sqrt{r_0^2 + r^2 - 2\vec{r}.\vec{r}_0} = r_0\left(1 - 2\frac{\vec{r}.\vec{r}_0}{r_0^2} + \frac{r^2}{r_0^2}\right)^{\frac{1}{2}}$$

where $r_0 = |\vec{r}_0|$, $r = |\vec{r}|$, $\vec{r} \in R^2$

A binomial expansion of this result gives

$$|\vec{r}-\vec{r}_0| = r_0\left(1 - \frac{\vec{r}.\vec{r}_0}{r_0^2} + \frac{r^2}{2r_0^2} + \cdots\right) \tag{14}$$

which under the condition $\frac{r}{r_0} << 1$ reduces to

$$|\vec{r}-\vec{r}_0| \approx r_0 - \hat{n}_0.\vec{r}$$

where $\hat{n}_0 = \frac{\vec{r}_0}{r_0}$

It is sufficient to let $\frac{1}{|\vec{r}-\vec{r}_0|} \cong \frac{1}{r_0}$ $\qquad r_0 >> r$

Because small changes in $\hat{n}.\vec{r}$ compared with r_0 are not significant in an expression of this type. However, with the exponential function $\exp(ik(r_0 - \hat{n}_0.\vec{r}))$ a relatively small change in the value of $(r_0 - \hat{n}_0.\vec{r})$ compared with r_0 will still cause this term to oscillate rapidly, particularly if the value if k is large. We therefore write

$$\exp(ik|\vec{r}-\vec{r}_0|) = \exp(ikr_0)\exp(-ik\hat{n}_0.\vec{r})$$

The asymptotic form of the two-dimensional Green's function is then given by

$$g(\vec{r}\mid\vec{r}_0,k) = \frac{\exp(i\frac{\pi}{4})}{\sqrt{8\pi}}\frac{1}{\sqrt{kr_0}}\exp(ikr_0)\exp(-ik\vec{n}_0.\vec{r}) \qquad kr_0 >> 1$$

In three dimensions, the result is

$$g(\vec{r}\mid\vec{r}_0,k) = \frac{1}{4\pi r_0}\exp(ikr_0)\exp(-ik\vec{n}_0.\vec{r}) \qquad \vec{r} \in R^3$$

When the source is brought closer to the observer, the wave front ceases to be a plane wave front. In this case, the Fraunhofer approximation is inadequate and another approximation for the Green's function must be used. This is known as the Fresnel approximation and is based on incorporating the next term in the binomial expansion of $|\vec{r}-\vec{r}_0|$, namely the quadratic term $r^2/2r_0^2$ in equation (14). In this case, it is assumed that $r^2/r_0^2 << 1$ rather than $r/r_0 << 1$ so that all the terms in the binomial expansion of $|\vec{r}-\vec{r}_0|$ that occur after the quadratic term can be neglected. As before, $|\vec{r}-\vec{r}_0|^{-1}$ is approximated by $1/r_0$ but the exponential term now possesses an additional feature, namely a quadratic phase factor. In this case, the two-and three-dimensional Green's functions are given by (respectively)

$$g(\vec{r}\mid\vec{r}_0,k)=\frac{\exp(i\dfrac{\pi}{4})}{\sqrt{8\pi}}\frac{1}{\sqrt{kr_0}}\exp(ikr_0)\exp(-ik\vec{n}_0.\vec{r})\exp(i\frac{r^2}{2r_0})\qquad kr_0>>1$$

and

$$g(\vec{r}\mid\vec{r}_0,k)=\frac{1}{4\pi r_0}\exp(ikr_0)\exp(-ik\vec{n}_0.\vec{r})\exp(i\frac{r^2}{2r_0})$$

This type of approximation is used in the study of systems in which the divergence of the field is a measurable quantity. If the source is moved even closer to the observer then neither the Fraunhofer nor the Fresnel approximations will apply.

2.2. Green's Function Solution to the three dimensional Inhomogeneous Wave Equation

In the previous section, the free space Green's functions for the inhomogeneous time-independent wave equation were considered in one, two and three dimensions [1, 3]. In this section, we turn our attention to the more general problem of developing a solution for the wave field $u(\vec{r},k)$ generated by an arbitrary and time-independent source function $f(\vec{r})$. Working in three dimensions, our aim is to solve

$$(\nabla^2+k^2)u(\vec{r},k)=-f(\vec{r})\qquad\qquad \vec{r}\in V$$

for u where V is the volume of the source function which is of compact support. Note that we define the source term as $-f$ rather than $+f$. This is done so that there is consistency with the definition of the Green's function which is defined in terms of $-\delta$ by convention. We start by writing the equation for a Green's function, i.e.

$$(\nabla^2+k^2)g(\vec{r}\mid\vec{r}_0,k)=-\delta^3(\vec{r}-\vec{r}_0)$$

if we now multiply both sides of the first of the first equation by g and both sides of the second equation by u, then by subtracting the two result and integrating over V we obtain

$$u(\vec{r}_0,k)=\int_v f(\vec{r})g(\vec{r}\mid\vec{r}_0,k)d^3\vec{r}+\int_v\left[g(\vec{r}\mid\vec{r}_0,k)\nabla^2u(\vec{r},k)-u(\vec{r},k)\nabla^2g(\vec{r}\mid\vec{r}_0,k)\right]d^3\vec{r}$$

Observe, that this expression is not a proper solution for u because this function occurs in both the left- and right-hand sides. We require a solution for u in terms of known quantities on the right-hand side of the above equation. To this end, we can simplify the second term by using Green's theorem:

$$\int_v(g\nabla^2u-u\nabla^2g)d^3\vec{r}=\oint_S(g\nabla u-u\nabla g)\cdot\hat{n}d^2\vec{r}$$

Here, S define the surface enclosing the volume V and $d^2\vec{r}$ is an element of this surface. The unit vector $\hat{n}$ points out of the surface and is perpendicular to the surface element $d^2\vec{r}$.

Green's theorem provided a solution for the wave field u at $\vec{r}_0$ of the form

$$u(\vec{r}_0,k)=\int_v fgd^3\vec{r}+\oint_S(g\nabla u-u\nabla g)\cdot\hat{n}d^2\vec{r}\qquad(15)$$

2.2.1 The Dirichlet and Neumann Boundary Conditions

Although Green's theorem allows us to simplify the solution for u, we still do not have a proper solution for u since this field variable is present on both the left- and right-hand sides of equation (15). However, as a result of applying Green's theorem we now only need to specify u and ∇u on the surface S.

Therefore, if we know, a priori, the behavior of u and ∇u on S we can compute u at any other observation point $\vec{r}_0$ from equation (15).

In general, the type of conditions that may be applied depends on the applications that are involved. In practice, two types of boundary conditions are commonly considered. The first one is known as the homogeneous Dirichlet boundary condition which states that u is zero on S and the second one is known as the homogeneous Neumann condition which states that ∇u is zero on S. When u satisfies these homogeneous boundary conditions, the solution for u is given by

$$u(\vec{r}_0,k)=\int_v f(\vec{r})g(\vec{r}\mid\vec{r}_0,k)d^3\vec{r}$$

Because

$$\oint_S(g\nabla u-u\nabla g)\cdot\hat{n}d^2\vec{r}=0$$

2.3. Green's Function Solutions to the Inhomogeneous Helmholtz equation: An Introduction to Scattering Theory

2.3.1 Basic Solution to the Inhomogeneous Helmholtz Equation

The inhomogeneous Helmholtz equation is

$$(\nabla^2 + k^2)u(\vec{r},k) = -k^2\gamma(\vec{r})u(\vec{r},k)$$

where γ is an inhomogeneous that is responsible for scattering the wave field u and is therefore sometimes referred to as a scatter– usually considered to be of compact support [1, 9, 10].

The same Green's function method that has already been presented in Section 4 can be used to solve the inhomogeneous Helmholtz equation. The basic solution is (under the assumption that γ is of compact support $\vec{r} \in V$)

$$u(\vec{r}_0,k) = k^2\int_v g\gamma u \, d^3\vec{r} + \oint_S (g\nabla u - u\nabla g)\cdot\hat{n}\, d^2\vec{r}$$

Once again, to compute the surface integral, a condition for the behavior of u on the surface S of γ must be chosen. Consider the case where the incident wave field u_i is a simple plane wave of unit amplitude $\exp(ik.\vec{r})$ satisfying the homogeneous wave equation

$$(\nabla^2 + k^2)u_i(\vec{r},k) = 0$$

By choosing the condition $u(\vec{r},k) = u_i(\vec{r},k)$ on the surface of S , we obtain the result

$$u(\vec{r}_0,k) = k^2\int_v g\gamma u \, d^3\vec{r} + \oint_S (g\nabla u_i - u_i\nabla g)\cdot\hat{n}\, d^2\vec{r}$$

Now, using Green's theorem to convert the surface integral back into a volume integral, we have

$$\oint_S (g\nabla u_i - u_i\nabla g)\cdot\hat{n}\, d^2\vec{r} = \oint_v (g\nabla^2 u_i - u_i\nabla^2 g)\, d^3\vec{r}$$

Noting that

$$\nabla^2 u_i = -k^2 u_i \qquad , \qquad \nabla^2 g = -\delta^3 - k^2 g$$

we obtain

$$\oint_v (g\nabla^2 u_i - u_i\nabla^2 g)\, d^3\vec{r} = \int_v \delta^3 u_i d^3\vec{r} = u_i$$

Hence, by choosing the field u to be equal to the incident wave field u_i on the surface of S , we obtain a solution of the form

$$u = u_i + u_s$$

where

$$u_s = k^2\int_v g\gamma u \, d^3\vec{r}$$

The wave field u_s is often referred to as the *scattered wave field*.

2.3.2 The Born Approximation

From the last result it is clear that in order to compute the scattered field u_s, we must define u inside the volume integral. Unlike the surface integral, a boundary condition will not help here because it is not sufficient to specify the behavior of u at a boundary. In this case, the behavior of u throughout V needs to be known. In general, it is not possible to do this (i.e. to compute the scattered wave field exactly – at least to date) and we are forced to choose a model for u inside V that is compatible with a particular physical problem in the same way that an appropriate set of boundary conditions are required to evaluate the surface integral. The simplest model for the internal field is based on assuming that u behaves like u_i for $\vec{r} \in V$. The scattered field is then given by

$$u_s(\vec{r}_0,k) = k^2\int_v g(\vec{r}\,|\,\vec{r}_0,k)\,\gamma(\vec{r})\,u_i(\vec{r},k)\,d^3\vec{r}$$

This assumption provides an approximate solution for the scattered field and is know as the *Born approximation.*

2.3.3 Asymptotic Born Scattering

By measuring, we can attempt to invert the relevant integral equation and hence recover or reconstruct γ. This type of problem is know as the inverse scattering problem, and solutions to this problem are called inverse scattering solutions. This subject is one of the most fundamental and difficult problems to solve in mathematical physics and is the subject of continuing research in the area of inverse problems in general. The simplest type of inverse scattering problem occurs when a Born scattered wavefield is measured in the far field or Fraunhofer.

From previous results, working in three dimensions, we know that when the incident field is a (unit) plane wave

$$u_i = \exp(ik\hat{n}_i.\vec{r})$$

where $\hat{n}_i$ points in the direction of the incident field, the Born scattered field observed at $\vec{r}_s$ becomes

$$u_s(\hat{n}_s, \hat{n}_i, k) = \frac{k^2}{4\pi r_s} \exp(ikr_s) \int_V \exp(-ik(\hat{n}_s - \hat{n}_i) \cdot \vec{r}) \gamma(\vec{r}) d^3\vec{r} \qquad \vec{r} \in V$$

where $\hat{n}_s (= \vec{r}_s / r_s)$ denotes the direction in which u_s propagates. From this result, it is clear, that the function γ can be recovered from u_s by three-dimensional Fourier inversion.

The scattered field produced by a two-dimensional Born scattered in the far field is given by

$$u_s(\hat{n}_i, \hat{n}_s, k) = \frac{\exp(i\pi/4)}{\sqrt{8\pi}} \frac{k^2}{\sqrt{kr_s}} \cdot \exp(ikr_s) \int_A \exp[-ik(\vec{n}_s - \vec{n}_i) \cdot \vec{r}] \gamma(\vec{r}) d^2\vec{r} \qquad \vec{r} \in A$$

In one-dimensional, the equivalent result is (for a right-traveling wave)

$$u_s(x_s, k) = \frac{ik}{2} \exp(ikx_s) \int_L \gamma(x) dx \qquad x \in L$$

2.4. Green's Functions for Time-dependent Inhomogeneous Wave Equations

We shall consider the three-dimensional problem first but stress that the methods of solution discussed here apply directly to problems in one and two dimension [1, 6, 8, 10]. Thus, consider the case in which a time varying source function $f(\vec{r}, t)$ produces a wave field u which is taken to be the solution to the equation.

$$\left(\nabla^2 + \frac{1}{c^2} \frac{\partial^2}{\partial t^2}\right) u(\vec{r}, t) = -f(\vec{r}, t)$$

As with the time-independent problem, the Green's function for this equation is defined as the solution to the equation to the equation obtained by replacing $f(\vec{r}, t)$ with $\delta^3(\vec{r} - \vec{r}_0)\delta(t - t_0)$, that is the solution to the equation

$$\left(\nabla^2 + \frac{1}{c^2} \frac{\partial^2}{\partial t^2}\right) G(\vec{r} \mid \vec{r}_0, t \mid t_0) = -\delta^3(\vec{r} - \vec{r}_0)\delta(t - t_0) \qquad (16)$$

where G is used to denote the time-dependent Green's function, $\vec{r}_0$ is the position of the source and $t|t_0 \equiv t - t_0$. To obtain the equation for the time-independent Green's function, we write G and $\delta(t - t_0)$ as Fourier transforms,

$$G(\vec{r} \mid \vec{r}_0, t \mid t_0) = \frac{1}{2\pi} \int_{-\infty}^{\infty} g(\vec{r} \mid \vec{r}_0, w) \exp(iw(t - t_0)) dw$$

$$\delta(t - t_0) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \exp(iw(t - t_0)) dw$$

where w is the same angular frequency. Substituting these equations into equation (16) we then obtain

$$(\nabla^2 + k^2) g(\vec{r} \mid \vec{r}_0, k) = -\delta^3(\vec{r} - \vec{r}_0)$$

which is the same equation as that used previously to define the time-independent Green's function. Thus, once g has been obtained, the time-dependent Green's function can be derived by computing the Fourier integral given above. Using the expression for g derived earlier,

$$G(\vec{r} \mid \vec{r}_0, t \mid t_0) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{1}{4\pi |\vec{r} - \vec{r}_0|} \exp(ik|\vec{r} - \vec{r}_0|) \exp(iw(t - t_0)) dw = \frac{1}{4\pi |\vec{r} - \vec{r}_0|} \delta\left(t - t_0 + \frac{|\vec{r} - \vec{r}_0|}{c}\right)$$

2.5. Green's Functions and Optics: Kirchhoff Diffraction Theory

The phenomenon of diffraction is common to one-dimensional transverse (and hence scalar) waves such as water waves, true scalar waves such as in acoustics and vector waves as in optics. All three cases appear to exhibit the same magnitude of diffraction phenomena [1, 2, and 5].

In Section 4, the surface integral obtained using a Green's functions solution and Green's theorem was discarded under the assumption of homogeneous boundary conditions or that $u = u_i$ (the incident field) of the surface S. We were then left with a volume integral (i.e. volume scattering). In this section, we make explicit use of this surface integral to develop a solution to the homogeneous Helmholtz equation. This leads to the theory of surface scattering of which Kirchhoff diffraction theory is a special (but very important) case.

2.5.1 Kirchhoff Diffraction Theory

Consider a scalar wave field u described by the homogeneous Helmholtz equation

$$\left(\nabla^2 + k^2\right)u = 0$$

Let u_i be the field incident on a surface S and consider the following (Kirchhoff) boundary conditions

$$u = u_i \quad , \quad \frac{\partial u}{\partial \hat{n}} = \frac{\partial u_i}{\partial \hat{n}}$$

The results which derive from a solution to the Helmholtz equation based on these boundary conditions are called Kirchhoff diffraction theory.

Consider the Green's function g which is the solution to

$$\left(\nabla^2 + k^2\right)g = -\delta^3\left(\vec{r} - \vec{r}_0\right)$$

given by

$$g\left(\vec{r} \mid \vec{r}_0, k\right) = \frac{1}{4\pi\left|\vec{r} - \vec{r}_0\right|}\exp\left(ik\left|\vec{r} - \vec{r}_0\right|\right)$$

We can construct two equations:

$$g\nabla^2 u + k^2 gu = 0 \quad , \quad u\nabla^2 g + k^2 ug = -u\delta^3 .$$

Subtracting these equations and integrating over a volume V we obtain a solution for the field u at $\vec{r}_0$,

$$u\left(\vec{r}_0, k\right) = \oint_S \left(g\frac{\partial u}{\partial \hat{n}} - u\frac{\partial g}{\partial \hat{n}}\right) d^2\vec{r}$$

Using the Kirchhoff boundary conditions we have

$$u\left(\vec{r}_0, k\right) = \oint_S \left(g\frac{\partial u_i}{\partial \hat{n}} - u_i\frac{\partial g}{\partial \hat{n}}\right) d^2\vec{r}$$

This equation referred to as the *Kirchhoff integral.*

To compute the diffracted field using the Kirchhoff integral, an expression for u_i must be introduced and the derivatives $\partial/\partial\hat{n}$ with respect to u_i and g of unit amplitude (with wave number $k \equiv \left|\vec{k}\right|, \hat{k} = \vec{k}/k$). Then

$$u_i = \exp\left(i\,\vec{k}\cdot\vec{r}\right)$$

$$\frac{\partial u_i}{\partial \hat{n}} = \hat{n}\cdot\nabla\exp\left(i\,\vec{k}\cdot\vec{r}\right) = i\,\vec{k}\cdot\hat{n}\ \exp\left(i\vec{k}\cdot\vec{r}\right) = ik\ \hat{n}\cdot\vec{k}\ \exp\left(i\vec{k}\cdot\vec{r}\right)$$

and the Kirchhoff diffraction formula reduces to the form

$$u\left(\vec{r}_0, k\right) = ik\int_S \exp\left(i\vec{k}\cdot\vec{r}\right)\left(\hat{n}\cdot\hat{k} - \hat{n}\cdot\hat{m}\right)g\left(\vec{r} \mid \vec{r}_0, k\right)d^2\vec{r} .$$

2.5.2 Fraunhofer Diffraction

Fraunhofer diffraction assumes that the diffracted wave field is observed a large distance away from the screen and as in previous sections is based on the asymptotic form of the Green's function. For this reason, Fraunhofer diffraction is sometime called diffraction in the "far field". The basic idea is to exploit the simplifications that can be made to the Kirchhoff diffraction integral by considering the case when $r_0 >> r$ where $r \equiv \left|\vec{r}\right|$ and $r_0 \equiv \left|\vec{r}_0\right|$. In this case,

$$\hat{n}\cdot\hat{k} - \hat{n}\cdot\hat{m} \approx \hat{n}\cdot\hat{k} + \hat{n}\cdot\hat{r}_0 \quad , \quad \frac{1}{\left|\vec{r} - \vec{r}_0\right|} \approx \frac{1}{r_0}$$

where $\hat{r}_0 = \frac{\vec{r}_0}{r_0}$.

With regard to the term $\exp\left(ik\left|\vec{r} - \vec{r}_0\right|\right)$,

$$\left|\vec{r} - \vec{r}_0\right| = r_0\left(1 - 2\frac{\vec{r}\cdot\vec{r}_0}{r_0^2} + \frac{r^2}{r_0^2}\right)^{\frac{1}{2}} \approx r_0 - \vec{r}\cdot\vec{r}_0$$

Thus, the Kirchhoff diffraction integral reduces to

$$u(\vec{r}_0, k) \approx \frac{ik\alpha}{4\pi r_0} \exp(ikr_0) \int_S \exp(i\vec{k} \cdot \vec{r}) \exp(-ik\ \vec{r}_0 \cdot \vec{r}) d^2\vec{r}$$

where $\alpha = \hat{n} \cdot \hat{k} + \hat{n} \cdot \vec{r}_0$.
This is the *Fraunhofer diffraction integral.*

2.5.3 Fresnel Diffraction

Fresnel diffraction is based on considering the binomial expansion of $|\vec{r} - \vec{r}_0|$ in the function $\exp(ik|\vec{r} - \vec{r}_0|)$ to second order and retaining the term $r^2/2r_0$;

$$|\vec{r} - \vec{r}_0| = r_0 - \vec{r} \cdot \vec{r}_0 + \frac{r^2}{2r_0} + \cdots \approx r_0 - \vec{r} \cdot \vec{r}_0 + \frac{r^2}{2r_0}$$

This approximation is necessary when the diffraction pattern is observed in what is called the *intermediate field* or *Fresnel zone* in which

$$u(\vec{r}_0, k) \approx \frac{ik\alpha}{4\pi r_0} \exp(ikr_0) \int_S \exp(i\vec{k} \cdot \vec{r}) \exp(-ik\ \vec{r}_0 \cdot \vec{r}) \exp\left(ik\frac{r^2}{2r_0}\right) d^2\vec{r}$$

This is the *Fresnel diffraction formula.*

2.6. Green's Function Solution to the Diffusion Equation

the homogeneous diffusion equation

$$\nabla^2 u(\vec{r}, t) = a\frac{\partial}{\partial t} u(\vec{r}, t)$$

where a is a constant differs in many aspects from the scalar wave equation and the Green's functions exhibit these differences [1]. The most important single feature is the asymmetry of the diffusion equation with respect to time. For the wave equation, if $u(\vec{r}, t)$ is a solution, so is $u(\vec{r}, -t)$. However, if $u(\vec{r}, t)$ is a solution of

$$\nabla^2 u(\vec{r}, t) = a\frac{\partial}{\partial t} u(\vec{r}, t)$$

the function $u(\vec{r}, -t)$ is not; it is a solution of the quite different equation,

$$\nabla^2 u(\vec{r}, -t) = -a\frac{\partial}{\partial t} u(\vec{r}, -t)$$

Thus, unlike the wave equation, the diffusion equation differentiates between past and future. This is because the diffusing field u represents the behavior of some average property of an ensemble of many particles which cannot in general return to their original state. Thus causality must be considered in the solution to the diffusion equation. This in turn leads to the use of the Laplace transform for solving the equation with respect to t (compared with the Fourier transform sued to solve the wave equation with respect to t).

2.6.1 General Solution to the Inhomogeneous Diffusion Equation

Working in three dimensions, let us consider the general solution to the equation

$$\left(\nabla^2 - a\frac{\partial}{\partial \tau}\right) u(\vec{r}, t) = -f(\vec{r}, t)$$

where f is a source of compact support ($\vec{r} \in V$) and define the Green's function as the solution to the equation

$$\left(\nabla^2 - a\frac{\partial}{\partial \tau}\right) G(\vec{r} \mid \vec{r}_0, t - t_0) = -\delta^3(\vec{r} - \vec{r}_0)\delta(t - t_0)$$

It is convenient to first take the Laplace transform of these equation with respect to $\tau = t - t_0$ to obtain

$$\nabla^2 \overline{U} - a(-u_0 + p\overline{U}) = -\bar{f} \tag{17}$$

$$\nabla^2 \overline{G} + a(-G_0 + p\overline{G}) = -\delta^3 \tag{18}$$

where

$$U(\vec{r}, p) = \int_0^\infty u(\vec{r} \mid \vec{r}_0, \tau) \exp(-p\tau) d\tau$$

$$\overline{G}(\vec{r} \mid \vec{r}_0, p) = \int_0^\infty G(\vec{r} \mid \vec{r}_0, \tau) \exp(-p\tau) d\tau$$

$$\bar{f}(\vec{r}, p) = \int_0^\infty f(\vec{r}, \tau) \exp(-p\tau) d\tau$$

$$u_0 = u(\vec{r}, \tau = 0) = 0 \quad , \quad G_0 = G(\vec{r} \mid \vec{r}_0, \tau = 0) = 0$$

Multiplying equation (17) by $\overline{G}$ and equation (18) by $\overline{U}$, subtracting the two results and integrating over V we obtain

$$\int_v \left(\overline{G}\nabla^2\overline{U} - \overline{U}\nabla^2\overline{G}\right)d^3\vec{r} + a\int_v u_0\overline{G}d^3\vec{r} = -\int_v f\overline{G}d^3\vec{r} + \overline{U}(\vec{r}_0, p)$$

Using Green's theorem and rearranging the result gives

$$\overline{U}(\vec{r}_0,\tau) = \int_v f(\vec{r},p)\overline{G}(\vec{r}\mid\vec{r}_0,p)d^3\vec{r} + a\int_v u_0(\vec{r})\overline{G}(\vec{r}\mid\vec{r}_0,p)d^3\vec{r} + \oint_S \left(\overline{g}\nabla\overline{U} - \overline{U}\nabla\overline{g}\right)\cdot\hat{n}d^2\vec{r}$$

Finally, taking the inverse Laplace transform and using the Convolution theorem for Laplace transforms, we can write

$$u(\vec{r}_0,\tau) = \int_0^\tau\int_v f(\vec{r},\tau')G(\vec{r}\mid\vec{r}_0,\tau-\tau')d^3\vec{r}\,d\tau' + a\int_v u_0(\vec{r})\overline{G}(\vec{r}\mid\vec{r}_0,\tau)d^3\vec{r}$$
$$+\int_0^\tau\oint_S\left[G(\vec{r}\mid\vec{r}_0,\tau')\nabla u(\vec{r},\tau-\tau') - u(\vec{r},\tau')\nabla G(\vec{r}\mid\vec{r}_0,\tau-\tau')\right]\cdot\hat{n}\,d^2\vec{r}\,d\tau'$$

2.7. Green's Function Solution to the Laplace and Poisson Equations

The two- and three-dimensional Laplace and Poisson equations are given by

$$\begin{aligned}\nabla^2 u &= 0\\ \nabla^2 u &= -f\end{aligned} \tag{19}$$

Respectively. We consider the Poisson equation first [1, 11]. The general approach is identical to that used to derive a solution to the inhomogeneous Helmholtz equation. Thus, working in three dimensions and

defining the Green's function to be the solution of

$$\nabla^2 g(\vec{r}\mid\vec{r}_0) = -\delta^3(\vec{r}-\vec{r}_0)$$

from equation (19) we obtain the following result

$$u = \oint_S (g\nabla u - u\nabla g)\cdot\hat{n}\,d^2\vec{r} + \int_v gf\,d^3\vec{r}$$

where we have used Green's theorem to obtain the surface integral on the right-hand side. The problem now is to find the Green's function for this problem. Clearly, since the solution to the equation

$$\left(\nabla^2 + k^2\right)g = -\delta^3(\vec{r}-\vec{r}_0)$$

is

$$g(\vec{r}\mid\vec{r}_0,k) = \frac{1}{4\pi|\vec{r}-\vec{r}_0|}\exp(ik|\vec{r}-\vec{r}_0|)$$

we should expect the Green's function for the three-dimensional Poisson equation (and the Laplace equation) to be of the form

$$g(\vec{r}\mid\vec{r}_0) = \frac{1}{4\pi|\vec{r}-\vec{r}_0|} \tag{20}$$

Thus, we obtain the following fundamental result: $\nabla^2\left(\frac{1}{4\pi R}\right) = -\delta^3(\vec{R})$.

With homogeneous boundary conditions, the solution to the Poisson equation is $u(\vec{r}_0) = \frac{1}{4\pi}\int_v \frac{f(\vec{r})}{|\vec{r}-\vec{r}_0|}d^3\vec{r}$

In two dimensions the solution is of the same form, but with a Green's function given by

$$g(\vec{r}\mid\vec{r}_0) = \frac{1}{2\pi}Ln(|\vec{r}-\vec{r}_0|)$$

The general solution to Laplace's equation is

$$u = \oint_S (g\nabla u - u\nabla g)\cdot\hat{n}\;d^2\vec{r}$$

with g given by equation (20).

3. RESULTS AND DISCUSSION

3.1. Variable Conductivity

Let the thermal conductivity in a rod unit length be function $k(x)$ which is positive and continuously differentiable. The steady temperature $g(x,\xi)$ in a rod with a concentrated unit source at ξ, with its left end at 0 temperature, and with its right end insulated satisfies

$$-\frac{d}{dx}\left(k(x)\frac{dg}{dx}\right)=\delta(x-\xi) \qquad 0<x,\xi<1$$
$$g(0,\xi)=g'(1,\xi)=0$$

An equivalent formulation is

$$-(kg')'=0 \qquad 0<x<\xi \ , \ \ \xi<x<1$$
$$g(0,\xi)=g'(1,\xi)=0$$

g continuous at $x=\xi$

$$k(\xi)\left(g'(\xi^{+},\xi)-g'(\xi^{-},\xi)\right)=-1$$

the jump condition on g' stemming from a heat balance for a thin slice of the rod containing the source. The functions

$$u_1(x)=\int_0^x\frac{1}{k(y)}dy \qquad , \qquad u_2(x)=1$$

are solutions of the homogeneous equation satisfying, respectively, the boundary conditions at the left and right endpoints. The matching conditions at $x=\xi$ give

$$g(x,\xi)=\begin{cases}\int_0^x\frac{1}{k(y)}dy & 0\le x<\xi\\ \int_0^\xi\frac{1}{k(y)}dy & \xi<x\le 1\end{cases}$$

3.2. Poisson's equation

Poisson's equation in 1D with homogeneous BCs serves to exemplify the general case. The operator in this example is $L=-d^2/dx^2$. For simplicity we take $x_1=0, x_2=a$. The homogeneous solutions $\Phi_{1,2}$ defined by (2) can be identified by inspection: $\Phi_1=x$ and $\Phi_2=(a-x)$. Then

$$W\equiv\Phi_1\Phi_2'-\Phi_1'\Phi_2=x(-1)-1(a-x)=-a\neq 0\,.$$

Consequently g becomes

$$g(x\mid\xi)=-\left\{\frac{H(\xi-x).(a-x).x}{a}+\frac{H(x-\xi).(a-x).\xi}{a}\right\}=-\frac{x_<(a-x)_>}{a}$$

The end-result, now reads

$$\Psi(x)=\frac{1}{a}\left\{x\int_x^a f(\xi)(a-\xi)\,d\xi+(a-x)\int_0^x f(\xi)\xi\,d\xi\right\}$$

3.3. Infinite domain

Consider the case when the source term is zero and the volume of interest is the infinite domain, so that the surface integral is zero. Then we have

$$u(\vec{r}_0,t)=a\int_v u_0(\vec{r})G(\vec{r}\mid\vec{r}_0,t)d^3\vec{r}$$

In one dimensional, this reduces to

$$u(x_0,t)=\sqrt{\frac{a}{4\pi t}}\int_{-\infty}^{\infty}\exp\left(-a\frac{(x-x_0)^2}{4t}\right)u_0(x)\,dx \qquad t>0$$

Thus we see that the field u at a time $t>0$ is given by the convolution of the field at time $t=0$ with the Gaussian function

$$\sqrt{\frac{a}{4\pi t}}\exp\left(\frac{-ax^2}{4t}\right).$$

REFERENCES

[1] G. Barton, Elements of Green's functions and propagation, Oxford, 1989

[2] G. Carrier, C. E. Pearson, Partial Differential Equations, New York, 1976

[3] G. Evans, Black ledge, P. Yardley, Analytic Methods for partial differential equation, New York,1999

[4] G. Roach, Green's function, Cambridge, 1982

[5] I. Stakgold, Green's functions and Boundary value problems, New York, 1998

[6] E.S. Panakhov, R. Yilmazer, On the Determination of the Hydrogen Atom Equation from Two Spectra, World Applied Sciences Journal 13 (10): 2203-2210, 2011

[7] M.E. Taylor,Wave Equations and Diffraction,In Elsevier Encyclopedia of Mathematical Physics,2006.

[8] M. S. Howe, Theory of Vortex Sound, Cambridge Texts in Applied Mathematics, 2003

[9] H. Izadneshan, Computing of Periodic Hamiltonian Matrices in Quantum Physics with Using PSM Method and WKB Approximation, World Applied Sciences Journal 11 (11): 1369-1374, 2010

[10] K.F. Riley, M.P. Hobson and S.J. Bence, Mathematical methods for physics and engineering, Cambridge University Press, ch. 19. ISBN 0-521-89067-5, 2002.

[11] M. Delkhosh, The Conversion a Bessel's Equation to a Self-Adjoint Equation and Applications, World Applied Sciences Journal 15 (12): 1687-1691, 2011

[12] G. Green, An Essay on the Application of mathematical Analysis to the theories of Electricity and Magnetism, Fellow of Gonville- and Cains-Colleges at Cambridge, 1828, Originally published as book in Nottingham, 1828. Reprinted in three parts in Journal fur die reine und angewandte Mathematik Vol. 39, 1 (1850) p. 73–89, Vol. 44, 4 (1852) p. 356–74, and Vol. 47, 3 (1854) p. 161–221. From there transcribed by Ralf Stephan, eMail: ralf@ark.in-berlin.de.

[13] D. M. Cannel, N. J. Lord, N. J. Lord, George Green, mathematician and physicist 1793-1841, The Mathematical Gazette (The Mathematical Gazette, Vol. 77, No. 478) 77 (478): 26–51, 1993.

[14] I. Grattan-Guinness, "Green, George (1793–1841)", Oxford Dictionary of National Biography, Oxford University Press, 2004.

[15] Tzer-Ming Chen, Ching-Chih Chen, Numerical solution for the hyperbolic heat conduction problems in the radial–spherical coordinate system using a hybrid Green's function method, International Journal of Thermal Sciences, Volume 49, Issue 7, Pages 1193–1196, 2010

[16] Zhigang Xu, The All-Source Green's Function of Linear Shallow Water Dynamic System: Its Numerical Constructions and Applications to Tsunami Problems, The Tsunami Threat - Research and Technology, 25.

[17] Ching-Chung Cheng , Investigations into Green's Function as Inversion-Free Solution of The Kriging Equation, With Geodetic Applications, Presented in Partial Fulfillment of the Requirements for the Degree Doctor of Philosophy in the Graduate School of The Ohio State University, 2004.

[18] Mori M.,Echigo T., Numerical Green's Function Method Based on the DE Transformation, Japan J. Indust. Appl. Math., 23, 193–205, 2006

[19] Chayim Rosen R., Numerical Green's Function Modeling of One-Dimensional Quantum Transport, A dissertation submitted for the degree of Master of Philosophy, Wolfson College University of Cambridge, 2007

[20] Blumer N., Numerically exact Green functions from Hirsch-Fye quantum Monte Carlo simulations, Johannes Gutenberg University, 2008

[21] Izadneshan H., Computing of Periodic Hamiltonian Matrices in Quantum Physics with Using PSM Method and WKB Approximation, World Applied Sciences Journal 11 (11): 1369-1374, 2010

[22] Sugihara M., Double exponential transformation in the Sinc-collocation method for two point boundary value problems. J. Comput. Appl. Math., 149, 239–250, 2002

[23] A. Nurmuhammad A., Muhammad M., Mori M. and Sugihara M., Double exponential transformation in the Sinc-collocation method for the boundary value problem of fourth-order ordinary differential equation. J. Comput. Appl. Math., 182, 32–50, 2005

[24] Delkhosh M., Analytic Solutions of Some Self-adjoint Equations by Using Variable Change Method And it's Applications, Journal of Applied Mathematics, Volume 2012, Article ID 180806, doi:10.1155/2012/180806

[25] Delkhosh M., Delkhosh M. and Jamal M., Introduction to Green's Function and its Numerical Solution, Middle-East Journal of Scientific Research 11 (7): 974-981, 2012

[26] Javadpour S.H, Ahmadi A.R., An Introduction to Ordinary and Partial Differential Equations, Iran, Alavi, 1993.