J. Basic. Appl. Sci. Res., 2(9)9739-9744, 2012
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ISSN 2090-4304
Journal of Basic and Applied Scientific Research
www.textroad.com

Solutions of Fifth-Order Boundary Value Problems Using Polynomial Spline in Off Step Points

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ABSTRACT

Polynomial spline in off step points is used to solve fifth order linear boundary value problems. Boundary formulas are developed. We compare our results with the results produced by B-spline method and Non-polynomial spline method. However, it is observed that our approach produce better numerical solutions in the sense that max $|e_i|$ is a minimum.

KEYWORDS: Fifth-order boundary-value problem; Polynomial spline functions; boundary value formulae; Numerical results.

1 INTRODUCTION

This theory has developed into an interesting branch of applicable mathematics, which contains a wealth of new ideas for inspiration and motivation to do research. This type of problems arises in the mathematical modeling of viscoelastic flows [1,2]. Siddiqi and Twizell[3-6] presented the solutions of 6th,8th,10th and 12th order boundary value problems using the 6th,8th,10th and 12th degree spline, respectively. Rashidinia et al.[7] presented the solutions of fifth-order boundary value problems using Non-Polynomial spline. Farajeyan and Rafati.[8] presented the solutions of tenth-order boundary-value problems in off step points.

In this paper we used polynomial spline approximation in off step points to develop a family of new numerical methods to smooth approximations to the solution of fifth-order differential equation.The method developed is observed to be better than that developed by Siddiqi et al [13],[10] and Caglar et al.[9] , as discussed in Examples 1,2 and 3.

The scientific contribution of this paper are:

1) The solution of boundary value problems.
2) Formulation of boundary conditions
3) Increase accuracy and reduce error

We consider fifth -order boundary-value problem of type

$$y^{(5)}(x)+f(x)y(x)=g(x)\ ,\quad x\in[a,b] \tag{1}$$

$$y(a)=\alpha_0, y^{(1)}(a)=\alpha_1, y^{(2)}(a)=\alpha_2,$$
$$y(b)=\beta_0, y^{(1)}(b)=\beta_1 \tag{2}$$

Where α_i,β_i for $i=0,1,2$ are finite real constants and the functions $f(x)$ and $g(x)$ are continuous on [a,b]. In this paper, in Section 2, the new polynomial spline methods are developed for solving equation (1) along with boundary condition(2).The boundary formulas are develop in Section 3. In Section 4, the polynomial spline solution of the BVP (1),(2) is determined and in Section 5 numerical experiment, discussion and comparison with other known methods, are given.

2 NUMERICAL METHODS

Let $S_i(x)$ be the polynomial spline defined on [a,b] as:

$$S_i(x)=a_i(x-x_i)^5+b_i(x-x_i)^4+c_i(x-x_i)^3+d_i(x-x_i)^2+e_i(x-x_i)+f_i \tag{3}$$

$x\in[x_{i-\frac{1}{2}},x_{i+\frac{1}{2}}], i=0,1,2,..,n-1$ and $x_0=a,\ x_n=b,$

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Where $h=\dfrac{b-a}{n}$ and $x_{i-\frac{1}{2}}=a+(i-\frac{1}{2})h$, $i=1,2,3,..,n$

The spline S is defined in terms of its 1th and 2th derivatives and we denote these values at knots as:

$$S_i(x_{i-\frac{1}{2}})=y_{i-\frac{1}{2}},\ S_i^{(1)}(x_{i-\frac{1}{2}})=m_{i-\frac{1}{2}},\ S_i^{(2)}(x_{i-\frac{1}{2}})=M_{i-\frac{1}{2}},\ S_i^{(5)}(x_{i-\frac{1}{2}})=\frac{L_{i-\frac{1}{2}}+L_{i+\frac{1}{2}}}{2},$$

$$S_i(x_{i+\frac{1}{2}})=y_{i+\frac{1}{2}},\ S_i^{(1)}(x_{i+\frac{1}{2}})=m_{i+\frac{1}{2}}$$

For $i=1,2,...,n.$ (4)

Assuming $y(x)$ to be the exact solution of the boundary value problem (1)

and y_i be an approximation to $y(x_i)$, obtained by the spline $S_i(x)$, we can obtained the coefficients in (3) in the following form

$$a_i=\frac{1}{240}(L_{i-\frac{1}{2}}+L_{i+\frac{1}{2}}),$$

$$b_i=-\frac{1}{1440h^4}[-h^5(L_{i-\frac{1}{2}}+L_{i+\frac{1}{2}})-1440hm_{i-\frac{1}{2}}-240h^2(2M_{i-\frac{1}{2}}+M_{i+\frac{1}{2}})-1440(y_{i-\frac{1}{2}}+y_{i+\frac{1}{2}})],$$

$$c_i=-\frac{1}{288h}[h^3(L_{i-\frac{1}{2}}+L_{i+\frac{1}{2}})+48(M_{i-\frac{1}{2}}-M_{i+\frac{1}{2}})],$$

$$d_i=-\frac{1}{960h^2}[h^5(L_{i-\frac{1}{2}}+L_{i+\frac{1}{2}})+1440hm_{i-\frac{1}{2}}+240h^2M_{i-\frac{1}{2}}+1440(y_{i-\frac{1}{2}}-y_{i+\frac{1}{2}})],$$

$$e_i=-\frac{1}{11520h}[-7h^5(L_{i-\frac{1}{2}}+L_{i+\frac{1}{2}})-480h^2(M_{i-\frac{1}{2}}-M_i+\frac{1}{2})+11520(y_{i-\frac{1}{2}}-y_{i+\frac{1}{2}})],$$

$$f_i=-\frac{1}{4608}[h^5(L_{i-\frac{1}{2}}+L_{i+\frac{1}{2}})+1440hm_{i-\frac{1}{2}}+48h^2(4M_{i-\frac{1}{2}}-M_{i+\frac{1}{2}})+3744y_{i-\frac{1}{2}}+864y_{i+\frac{1}{2}}].$$

Continuity condition of the second, third and fourth derivatives at $(x_{i-\frac{1}{2}},x_{i+\frac{1}{2}})$, that is $S_{i-1}^{(\lambda)}(x_{i-\frac{1}{2}})=S_i^{(\lambda)}(x_{i-\frac{1}{2}})$, Where $\lambda=1,3,4$, yields the following equations:

$$-\frac{h^4}{720}(L_{i-\frac{3}{2}}+L_{i-\frac{1}{2}})-(m_{i-\frac{3}{2}}+m_{i-\frac{1}{2}})-\frac{h}{6}(M_{i-\frac{3}{2}}-M_{i-\frac{1}{2}})-\frac{2}{h}(y_{i-\frac{3}{2}}-y_{i-\frac{1}{2}})=0, \quad (5)$$

$$-\frac{1}{60h^3}[h^5(3L_{i-\frac{3}{2}}+L_{i-\frac{1}{2}}-2L_{i+\frac{1}{2}})+720h(m_{i-\frac{3}{2}}-m_{i-\frac{1}{2}})+60h^2(3M_{i-\frac{3}{2}}+8M_{i-\frac{1}{2}}+M_{i+\frac{1}{2}})+720(y_{i-\frac{3}{2}}-y_{i-\frac{1}{2}})]=0, \quad (6)$$

$$-\frac{1}{30h^4}[h^5(8L_{i-\frac{3}{2}}+15L_{i-\frac{1}{2}}+7L_{i+\frac{1}{2}})+720h(m_{i-\frac{3}{2}}+m_{i-\frac{1}{2}})+120h^2(2M_{i-\frac{3}{2}}-M_{i-\frac{1}{2}}-M_{i+\frac{1}{2}})+720(y_{i-\frac{3}{2}}-2y_{i-\frac{1}{2}}+y_{i+\frac{1}{2}})]=0. \quad (7)$$

In order to eliminate m's and M's in the above equation (5-7) get nine additional equation i is replaced by $i-1,i+1,i+2$ in each of the Eqs. (5),(6) and (7).After lengthy calculations, the following recurrence relation is obtained:

$$-y_{i-\frac{5}{2}}+5y_{i-\frac{3}{2}}-10y_{i-\frac{1}{2}}+10y_{i+\frac{1}{2}}-5y_{i+\frac{3}{2}}+y_{i+\frac{5}{2}}=\frac{h^5}{240}[L_{i-\frac{5}{2}}+27L_{i-\frac{3}{2}}+92L_{i-\frac{1}{2}}+92l_{i+\frac{1}{2}}+27L_{i+\frac{3}{2}}+L_{i+\frac{5}{2}}]$$

$i=3,4,...,n-3$ (8)

3 Development of the boundary formulas

Liner system equation (8) consist of (n − 1) unknown, so that to obtain unique solution we need five more equations to be associate with equation (8) so that we can develop the boundary formulas of different orders, but for sake of briefness here we develop the fifth order boundary formulas so that we define the following identity:

$$y_0-\frac{3675}{2816}y_{\frac{1}{2}}+\frac{1225}{2816}y_{\frac{3}{2}}-\frac{441}{2816}y_{\frac{5}{2}}+\frac{75}{2816}y_{\frac{7}{2}}+\frac{105}{352}hy_0'=h^5\left(\frac{3185}{270336}y_{\frac{1}{2}}^{(5)}+\frac{1225}{270336}y_{\frac{7}{2}}^{(5)}\right)+t_1, \tag{9}$$

$$y_0+\frac{231915}{22912}y_{\frac{1}{2}}-\frac{19835}{716}y_{\frac{3}{2}}+\frac{314919}{11456}y_{\frac{5}{2}}-\frac{38985}{2864}y_{\frac{7}{2}}+\frac{61935}{22912}y_{\frac{9}{2}}+\frac{2325}{716}hy_0'=h^5\left(\frac{147593}{91648}y_{\frac{1}{2}}^{(5)}+y_{\frac{9}{2}}^{(5)}\right)+t_2, \tag{10}$$

$$\frac{653398333}{27804032}y_{\frac{1}{2}}-\frac{2694675225}{62559072}y_{\frac{3}{2}}+\frac{331090499}{13902016}y_{\frac{5}{2}}-\frac{24885835}{6951008}y_{\frac{7}{2}}-\frac{666095315}{250236288}y_{\frac{9}{2}}+\frac{16069955}{1303314}hy_0'+$$
$$\frac{8429465}{1737752}h^2y_0''+y_{\frac{11}{2}}+y_0=h^5\left(y_{\frac{1}{2}}^{(5)}+y_{\frac{11}{2}}^{(5)}\right)+t_3, \tag{11}$$

$$-\frac{1808365}{664448}y_{n-\frac{9}{2}}+\frac{1142715}{83056}y_{n-\frac{7}{2}}-\frac{9299349}{332224}y_{n-\frac{5}{2}}-\frac{5972265}{20764}y_{n-\frac{3}{2}}-\frac{8511585}{664448}y_{n-\frac{1}{2}}+y_n+\frac{55455}{20764}hy_n'=$$
$$h^5\left(y_{n-\frac{9}{2}}^{(5)}+\frac{4240507}{267792}y_{n-\frac{1}{2}}^{(5)}\right)+t_{n-2}, \tag{12}$$

$$\frac{75}{2816}y_{n-\frac{7}{2}}-\frac{441}{2816}y_{n-\frac{5}{2}}+\frac{1225}{2816}y_{n-\frac{3}{2}}-\frac{3675}{2816}y_{n-\frac{1}{2}}+y_n+\frac{105}{352}hy_n'=h^5\left(\frac{-1225}{270336}y_{n-\frac{7}{2}}^{(5)}-\frac{3185}{270336}y_{n-\frac{1}{2}}^{(5)}\right)+t_{n-1}, \tag{13}$$

Where:

$$t_1=-\frac{4585}{360448}h^7y_0^{(7)},\ t_2=-\frac{22718127}{5132288}h^7y_0^{(7)},\ t_3=-\frac{20585178253}{3114051584}h^7y_0^{(7)},$$
$$t_{n-2}=-\frac{657489453}{14883652}h^7y_n^{(7)},\ t_{n-1}=-\frac{4585}{360448}h^7y_n^{(7)}. \tag{14}$$

4 polynomial spline solution

Using the system defined by (9) − (13) and (8) along with the consideration of BVP (1), the following system in matrix form is obtained:

$$(A+h^5BF)Y=C$$

Where $Y=[y_{\frac{1}{2}},y_{\frac{3}{2}},...,y_{n-\frac{1}{2}}]^T$ and $C=[c_{\frac{1}{2}},c_{\frac{3}{2}},...,c_{n-\frac{1}{2}}]^T$ and

$$A=\begin{bmatrix}
-\frac{3675}{2816} & \frac{1225}{2816} & -\frac{441}{2816} & \frac{75}{2816} & 0 & 0 & 0 & . & . & . & 0\\
\frac{231915}{22912} & -\frac{19835}{716} & \frac{314919}{11456} & -\frac{38985}{2864} & \frac{61935}{22912} & 0 & 0 & . & . & . & 0\\
\frac{653398333}{27804032} & -\frac{2694675225}{62559072} & \frac{331090499}{13902016} & -\frac{24885835}{6951008} & -\frac{666095315}{250236288} & 0 & 0 & . & . & . & 0\\
-1 & 5 & -10 & 10 & -5 & 1 & & & & & \\
0 & -1 & 5 & -10 & 10 & -5 & 1 & & & & \\
0 & .. & .. & .. & .. & .. & .. & .. & .. & .. & 0\\
0 & . & . & . & . & . & . & . & . & . & 0\\
0 & . & . & . & . & . & . & . & . & . & 0\\
0 & .. & .. & .. & 0 & -1 & 5 & -10 & 10 & -5 & 1\\
0 & .. & .. & .. & 0 & 0 & -\frac{1808365}{664448} & \frac{1142715}{83056} & -\frac{9299349}{332224} & -\frac{5972265}{20764} & -\frac{8511585}{664448}\\
0 & .. & .. & .. & 0 & 0 & 0 & \frac{75}{2816} & -\frac{441}{2816} & \frac{1225}{2816} & -\frac{3675}{2816}
\end{bmatrix}$$

$$B=\begin{bmatrix} \frac{3185}{270336} & b_1' & b_2' & \frac{1225}{270336} & 0 & 0 & 0 & .. & .. & 0 \\ \frac{147593}{91648} & b_1'' & b_3'' & b_4'' & 1 & 0 & 0 & .. & .. & 0 \\ 1 & b_1''' & b_2''' & b_3''' & b_4''' & 1 & 0 & .. & .. & 0 \\ \frac{1}{240} & \frac{27}{240} & \frac{92}{240} & \frac{92}{240} & \frac{27}{240} & \frac{1}{240} & & & & \\ 0 & \frac{1}{240} & \frac{27}{240} & \frac{92}{240} & \frac{92}{240} & \frac{27}{240} & \frac{1}{240} & & & \\ 0 & .. & .. & .. & .. & .. & .. & .. & .. & 0 \\ 0 & . & . & . & . & . & . & . & . & 0 \\ 0 & .. & .. & 0 & \frac{1}{240} & \frac{27}{240} & \frac{92}{240} & \frac{92}{240} & \frac{27}{240} & \frac{1}{240} \\ 0 & .. & .. & 0 & 0 & 1 & b_3^* & b_2^* & b_1^* & \frac{4240507}{267792} \\ 0 & .. & .. & 0 & 0 & 0 & \frac{-1225}{270336} & b_2^{\bullet} & b_1^{\bullet} & \frac{-3185}{270336} \end{bmatrix}$$

$$F = diag(f_i)\ ,\ i = 1,2,3,...,n-1.$$

The vector C is defined by

$$c_{\frac{1}{2}} = h^5\left(\frac{3185}{270336} g_{\frac{1}{2}} + \frac{1225}{270336} g_{\frac{7}{2}}\right) - \frac{105}{352} h y_0^{(1)} - y_0,$$

$$c_{\frac{3}{2}} = h^5\left(\frac{147593}{91648} g_{\frac{1}{2}} + g_{\frac{9}{2}}\right) - \frac{2325}{716} h y_0^{(1)} - y_0,$$

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$$c_{i+\frac{1}{2}} = \frac{h^5}{240}\left(g_{i-\frac{5}{2}} + 27 g_{i-\frac{3}{2}} + 92 g_{i-\frac{1}{2}} + 92 g_{i+\frac{1}{2}} + 27 g_{i+\frac{3}{2}} + g_{i+\frac{5}{2}}\right), \quad i = 3,4,...,(n-3)$$

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$$c_{n-\frac{3}{2}} = h^5\left(g_{n-\frac{9}{2}} + \frac{4240507}{2657792} g_{n-\frac{1}{2}}\right) - \frac{55455}{20764} h y_n^{(1)} - y_n,$$

$$c_{n-\frac{1}{2}} = \frac{-h^5}{270336}\left(1225 g_{n-\frac{7}{2}} + 3185 g_{n-\frac{1}{2}}\right) + \frac{105}{352} h y_n^{(1)} - y_n.$$

5 NUMERICAL RESULTS

Example 1. We Consider the following boundary-value problem

$$y^{(5)}(x) + xy(x) = (1-x)Cosx - 5Sinx + xSinx - x^{(2)}Sinx \quad , \quad 0 \le x \le 1$$

$$y(0) = y(1) = 0,\ y'(0) = 1,\ y'(1) = -Sin(1),\ y''(0) = -2 \tag{15}$$

The analytic solution of the above system is $y(x) = (1-x)Sinx$. It is evident from Table 1 that the maximum errors in absolute values are less than those presented by [13].

Example 2. We Consider the following boundary-value problem

$$y^{(5)}(x) + y(x) = -4e^x(x^2Cosx - 2xCosx - 9Cosx) - e^x(3x^2Sinx + 34xSinx + 3Sinx)\ ,\ 0 \le x \le 1$$

$$y(0) = y(1) = 0,\ y'(0) = 1,\ y'(1) = 0,\ y''(0) = -2 \tag{16}$$

The analytic solution of the above system is $y(x) = (1-x^2)e^x Sinx$. It is evident from Table 2 that the maximum errors in absolute values are less than those presented by [13].

Example 3. We Consider the following boundary-value problem

$$y^{(5)}(x) - y(x) = -(15+10x)e^{x}, \qquad 0 \le x \le 1$$
$$y(0) = y(1) = 0, y'(0) = 1, y'(1) = -e, y''(0) = 0 \tag{17}$$

The analytic solution for this problem is $y(x) = x(1-x)e^{x}$. This example have be solved by our presented method with value of, $h = \frac{1}{10}, \ldots, \frac{1}{80}$, the computed solution are compared with exact solution , The maximum absolute errors associated with y_i for the system (17) are summarized in Table (3) and compared with [9,10].

Table1: Observed maximum absolute errors for example 1.

h	[13]	Our methods
$1/12$	9.9638(−5)	3.188(−7)
$1/24$	1.3881(−5)	3.768(−8)
$1/48$	1.8200(−6)	6.138(−8)
$1/96$	2.3269(−7)	1.446(−9)
$1/12$	2.9278(−8)	3.967(−9)

Table2: Observed maximum absolute errors for example 2.

h	[13]	Our methods
$1/12$	9.0155(−4)	6.453(−5)
$1/24$	7.0710(−5)	2.077(−6)
$1/48$	5.4310(−6)	5.383(−7)
$1/96$	4.4069(−7)	1.483(−8)
$1/12$	4.0739(−8)	3.299(−9)

Table3: Observed maximum absolute errors for example 3.

h	Our method	[10]	[9]
$1/10$	5.466(−5)	1.286(−4)	1.570(−1)
$1/20$	8.496(−6)	2.791(−5)	7.47(−2)
$1/40$	8.653(−7)	9.398(−6)	2.08(−2)
$1/80$	4.856(−7)	-	-

Conclusion

We approximate solution of the fifth-order linear boundary-value problems by using polynomial spline. The new methods enable us to approximate the solution at every point of the range of integration . The method is compared with that developed by et al [13],[10] and Caglar et al.[9] considering the same examples.Tables 1-3 shows that our methods produced better result the sense that max$|e_i|$= max$|y(x_i) - y_i|$ is in comparison with the methods in [9,10,13].

REFERENCES

[1] Karageorghis, A. Philips, T.N. Davise, A.R., "Spectral collocation methods for the primary two-point boundary-value problem in modelling viscoelastic flows" ,Int .J. Numer. Methods Engng,Vol.26,pp.805-813, 1988.

[2] Davies, A.R. Karageorghis, A. Philips, T.N.,"Spectral Galerkin methods for the primary two-point boundary-value problem in modelling viscoelastic flows",Int.J.Numer.Methods Engng,Vol.26,pp.647-662, 1988.

[3] Siddiqi, S.S. Twizell, E.H.," Spline solution of linear sixth-order boundary value problems",Int.J. Comput. Math,Vol.60,No.3,pp.295-304,1996.

[4] Siddiqi, S.S. Twizell, E.H.," Spline solution of linear eighth-order boundary value problems", Comput.Methods Appl .Mech. Engng ,Vol.131,pp.309-325, 1996.

[5] Siddiqi, S.S. Twizell, E.H.," Spline solution of linear tenth-order boundary value problems ",Int.J. Comput. Math,Vol.60,pp.395-304, 1996.

[6] Siddiqi, S.S. Twizell, E.H.," Spline solution of linear twelfth-order boundary value problems",Int.J.Comput.Appl.Math,Vol.78,pp.371-390,1997.

[7] Rashidinia,J. Jalilian,R. Farajeyan.K.," solutions of fifth-order boundary value problems using Non-Polynomial spline ", Journal of Information and Computing Science, Vol.4,pp. 265-274, 2009.

[8] Farajeyan,K. Rafati,M.N.," Numerical solution of tenth-order boundary-value problems in off step points", J. Basic.Appl.Sci. Res,Vol.2,No.6,pp.6235-6244,2012.

[9] Caglar, H.N. Caglar, S.H. Twizell, E.H.,"The numerical solution of fifth-order boundary -value problems with sixth-degree B-spline functions", Appl. Math. Lett,Vol.12,pp.25-30, 1999.

[10] Siddiqi, S.S. Akram, G.," solution of fifth-order boundary value problems using non-polynomial spline technique",Appl.Math.Comput,Vol.175,No.2,pp.1574-1581, 2006.

[11] Siraj-Ul-Islam, Khan, M.A. ," A numerical method based on polynomial sextic spline functions for the solution of special fifth-order boundaryvalue problems",Appl. Math. Comput.Vol.181,pp.336-361, 2006.

[12]Khan, M.A. Siraj-ul-Islam, Tirmizi, I.A. Twizell, E.H.," Saadat Ashraf, class of methods based on non polynomial sextic spline functions for the solution of a special fifth-order boundary-value problems", J.Math.Anal.Appl,Vol.321,pp.651-660, 2006.

[13]Siddiqi, S.S. Akram, G.Elahi, A.," Quartic spline solution of linear fifth-order boundary value problems", Appl. Math. Comput.Vol.196,No.1,pp.214-220,2008.