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A Five-Point C^4 Non-Stationary Subdivision Scheme

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ABSTRACT

The paper proposes, a five-point binary non-stationary approximating subdivision scheme, which can generate the family of C^4 limiting curves. The asymptotic equivalence method has been applied in order to determine the smoothness and convergence of that scheme. The proposed scheme can be considered as the non-stationary counter part of the first binary 5-point approximating stationary scheme. Some examples are illustrated for comparison, with the existing 5-point binary approximating schemes, to show the usefulness of proposed scheme.
It is also being concluded that the existing 5-point binary stationary schemes only can generate C^1 and C^4 limiting curves for some particular values while proposed scheme can generate the family of limiting curves. Moreover, the proposed scheme has ability to reproduce conic sections (in particular circles and ellipses), trigonometric polynomials and trigonometric splines.
KEYWORDS: binary, non-stationary, approximating subdivision scheme, mask, convergence and smoothness.

1. INTRODUCTION

In today's world no one can deny the importance of geometric modelling, computer graphics and computer aided geometric designing (CAGD), as they play vital role in almost every field of life like computer applications, medical image processing, scientific visualization, reverse engineering, robotics etc.

For the goal of creating smooth curves or surfaces, subdivision scheme are one of the pleasing method these days and are easy to operate. Their advantage lies in the simplicity and small number of floating-point operations. A subdivision scheme defines a curve from initial control polygon or a surface from an initial control mesh by subdividing them according to some refining rules, recursively.

In the field of stationary subdivision schemes, the first recursively corner cutting piecewise linear approximating scheme was presented, in mid of nineteen's, by De Rham [7]. Afterwards, another corner cutting linear approximating scheme was developed by Chaikin [3]. Both of these schemes generate the limit curves of C^1 continuity but the difference between them is that the curvature of de Rahm's scheme diverges and Chaikin's piecewise continuous. A lot of work has been done on stationary schemes as compared to non-stationary schemes. Some of stationary work related to approximating scheme is highlighted over here. Hassan et al. [12] introduced a 3-point binary and ternary approximating schemes, which generate C^3 and C^2 limiting curves, respectively. They also presented 3-point interpolating schemes, which generates C^1 limiting curves. The 4-point binary approximating scheme was introduced by Dyn et al. [9] which generates $C2$ limiting curves, it reproduces cubic polynomials and has a basic limit function with support [-4 3].

In the field of non-stationary subdivision schemes, Jena et al. [13] introduced a 4-point binary interpolatory non-stationary C^1 subdivision scheme which was the generalization of four point stationary subdivision scheme developed by Deslauriers and Dubuc [8]. Chen et al. [4] introduced a novel non-stationary subdivision scheme based on the subdivision generation analysis of B-spline curves and surfaces. In 2007, Beccari et al. [2] presented a 4-point binary non-stationary interpolating subdivision scheme, using tension parameter, which can produce certain families of conics and cubic polynomials. They also developed a 4-point ternary interpolating non-stationary subdivision scheme, in the same year, that generates C^2 continuous limit curves showing considerable variation of shapes with a tension parameter [1]. In 2009, Daniel and Shunmugaraj [6] introduced a non-stationary 2-point approximating scheme that generates C^1 limiting curve and two 3-point binary schemes that generate C^2 and C^3 limiting curves, respectively. The masks of these schemes were defined in terms of trigonometric B-spline basis functions. Conti and Romani [5] presented a new family of 6-point interpolatory non-stationary subdivision scheme using cubic exponential B-spline symbol generating functions that can reproduce conic sections. In this paper, a five-point non-stationary approximating scheme has been introduced, which is capable to reproduce the family of curves for geometric modelling and designing.

In the following, the structure of five-point binary approximating subdivision scheme is presented

$$\left.\begin{aligned} p_{2i}^{k+1} &= a_0^k p_{i-2}^k + a_1^k p_{i-1}^k + a_2^k p_i^k + a_3^k p_{i+1}^k + a_4^k p_{i+2}^k \\ p_{2i+1}^{k+1} &= a_4^k p_{i-2}^k + a_3^k p_{i-1}^k + a_2^k p_i^k + a_1^k p_{i+1}^k + a_0^k p_{i+2}^k \end{aligned}\right\} \tag{1.1}$$

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where the co-efficient $\left\{a_j^k\right\}_{j=0,1,2,3,4}$ are chosen to satisfy the relation $\sum_{j=0}^{4} a_j^k = 1.$

This paper is organized as follows, in section 2 the basic notion and definitions of binary subdivision scheme are considered. The proposed binary scheme is being presented in section 3 and its convergence analysis is discussed in section 4. Some figures are depicted, in section 5, for comparison. In section 6, the symmetry of basic limit function of the scheme is discussed. The conclusion is drawn in section 7.

2. Preliminaries

In subdivision scheme, the set of control points $P^k = \left\{ p_i^k \in Z,\ i \in Z^m \right\}$ (where m = 1, for the curve case and m = 2, for the surface case) of polygon at *kth* level is mapped to a refined polygon to generate the new set of control points $P^k = \left\{ p_i^k \in Z,\ i \in Z^m \right\}$at the *(k+1)*st level by applying the following subdivision rule,

$$P^{k+1} = \left\{ S_{a^k} P^k \right\} = \sum_{j \in Z} a_{i-2j}^k p_j^k, \qquad \forall i \in Z, \tag{2.1}$$

where the set $\left\{ a_i^k : i \in Z^m,\ a_i^k \neq 0 \right\}$ is finite for every $k \in Z_+$. If the masks of the scheme are independent of k, then the scheme is called stationary $\left\{ S_a \right\}$, otherwise it is called non-stationary $\left\{ S_{a^k} \right\}$. The equation (2.1) is called the compact form of scheme (1.1) in single equation.

Definition 2.1 [10] The mask of non-stationary scheme $\left\{ S_{a^k} \right\}$ at *kth* level is a^k, then the set $\left\{ i \in Z,\ a_i^k \neq 0 \right\}$ is called the support of the mask a^k.

Definition 2.2 [10] A non-stationary subdivision scheme $\left\{ S_{a^k} \right\}$ is said to be convergent if for every initial data $P^0 \in l^\infty$ there exists a limit function $f \in C^m(R)$ such that

$$\lim_{k \to \infty} \sup_{i \in Z} \left| P_i^k - f\left(2^{-k} i\right) \right| = 0,$$

and f is not identically zero for some initial data P^0.

Definition 2.3 [10] Two binary subdivision schemes $\left\{ S_{a^k} \right\}$ and $\left\{ S_{b^k} \right\}$ are said to be asymptotically equivalent if

$$\sum_{k=1}^{\infty} \left\| S_{a^k} - S_{b^k} \right\|_\infty < \infty,$$

Where

$$\left\| S_{a^k} \right\|_\infty = \max \left\{ \sum_{i \in Z} \left| a_{2i}^k \right|, \sum_{i \in Z} \left| a_{2i+1}^k \right| \right\}.$$

Theorem 2.4 [13] The non-stationary binary scheme $\left\{ S_{a^k} \right\}$ and stationary scheme $\left\{ S_a \right\}$ are said to be asymptotically equivalent if they have finite mask of the same support. If stationary scheme $\left\{ S_a \right\}$ is C^m and

$$\sum_{k=0}^{\infty} 2^{mk} \left\| S_{a^k} - S_a \right\|_\infty < \infty,$$

then non-stationary binary scheme $\left\{ S_{a^k} \right\}$ is also C^m.

Definition 2.5 [14] Let $m > n > 0$ and $0 < \alpha < \frac{\pi}{n}$, then Uniform Trigonometric B-splines $\left\{ T_j^n(x;\alpha) \right\}_{j=1}^{m}$ of order n associated with the knot sequence $\Delta := \left\{ t_i = i\alpha : i = 0,1,2,\ldots,m+n \right\}$ with the mesh size α are defined by the recurrence relation,

$$T_0^1(x;\alpha) = \begin{cases} 1, & x \in [0,\alpha) \\ 0, & otherwise \end{cases}$$

for $1 < r \le n$,

$$T_0^r(x;\alpha) = \frac{1}{\sin\left((r-1)\alpha\right)}\left\{\sin x T_0^{r-1}(x;\alpha) + \sin(t_r - x) T_0^{r-1}(x-\alpha;\alpha)\right\} \tag{3.1}$$

The trigonometric B-spline $T_j^n(x;\alpha)$ is supported on [t_j , t_{j+n}] and it is the interior of its support. Moreover, $\left\{T_j^n\right\}_{j=1}^m$ are linearly independent set o the interval [t_{n-1}, t_{m+1}]. Uniform trigonometric spline $S(x)$, on the interval [t_{n-1}, t_{m+1}], has a unique representation of the form $S(x) = \sum_{j=0}^{m} p_j T_j^n(x;\alpha),\ p_j \in R$.

3. The Scheme

In this section, a five-point binary approximating non-stationary subdivision scheme is presented and masks of that scheme can be calculated from the quintic uniform trigonometric B-spline, for any value of k , using the relation

$$\gamma_0^k(\alpha) = T_0^5\left((4-i)\frac{\alpha}{2^k} + \frac{\alpha}{2^{k+2}};\frac{\alpha}{2^k}\right) \quad i = 0,1,\ldots,4\,, \tag{3.1}$$

where $T_0^5\left(x;\frac{\alpha}{2^k}\right)$ with mesh size $\left(\frac{\alpha}{2^k}\right)$ can be calculated from above relation. The proposed scheme is defined, for some value of $\alpha \in]0,\frac{\pi}{3}[$, as

$$\left.\begin{aligned} p_{2i}^{k+1} &= \eta_0^k p_{i-2}^k + \eta_1^k p_{i-1}^k + \eta_2^k p_i^k + \eta_3^k p_{i+1}^k + \eta_4^k p_{i+2}^k \\ p_{2i+1}^{k+1} &= \eta_4^k p_{i-2}^k + \eta_3^k p_{i-1}^k + \eta_2^k p_i^k + \eta_1^k p_{i+1}^k + \eta_0^k p_{i+2}^k \end{aligned}\right\}. \tag{3.2}$$

Where

$$\eta_0^k = \frac{\sin^4 \frac{3\alpha}{2^{k+2}}}{\sin\frac{\alpha}{2^k}\sin\frac{2\alpha}{2^k}\sin\frac{3\alpha}{2^k}\sin\frac{4\alpha}{2^k}},$$

$$\eta_1^k = \frac{\sin^3\frac{3\alpha}{2^{k+2}}\sin\frac{13\alpha}{2^{k+2}} + \sin^2\frac{3\alpha}{2^{k+2}}\sin\frac{7\alpha}{2^{k+2}}\sin\frac{9\alpha}{2^{k+2}} + \sin\frac{3\alpha}{2^{k+2}}\sin\frac{5\alpha}{2^{k+2}}\sin^2\frac{7\alpha}{2^{k+2}} + \sin\frac{\alpha}{2^{k+2}}\sin^3\frac{7\alpha}{2^{k+2}}}{\sin\frac{\alpha}{2^k}\sin\frac{2\alpha}{2^k}\sin\frac{3\alpha}{2^k}\sin\frac{4\alpha}{2^k}},$$

$$\eta_2^k = \frac{\sin^2\frac{3\alpha}{2^{k+2}}\sin^2\frac{9\alpha}{2^{k+2}} + \sin\frac{3\alpha}{2^{k+2}}\sin\frac{5\alpha}{2^{k+2}}\sin\frac{7\alpha}{2^{k+2}}\sin\frac{9\alpha}{2^{k+2}} + \sin\frac{\alpha}{2^{k+2}}\sin\frac{5\alpha}{2^{k+2}}\sin^2\frac{7\alpha}{2^{k+2}}\sin\frac{9\alpha}{2^{k+2}} + \sin\frac{3\alpha}{2^{k+2}}\sin^2\frac{5\alpha}{2^{k+2}}\sin\frac{11\alpha}{2^{k+2}}}{\sin\frac{\alpha}{2^k}\sin\frac{2\alpha}{2^k}\sin\frac{3\alpha}{2^k}\sin\frac{4\alpha}{2^k}}$$

$$+\frac{\sin\frac{\alpha}{2^{k+2}}\sin\frac{5\alpha}{2^{k+2}}\sin\frac{7\alpha}{2^{k+2}}\sin\frac{11\alpha}{2^{k+2}} + \sin^2\frac{\alpha}{2^{k+2}}\sin^2\frac{11\alpha}{2^{k+2}}}{\sin\frac{\alpha}{2^k}\sin\frac{2\alpha}{2^k}\sin\frac{3\alpha}{2^k}\sin\frac{4\alpha}{2^k}}$$

$$\eta_3^k = \frac{\sin\frac{3\alpha}{2^{k+2}}\sin^3\frac{5\alpha}{2^{k+2}} + \sin\frac{\alpha}{2^{k+2}}\sin^2\frac{5\alpha}{2^{k+2}}\sin\frac{7\alpha}{2^{k+2}} + \sin^2\frac{\alpha}{2^{k+2}}\sin\frac{5\alpha}{2^{k+2}}\sin\frac{11\alpha}{2^{k+2}} + \sin^3\frac{\alpha}{2^{k+2}}\sin\frac{15\alpha}{2^{k+2}}}{\sin\frac{\alpha}{2^k}\sin\frac{2\alpha}{2^k}\sin\frac{3\alpha}{2^k}\sin\frac{4\alpha}{2^k}},$$

$$\eta_4^k = \frac{\sin^4\frac{\alpha}{2^{k+2}}}{\sin\frac{\alpha}{2^k}\sin\frac{2\alpha}{2^k}\sin\frac{3\alpha}{2^k}\sin\frac{4\alpha}{2^k}}.$$

The proposed scheme can be considered as the non-stationary counterpart of the stationary five-point binary approximating scheme, which was introduced by Siddiqi and Ahmad [15] and its subdivision rules to refine the control polygon are defined as

$$\left.\begin{aligned} p_{2i}^{k+1} &= \tfrac{27}{2048} p_{i-2}^k + \tfrac{499}{1536} p_{i-1}^k + \tfrac{1723}{3072} p_i^k + \tfrac{155}{1536} p_{i+1}^k + \tfrac{1}{6144} p_{i+2}^k \\ p_{2i+1}^{k+1} &= \tfrac{1}{6144} p_{i-2}^k + \tfrac{155}{1536} p_{i-1}^k + \tfrac{1723}{3072} p_i^k + \tfrac{499}{1536} p_{i+1}^k + \tfrac{27}{2048} p_{i+2}^k \end{aligned}\right\}. \tag{3.3}$$

Since the weights of the proposed scheme are bounded by the coefficient of the mask of the scheme (3.3). So, we can write

$$\eta_0^k \to \frac{27}{2048},\ \eta_1^k \to \frac{499}{1536},\ \eta_2^k \to \frac{1723}{3072},\ \eta_3^k \to \frac{155}{1536} \text{ and } \eta_4^k \to \frac{1}{6144}.$$

The proof of $\eta_0^k \to \frac{27}{2048}$ follows from the Lemma (4.1) and the proofs of $\eta_1^k \to \frac{499}{1536}$, $\eta_2^k \to \frac{1723}{3072}$, $\eta_3^k \to \frac{155}{1536}$ and $\eta_4^k \to \frac{1}{6144}$ can be obtained similarly. Moreover, the support size of non-stationary scheme is similar to that of binary approximating schemes developed by quintic polynomials with the same approximation order.

4. Convergence Analysis

Following [10,13] the theory of asymptotic equivalence is used to investigate the convergence and smoothness of the scheme. Some estimations of $\eta_i^k, i = 0,1,2,3,4$ are used in order to prove the convergence of the proposed scheme, are given in the following Lemmas. To prove the Lemmas, the following three inequalities are used.

$$\frac{\sin a}{\sin b} \geq \frac{a}{b} \quad \text{for} \quad 0 < a < b < \frac{\pi}{2},$$

$$\theta \csc\theta \leq t \csc t \quad \text{for} \quad 0 < \theta < t < \frac{\pi}{2}$$

and

$$\cos x \leq \frac{\sin x}{x} \quad \text{for} \quad 0 < x < \frac{\pi}{2}.$$

Lemma 4.1. For $k \geq 0$ and $0 < \alpha < \frac{\pi}{2}$.

(i) $$\frac{27}{2048} \leq \eta_0^k \leq \frac{27}{2048} \frac{1}{\cos^4\left(\frac{12\alpha}{2^k}\right)}$$

(ii) $$\frac{499}{1536} \leq \eta_1^k \leq \frac{499}{1536} \frac{1}{\cos^4\left(\frac{12\alpha}{2^k}\right)}$$

(iii) $$\frac{1723}{3072} \leq \eta_2^k \leq \frac{1723}{3072} \frac{1}{\cos^4\left(\frac{12\alpha}{2^k}\right)}$$

(iv) $$\frac{155}{1536} \leq \eta_3^k \leq \frac{155}{1536} \frac{1}{\cos^4\left(\frac{12\alpha}{2^k}\right)}$$

(v) $$\frac{1}{6144} \leq \eta_4^k \leq \frac{1}{6144} \frac{1}{\cos^4\left(\frac{12\alpha}{2^k}\right)}$$

Proof. To prove the inequality (i), we can write

$$\eta_0^k = \frac{\sin^4\left(\frac{3\alpha}{2^{k+2}}\right)}{\sin\left(\frac{\alpha}{2^k}\right)\sin\left(\frac{2\alpha}{2^k}\right)\sin\left(\frac{3\alpha}{2^k}\right)\sin\left(\frac{4\alpha}{2^k}\right)} \geq \frac{\left(\frac{3\alpha}{2^{k+2}}\right)^4}{\left(\frac{\alpha}{2^k}\right)\left(\frac{2\alpha}{2^k}\right)\left(\frac{3\alpha}{2^k}\right)\left(\frac{4\alpha}{2^k}\right)} = \frac{27}{2048}$$

and

$$\begin{aligned}
\eta_0^k &\leq \frac{81\alpha^4}{2^{4k+8}} \csc\left(\frac{\alpha}{2^k}\right)\csc\left(\frac{2\alpha}{2^k}\right)\csc\left(\frac{3\alpha}{2^k}\right)\csc\left(\frac{4\alpha}{2^k}\right) \\
&\leq \frac{81\alpha^4}{2^{4k+8}}\, 864 \csc^4\left(\frac{12\alpha}{2^k}\right) \\
&\leq \frac{81\alpha^4}{2^{4k+8}}\, 864 \frac{1}{\left(\frac{12\alpha}{2^k}\right)^4 \cos^4\left(\frac{12\alpha}{2^k}\right)} \\
&= \frac{27}{2048} \frac{1}{\cos^4\left(\frac{12\alpha}{2^k}\right)}.
\end{aligned}$$

The proofs of (ii), (iii), (iv) and (v) can be obtained similarly.

Lemma 4.2 For some constants C_0, C_1, C_2, C_3 and C_4 independent of k, we have

(i) $\left|\eta_0^k - \frac{27}{2048}\right| \le C_0 \frac{1}{2^{2k}}$ (ii) $\left|\eta_1^k - \frac{499}{1536}\right| \le C_1 \frac{1}{2^{2k}}$

(iii) $\left|\eta_2^k - \frac{1723}{3072}\right| \le C_2 \frac{1}{2^{2k}}$ (iv) $\left|\eta_3^k - \frac{155}{1536}\right| \le C_3 \frac{1}{2^{2k}}$

(v) $\left|\eta_4^k - \frac{1}{6144}\right| \le C_4 \frac{1}{2^{2k}}$

Proof. To prove the inequality (i), Lemma (4.1) is being used

$$\text{(i)} \quad \left|\eta_0^k - \frac{27}{2048}\right| \le \frac{27}{2048}\left(\frac{1-\cos^4\left(\frac{12\alpha}{2^k}\right)}{\cos^4\alpha}\right)$$
$$\le \frac{27}{2048}\left(\frac{2\sin^2\left(\frac{12\alpha}{2^k}\right)}{\cos^4\alpha}\right)$$
$$\le \frac{7776\alpha^2}{2048\cos^4\alpha}\,\frac{1}{2^{2k}}.$$

The proofs of (ii), (iii), (iv) and (v) can be obtained similarly.

Lemma 4.3 The Laurent polynomial $a^k(z)$ of the scheme $\{S_{a^k}\}$ at the k^{th} level can be written as $a^k(z) = \left(\frac{1+z}{2}\right) b^k(z)$, where

$$b^k(z) = 2\left\{\begin{array}{l}\eta_0^k z^{-4} + \left(\eta_0^k - \eta_4^k\right) z^{-3} + \left(\eta_3^k + \eta_4^k - \eta_0^k\right) z^{-2} + \left(\eta_0^k + \eta_1^k - \eta_3^k - \eta_4^k\right) z^{-1} + \left(-\eta_0^k - \eta_1^k + \eta_2^k + \eta_3^k + \eta_4^k\right) \\ + \left(\eta_0^k + \eta_1^k - \eta_3^k - \eta_4^k\right) z + \left(\eta_3^k + \eta_4^k - \eta_0^k\right) z^2 + \left(\eta_0^k - \eta_4^k\right) z^3 + \eta_4^k z^4\end{array}\right\}.$$

Proof. Since,

$$a^k(z) = \eta_4^k z^{-5} + \eta_0^k z^{-4} + \eta_3^k z^{-3} + \eta_1^k z^{-2} + \eta_2^k z^{-1} + \eta_2^k + \eta_1^k z + \eta_3^k z^2 + \eta_0^k z^3 + \eta_4^k z^4.$$

Therefore using $\eta_0^k + \eta_1^k + \eta_2^k + \eta_3^k + \eta_4^k = 1$, $a^k(z) = \left(\frac{1+z}{2}\right) b^k(z)$ can be proved.

Lemma 4.4 The Laurent polynomial $a(z)$ of the scheme $\{S_a\}$ at the k^{th} level can be written as $a(z) = \left(\frac{1+z}{2}\right) b(z)$, where

$$b(z) = \frac{1}{3072}z^{-4} + \frac{5}{192}z^{-3} + \frac{45}{256}z^{-2} + \frac{91}{192}z^{-1} + \frac{995}{1536} + \frac{91}{192}z + \frac{45}{256}z^2 + \frac{5}{192}z^3 + \frac{1}{3072}z^4$$

and the subdivision scheme $\{S_b\}$ corresponding to the symbol $b(z)$ is C^3.

Proof. To prove that the subdivision scheme $\{S_b\}$ corresponding to the symbol $b(z)$ is C^3, we have

$$d(z) = \frac{8b(z)}{(1+z)^3} = \frac{1}{96}\left[\frac{1}{4} + 19z + \frac{115}{2}z^2 + 19z^3 + \frac{1}{4}z^4\right]$$

Since the norm of the subdivision scheme $\{S_d\}$ is

$$\|S_d\|_\infty = \max\left\{\sum_{i\in\mathbb{Z}}\left|d_{2i}^k\right|, \sum_{i\in\mathbb{Z}}\left|d_{2i+1}^k\right|\right\}$$
$$= \max\left\{\frac{29}{48}, \frac{19}{48}\right\} < 1.$$

Thus, the stationary scheme is C^3.

Theorem 4.1 The scheme defined by (3.2) converges and has smoothness C^4.

Proof. To prove the proposed scheme to be C^4, it is sufficient to show that the scheme corresponding to $b^k(z)$ is C^3 (Theorem (8) given by Dyn and Levin [11]) for binary scheme. Since $\{S_b\}$ is C^3 by Lemma (4.4), so in view of Theorem 2.4, it is sufficient to show for the convergence of binary non-stationary scheme $\{S_{b^k}\}$ that,

$$\sum_{k=0}^{\infty} 2^{3k} \left\| S_{b^k} - S_b \right\|_\infty < \infty .$$

Where

$$\left\| S_{b^k} - S_b \right\|_\infty = \max\left\{ \sum_{j\in Z}^{\infty} \left| b^k{}_{i+2j} - b_{i+2j} \right| : i = 0,1 \right\}.$$

From Lemmas (4.3) and (4.4), it can be written as

$$\sum_{j\in Z}^{\infty} \left| b^k{}_{2j} - b_{2j} \right| = 4\left|\eta_4^k - \frac{1}{6144}\right| + 4\left|\eta_3^k + \eta_4^k - \eta_0^k - \frac{45}{512}\right| + 2\left|-\eta_0^k - \eta_1^k + \eta_2^k + \eta_3^k + \eta_4^k - \frac{995}{3072}\right|$$
$$= 6\left|\eta_0^k - \frac{27}{2048}\right| + 6\left|\eta_1^k - \frac{499}{1536}\right| + 2\left|\eta_2^k - \frac{1723}{3072}\right| + 6\left|\eta_3^k - \frac{155}{1536}\right| + 10\left|\eta_4^k - \frac{1}{6144}\right|$$

and similarly, it may be noted that

$$\sum_{j\in Z}^{\infty} \left| b^k{}_{2j+1} - b_{2j+1} \right| = 8\left|\eta_0^k - \frac{27}{2048}\right| + 4\left|\eta_1^k - \frac{499}{1536}\right| + 4\left|\eta_3^k - \frac{155}{1536}\right| + 8\left|\eta_4^k - \frac{1}{6144}\right|$$

From (i), (ii), (iii), (iv) and (v) of Lemma (4.2), it can be written, $\sum_{k=0}^{\infty} 2^{3k}\left|\eta_0^k - \frac{27}{2048}\right| < \infty$, $\sum_{k=0}^{\infty} 2^{3k}\left|\eta_1^k - \frac{499}{1536}\right| < \infty$, $\sum_{k=0}^{\infty} 2^{3k}\left|\eta_2^k - \frac{1723}{3072}\right| < \infty$, $\sum_{k=0}^{\infty} 2^{3k}\left|\eta_3^k - \frac{155}{1536}\right| < \infty$ and $\sum_{k=0}^{\infty} 2^{3k}\left|\eta_4^k - \frac{1}{6144}\right| < \infty$. Thus, one can be followed $\sum_{k=0}^{\infty} 2^{3k}\left\| S_{b^k} - S_b \right\|_\infty < \infty$. Hence $\{S_{b^k}\}$ is C^3, so the proposed scheme $\{S_{a^k}\}$ is C^4.

5. Comparison

The comparison of the proposed scheme has been shown with the 5-point binary approximating stationary scheme, developed by Siddiqi et al. in [15] and [16].

It is concluded that the proposed scheme has the ability to regenerate or reproduce the conic sections like circles and ellipses etc. (see Fig. 1), where as the existing schemes do not have this ability. To draw the circle, we take hexagonal as control polygon. It is observed that the limit curves of proposed scheme almost overlap with unit circle (for numerical values see, Table 1). It also happens, for ellipse case.

The control polygons are represented by dotted lines and limit curves are taken after third iteration in each of the figure. The continuous lines represent the limit curves of the schemes [16] and [15] in 1st and 2nd columns of figure 2. While the limit curves of proposed scheme (3.2) are depicted in 3rd column of figure 2, taking $\alpha = \frac{\pi}{180}, \alpha = \frac{\pi}{20}, \alpha = \frac{\pi}{15}$ and $\alpha = \frac{\pi}{12}$. The comparison shows, the proposed scheme has the ability to reproduce the family of C^4 limiting curves whereas the existing 5-point stationary schemes do not have the ability to generate the family of curves (see also, Table 2).

Remark: The limit curve of proposed scheme, taking $\alpha = \frac{\pi}{180}$, coincide with the limit curves of the scheme, introduced in [15].

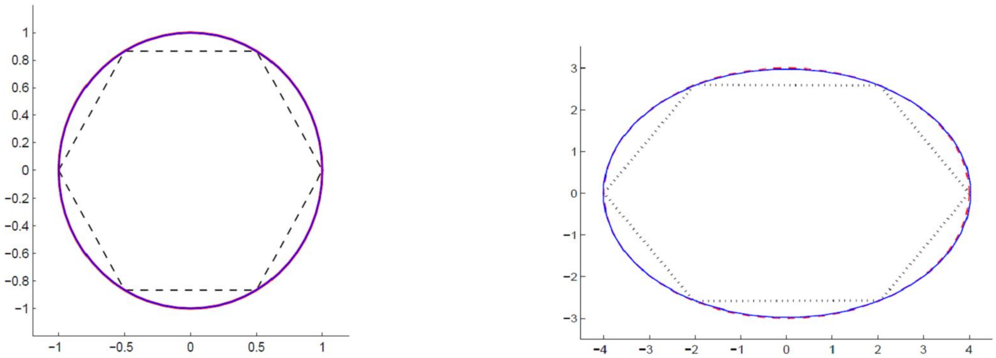


Figure 1: The reproduction of the unit circle and ellipse by the proposed scheme (3.2). The dotted lines represent the initial polygon.

X	Unit Circle	Proposed Scheme (3.2)	Error
0.0	1.000	0.9980	2.0e-3
0.1	0.9948	0.9933	1.5e-3
0.2	0.9794	0.9795	1.0e-4
0.3	0.9527	0.9526	1.0e-4
0.4	0.9161	0.9144	1.7e-3
0.5	0.8666	0.8636	2.4e-3
0.6	0.7992	0.7974	1.8e-3
0.7	0.7123	0.7112	1.1e-3
0.8	0.5990	0.5966	2.4e-3
0.9	0.4334	0.4354	2.0e-3

Table 1 - Numerical result of the proposed scheme with unit circle

6. Basic Limit Function and Properties

The basic limit function of the proposed scheme is the limit function for the data and is symmetric about y-axis

$$p_i^0 = \begin{cases} 1, & i = 0, \\ 0, & i \neq 0 \end{cases}$$

The symmetry of the basic limit function can be proved, in view of [2], in the following theorem.

Theorem 6.1. [2] The basic limit function F is symmetric about the Y-axis.

Proof. Let F denote the basic limit function and define $D_n := \left\{ \frac{i}{2^n}; i \in Z \right\}$ such that restriction of F to D_n satisfies $F\left(\frac{i}{2^n}\right) = p_i^n$ for all $i \in Z$. The symmetry of basic limit function is proved using induction on *n*.

It can be observed that $F(i) = F(-i)\ \forall i \in Z$, thus $F\left(\frac{i}{2^n}\right) = F\left(\frac{-i}{2^n}\right)\ \forall i \in Z$ and $n = 0$.

Assume that $F\left(\frac{i}{2^n}\right) = F\left(\frac{-i}{2^n}\right)\ \forall i \in Z$ and then $p_i^n = p_{-i}^n\ \forall i \in Z$. It may be observed that,

$$\begin{aligned} F\left(\frac{2i}{2^{n+1}}\right) &= p_{2i}^{n+1} \\ &= \eta_0^n p_{i-2}^n + \eta_1^n p_{i-1}^n + \eta_2^n p_i^n + \eta_3^n p_{i+1}^n + \eta_4^n p_{i+2}^n \\ &= \eta_4^n p_{-i-2}^n + \eta_3^n p_{-i-1}^n + \eta_2^n p_{-i}^n + \eta_1^n p_{-i+1}^n + \eta_0^n p_{-i+2}^n \\ &= p_{-2i}^{n+1} = F\left(-\frac{2i}{2^{n+1}}\right) \end{aligned}$$

and similarly

$$F\left(\frac{2i+1}{2^{n+1}}\right) = p_{2i+1}^{n+1} = p_{-2i-1}^{n+1} = F\left(-\frac{2i+1}{2^{n+1}}\right).$$

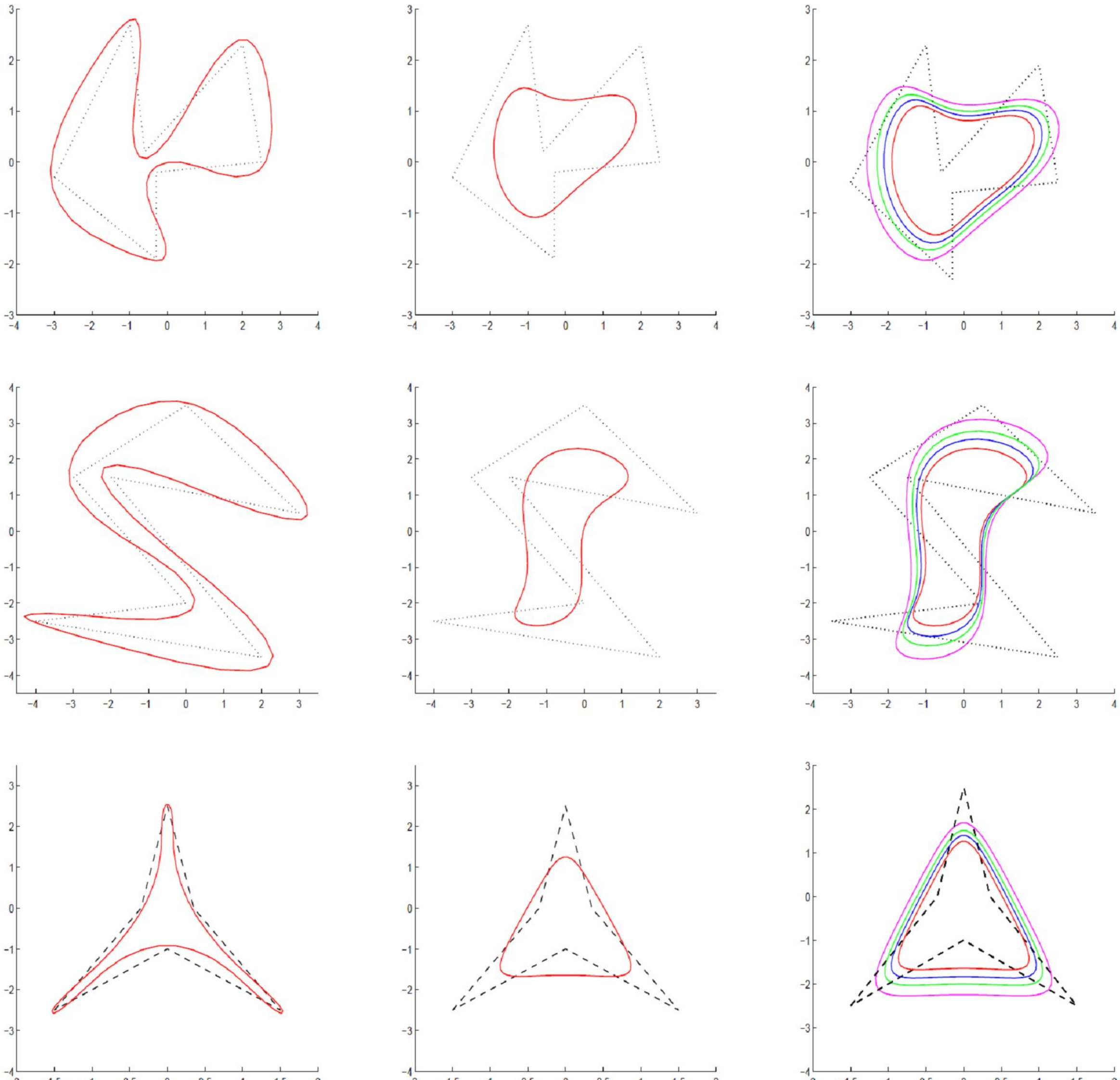
Hence $F\left(\frac{i}{2^n}\right) = F\left(\frac{-i}{2^n}\right)$ $\forall i \in Z$ and $n = Z_+$, thus from the continuity of F, $F(x) = F(-x)$ holds for all $x \in R$,

Scheme	Type	Continuity	Support	Range
Binary [16]	Stationary	C^1	[-5,4]	For some particular value
Binary [15]	Stationary	C^4	[-5,4]	For some particular value
Binary (Proposed)	Non-Stationary	C^4	[-5,4]	$0 < \alpha < \pi/3$

which completes the required result.

Table 2 - Comparison of the 5-point approximating schemes

The comparison shows that the proposed scheme generates the family of C^4 limiting curves, for certain range of parameter. On the other hand, existing 5-point binary schemes only can generate C^1 and C^4 limiting curves for some particular values.



Scheme [16] Scheme [15] Proposed Scheme

Figure 2: The continuous lines show the behavior of limit curves of the schemes presented in [16], [15] and proposed scheme (3.2), (from inner to outer, taking $\alpha = \frac{\pi}{180}, \alpha = \frac{\pi}{20}, \alpha = \frac{\pi}{15}$ and $\alpha = \frac{\pi}{12}$, after three subdivision steps. The dotted lines represent the initial control polygon.

7. Conclusion

A 5-point binary approximating non-stationary subdivision scheme has been developed which generates the family of C^4 limiting curve with support size [−5, 4]. The construction of non-stationary scheme is associated with the uniform trigonometric b-spline basis function for $0 < \alpha < \pi/3$. The scheme is analyzed, using the theory of asymptotic equivalence. The comparison of the scheme, with existing 5-point approximating schemes have been depicted in different figures. It shows that the proposed scheme has ability to produce the family of limit curves, for different values of α. Moreover, conic sections, trigonometric splines and trigonometric polynomials can also be generated by the proposed scheme. Furthermore, it is also concluded, the scheme introduced by Siddiqi and Ahmad [15] can be considered the special case of the proposed scheme.

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