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Numerical Solution of Convection-Diffusion Integro-Differential Equations with a Weakly Singular Kernel

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ABSTRACT

Many mathematical formulations of physical phenomena contain integro-differential equations. In this paper a numerical method is developed to solve the convection-diffusion integro-differential equations with a weakly singular kernel using the cubic B-spline collocation method. These equations occur in many applications such as in the transport of air and ground water pollutants, oil reservoir flow, in the modeling of semiconductors *etc*. The proposed method is based on collocation of cubic B-spline over finite elements, so that the continuity of the dependent variable and its first two derivatives throughout the solution range is obtained. The backward Euler scheme is used in time direction and the cubic B-spline collocation method is used for the spatial derivative. Some numerical examples are considered to illustrate the efficiency of the method developed. It has been observed that the numerical results efficiently approximate the exact solutions.

KEYWORDS: Cubic B-spline, Collocation method, Integro-differential equation, Weakly singular kernel, Convection-diffusion equation.

Mathematics Subject Classification (2010): 35-XX, 35R09, 45KXX

1 INTRODUCTION

Many phenomena in various fields of engineering, biology, physics formulate the systems, including space and time variables, are modeled by partial differential equations.

When the effects of the memory of the system are considered, the model involves the integral term containing the unknown function. Therefore, the obtained partial integro-differential equation (PIDE) consists of partial differentiations and integral terms. Partial integro-differential equations can describe some physical situations such as compression of poroviscoelastic media, convection-diffusion problems, nuclear reactor dynamics, geophysics, plasma physics and electromagnetic theory.

The convection-diffusion equation is a parabolic partial differential equation, which describes physical phenomena where the energy is transformed inside a physical system due to two processes: convection and diffusion. The term convection means the movement of molecules within fluids, whereas, diffusion describes the spread of particles through random motion from regions of higher concentration to regions of lower concentration.

It is necessary to calculate the transport of fluid properties or trace constituent concentrations within a fluid for applications such as water quality modeling, air pollution, meteorology, oceanography and other physical sciences.

Solutions of integro-differential equations have recently attracted much attention of researchers.

Many mathematical formulations of physical phenomena such as, convection (advection)-diffusion, contain integro-differential equations. Integro-differential equations are usually difficult to solve analytically so, it is required to obtain an efficient approximate solution.

In this paper, the following convection-diffusion integro-differential equation with a weakly singular kernel is considered

$$u_t(x,t)+m\,u_x(x,t)-b\,u_{xx}(x,t)=\int_0^t K(t-s)u(x,s)\,ds+f(x,t), \quad x\in[0,L],\ t>0 \tag{1}$$

where, $m > 0$ and $b > 0$ are considered to be positive constants quantifying the advection (convection) and diffusion processes, respectively. The integral term is called memory term, the kernel is a weakly singular kernel.

$$K(t-s)=(t-s)^{-\alpha}, \qquad 0<\alpha<1$$

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Subject to the initial condition

$$u(x,0) = g_0(x), \quad 0 \le x \le L \tag{2}$$

and the boundary conditions

$$u(0, t) = f_0(t), \qquad u(L,t) = f_1(t), \qquad t \ge 0 \tag{3}$$

where, $g_0(x)$, $f_0(t)$, $f_1(t)$ are known functions and $f(x,t)$ is a given smooth function.
If the memory term is zero, the Eq. (1) reduces to more general inhomogeneous convection- diffusion equation given by

$$u_t(x,t) + m u_x(x,t) - b u_{xx}(x,t) = f(x,t), \quad x \in [0,L], \; t > 0 \tag{4}$$

The source term $f(x,t)$, accounts for an insertion or extraction of mass of the system as it evolves with time. Specifically, $f(x,t)$ represents the time rate of change of concentration due to external factors, such as a source or a sink.

2 LITERATURE REVIEW

The integro-differential Eq. (1) along with the constraints (2) and (3) is of primary importance in many physical systems, especially those involving fluid flow [1-2].

Eq. (1) is the one dimensional version of the partial integro-differential equations which describe convection-diffusion of quantities such as mass, heat, energy, vorticity *etc.* [1-3].

It can be seen that in Eq. (1), the kernel function has a weak singularity at the origin [4].

This is particularly interesting in viscoelasticity, because it might smooth the solution when the boundary data is discontinuous [5].

A. F. Soliman *et al.* [6, 7] used finite difference formula and collocation method for the numerical solution of partial integro-differential equation. Haixiang Zhang *et al.* [8] used quintic B-spline collocation method for solving fourth order partial integro-differential equation with a weakly singular kernel. A. F. Soliman *et al.* [9], used sixth-degree B-spline for the numerical solution of integro-partial differential equations. Xuehua Yang *et al.* [10], used quasi-wavelet based numerical method for the solution of fourth order partial integro-differential equations with a weakly singular kernel. Haixiang Zhang and Xuli Han [11], used quasi-wavelet collocation method for the solution of time-dependent fractional partial differential equation. A. Golbabai and K. Sayevand [12], used homotopy perturbation method for the solution of generalized fourth-order fractional diffusion–wave equations. Yasir Nawaz [13], used variational iteration and homotopy methods for the solution of fourth-order fractional integro-differential equations. The numerical solution of fourth-order partial integro-differential equation with a weakly singular kernel is proposed by Xuehua Yang *et al.* [14], using Crank-Nicolson scheme in time direction and quasi-wavelet method for spatial derivatives. Quasi-wavelet based numerical method is proposed by W. T. Long *et al.* [15], for the solution of a class of partial integro-differential equations. R. C. Mittal and R. K. Jain [16], discussed collocation method based on redefined cubic B-spline basis functions for solving convection-diffusion equation.

In this paper, the approximate solution of convection-diffusion integro-differential equation with a weakly singular kernel is proposed using cubic B-spline collocation method. The collocation method with B-spline basis functions represents an economical alternative, since it only requires the evaluation of the unknown parameters at the grid points.

The paper is organized into seven sections. The literature review is presented in section 2. Section 3 presents the detailed description about the cubic B-spline. The backward Euler Scheme, used to discretize the time derivative, involved in Eq. (1.1), is discussed in Section 4. In section 5, cubic B-spline collocation method is developed to solve convection-diffusion integro-differential equation. Numerical results are presented in section 6, while the conclusion is presented in section 7.

3 Cubic B-spline Collocation Method

Let $\Delta^* = \{0 = x_0 < x_1 < x_2 < \ldots < x_N = L\}$ be the partition of $[0, L]$. The special step length is denoted by h, $h = x_i - x_{i-1}$, $i = 1, 2, 3, \ldots, N$.

Let B_i be B-spline basis functions with knots at the points x_i, $i = 0, 1, \ldots, N$. Thus, an approximation $U^{n+1}(x)$ to the exact solution $u^{n+1}(x)$ at $n+1$ time level, can be expressed in terms of the cubic B- spline basis functions $B_i(x)$ as

$$U^{n+1}(x) = \sum_{i=-1}^{N+1} c_i(t) B_i(x) \qquad (5)$$

where, c_i are unknown time dependent quantities to be determined from the boundary conditions and the collocation form of the integro-differential equation.

The cubic B-spline $B_i(x)$, $i = -1, 0, \ldots, N+1$ can be defined as below

$$B_i(x) = \frac{1}{h^3} \begin{cases} (x - x_{i-2})^3, & x \in [x_{i-2}, x_{i-1}], \\ h^3 + 3h^2(x - x_{i-1}) + 3h(x - x_{i-1})^2 - 3(x - x_{i-1})^3, & x \in [x_{i-1}, x_i], \\ h^3 + 3h^2(x_{i+1} - x) + 3h(x_{i+1} - x)^2 - 3(x_{i+1} - x)^3, & x \in [x_i, x_{i+1}], \\ (x_{i+2} - x)^3, & x \in [x_{i+1}, x_{i+2}], \\ 0, & otherwise \end{cases}$$

The values of successive derivatives $B_i^{(r)}(x)$, $i = -1, \ldots, N+1; r = 0, 1, 2$ at nodes are listed in Table 1.

Table 1: Coefficient of cubic B-spline and its derivatives at knots x_i

	x_{i-2}	x_{i-1}	x_i	x_{i+1}	x_{i+2}	**Else**
$B_i(x)$	0	1	4	1	0	0
$B_i^{(1)}(x)$	0	$\frac{3}{h}$	0	$-\frac{3}{h}$	0	0
$B_i^{(2)}(x)$	0	$\frac{6}{h^2}$	$-\frac{12}{h^2}$	$\frac{6}{h^2}$	0	0

4 Discretization in time: a backward Euler scheme

The time derivative is discretized by the first-order backward Euler scheme. Let $t_n = nk$, where *k* is the time step, $u^n(x)$ is an approximation to the value of *u*(*x*, *t*) at a time point

$t = t_n$, *n* = 0, 1,….

Considering the temporal discrete process of Eq. (1) at time point $t = t_{n+1}$, the first expression in left side of Eq.(1) is approximated by

$$u_t(x, t_{n+1}) \approx \frac{u(x, t_{n+1}) - u(x, t_n)}{k} \qquad (6)$$

Substituting Eq.(6) in Eq.(1), it can be written as

$$\frac{u(x,t_{n+1})-u(x,t_n)}{k}+mu_x(x,t_{n+1})-bu_{xx}(x,t_{n+1})=\int_0^{t_{n+1}}(t_{n+1}-s)^{-\alpha}u(x,s)ds+f(x,t_{n+1}) \quad (7)$$

The integral term in the above equation can be calculated as under

$$\begin{aligned}\int_0^{t_{n+1}}(t_{n+1}-s)^{-\alpha}u(x,s)ds &= \int_0^{t_{n+1}}s^{-\alpha}u(x,t_{n+1}-s)ds\\ &=\sum_{j=0}^{n}\int_{t_j}^{t_{j+1}}s^{-\alpha}u(x,t_{n+1}-s)ds\\ &\approx\sum_{j=0}^{n}u(x,t_{n-j+1})\int_{t_j}^{t_{j+1}}s^{-\alpha}ds\\ &=\frac{k^{1-\alpha}}{1-\alpha}\sum_{j=0}^{n}u(x,t_{n-j+1})((j+1)^{1-\alpha}-j^{1-\alpha})\end{aligned} \quad (8)$$

Substituting Eq. (8) in Eq. (7), it can be written as

$$\frac{u(x,t_{n+1})-u(x,t_n)}{k}+mu_x(x,t_{n+1})-bu_{xx}(x,t_{n+1})=\frac{k^{1-\alpha}}{1-\alpha}\sum_{j=0}^{n}u(x,t_{n-j+1})((j+1)^{1-\alpha}-j^{1-\alpha})+f(x,t_{n+1})$$

The above equation can be rewritten as

$$u^{n+1}(x)+mku_x^{n+1}(x)-bku_{xx}^{n+1}(x)-\frac{k^{2-\alpha}}{1-\alpha}u^{n+1}(x)=u^n(x)+\frac{k^{2-\alpha}}{1-\alpha}\sum_{j=1}^{n}b_j\,u^{n-j+1}(x)+k\,f^{n+1}(x),\ \ n\ge 1 \quad (9)$$

where, $u^{n+1}(x)=u(x,t_{n+1}),\ \ f^{n+1}(x)=f(x,t_{n+1}),\ b_j=(j+1)^{1-\alpha}-j^{1-\alpha},\ j=1,\,2,\,3,...$

For the special case n = 0, which is the first time step, the scheme simply leads to

$$u^1(x)+mku_x^1(x)-bku_{xx}^1(x)-\frac{k^{2-\alpha}}{1-\alpha}u^1(x)=u^0(x)+k\,f^1(x) \quad (10)$$

5 Discretization in space: cubic B-spline collocation method

Consider a uniform mesh Δ with the grid points λ_{in} to discretize the region $\Omega=[0,L]\times[0,T]$. Each ¸λ_{in} is the grid point (x_i,t_n) where $x_i=ih,\ i=0,1,2,...,N$ and $t_n=nk,\ n=0,1,2,...,M,\ Mk=T$. h and k are the mesh sizes in the space and time directions, respectively. The space discretization of Eq. (9) is carried out using Eq. (5) and the collocation method is implemented by identifying the collocation points as nodes. So, for $i=0,1,2,...,N$ the following relation can be obtained as

$$[(c_{i-1}^{n+1}+4c_i^{n+1}+c_{i+1}^{n+1})+mk\frac{3}{h}(-c_{i-1}^{n+1}+c_{i+1}^{n+1})-bk\frac{6}{h^2}(c_{i-1}^{n+1}-2c_i^{n+1}+c_{i+1}^{n+1})-\frac{k^{2-\alpha}}{1-\alpha}(c_{i-1}^{n+1}+4c_i^{n+1}+c_{i+1}^{n+1})]$$

$$=[(c_{i-1}^{n}+4c_i^{n}+c_{i+1}^{n})+\frac{k^{2-\alpha}}{1-\alpha}\sum_{j=1}^{n}b_j(c_{i-1}^{n+1-j}+4c_i^{n+1-j}+c_{i+1}^{n+1-j})+kf_i^{n+1}]$$

Simplifying the above relation leads to the following system of (N + 1) linear equations in (N + 3) unknowns $c_{-1}^{n+1},c_0^{n+1},c_1^{n+1},...,c_N^{n+1},c_{N+1}^{n+1}$.

$$c_{i-1}^{n+1}(1-mk\frac{3}{h}-bk\frac{6}{h^2}-\frac{k^{2-\alpha}}{1-\alpha})+c_i^{n+1}(4+bk\frac{12}{h^2}-4\frac{k^{2-\alpha}}{1-\alpha})+c_{i+1}^{n+1}(1+mk\frac{3}{h}-bk\frac{6}{h^2}-\frac{k^{2-\alpha}}{1-\alpha})=F_i,$$
$$i=0,1,...,N \quad (11)$$

where,

$$F_i = (c_{i-1}^n + 4c_i^n + c_{i+1}^n) + \frac{k^{2-\alpha}}{1-\alpha}\sum_{j=1}^{n} b_j (c_{i-1}^{n+1-j} + 4c_i^{n+1-j} + c_{i+1}^{n+1-j}) + kf_i^{n+1}$$

To obtain the unique solution of the system (11), two additional constraints are required.

These constraints are obtained from the boundary conditions. Imposition of the boundary conditions enables to eliminate the parameters c_{-1} and c_{N+1} from the system (11).

In order to eliminate c_{-1} and c_{N+1}, boundary conditions are used as

$$u(x_0,t) = (c_{-1} + 4c_0 + c_1) = p_0(t)$$
$$u(x_N,t) = (c_{N-1} + 4c_N + c_{N+1}) = p_1(t)$$

$$c_{-1} = -4c_0 - c_1 + p_0(t)$$
$$c_{N+1} = -c_{N-1} - 4c_N + p_1(t)$$

After eliminating c_{-1} and c_{N+1}, the system (11) is reduced to a tri-diagonal system of (N +1) linear equations in (N +1) unknowns. The system can be rewritten in the following matrix form

$$AC^{n+1} = F, \quad n = 1,2,3,...$$

where, $C^{n+1} = [c_0^{n+1}, c_1^{n+1}, c_2^{n+1}, ..., c_N^{n+1}]^T$, $n = 1,2,3,...$

and

$$\mathrm{A} = \begin{bmatrix} (\frac{12mk}{h} + \frac{36bk}{h^2}) & \frac{6mk}{h} & & & & \\ p & q & r & & & \\ & p & q & r & & \\ & & \ddots & \ddots & \ddots & \\ & & & p & q & r \\ & & & & \frac{6mk}{h} & (\frac{12mk}{h} + \frac{36bk}{h^2}) \end{bmatrix}$$

where, $p = (1 - mk\frac{3}{h} - bk\frac{6}{h^2} - \frac{k^{2-\alpha}}{1-\alpha})$, $q = (4 + bk\frac{12}{h^2} - 4\frac{k^{2-\alpha}}{1-\alpha})$, $r = (1 + mk\frac{3}{h} - bk\frac{6}{h^2} - \frac{k^{2-\alpha}}{1-\alpha})$

In order to find the value of $C^2 = [c_0^2, c_1^2, c_2^2, ..., c_N^2]^T$, it is first needed to find the value of $C^1 = [c_0^1, c_1^1, c_2^1, ..., c_N^1]^T$. The value of C^1 is obtained, solving Eq. (10), as

$$c_{i-1}^1(1 - mk\frac{3}{h} - bk\frac{6}{h^2} - \frac{k^{2-\alpha}}{1-\alpha}) + c_i^1(4 + bk\frac{12}{h^2} - 4\frac{k^{2-\alpha}}{1-\alpha}) + c_{i+1}^1(1 + mk\frac{3}{h} - bk\frac{6}{h^2} - \frac{k^{2-\alpha}}{1-\alpha}) = F_i,$$
$$i = 0,1,...,N \qquad (13)$$

where,

$$F_i = (c_{i-1}^0 + 4c_i^0 + c_{i+1}^0) + kf_i^1$$

The above Eq. (13) is a system of (N+1) linear equations in (N+3) unknowns c_{-1}^1, c_0^1, ..., c_N^1, c_{N+1}^1.

To obtain the unique solution of the system, c_{-1} and c_{N+1} are eliminated using boundary conditions.
The time evolution of the approximate solution U^{n+1} is determined by the time evolution of the vector C^{n+1}. This is found by repeatedly solving the recurrence relationship, after the initial vector $C^0 = [c_0^0, c_1^0, c_2^0, \ldots, c_N^0]^T$, has been computed from the initial condition. The recurrence relationship is tri-diagonal and so can be solved using the Thomas algorithm.
The initial state vector C^0 can be determined from the initial condition $u(x,0) = g_0(x)$ which gives *(N+1)* equations in *(N+3)* unknowns. For determining these unknowns the following relations at the knots are used

$$U_x(0,0) = u_x(x_0,0),$$

$$U(x_i,0) = g_0(x_i), \quad i = 1,2,3,\ldots,N-1$$

$$U_x(L,0) = u_x(x_N,0)$$

which give a tri-diagonal system of equations in the following matrix

$$GC^0 = E$$

where, $$G = \begin{bmatrix} 4 & 2 & 0 & & & & \\ 1 & 4 & 1 & & & & \\ & 1 & 4 & 1 & & & \\ & & \ddots & \ddots & \ddots & & \\ & & & & 1 & 4 & 1 \\ & & & & 0 & 2 & 4 \end{bmatrix}$$

6 Numerical Results

In this section, the proposed method is tested on five problems.

Let $t_n = nk,\ n = 0,1,2,\ldots,M,\ h = \dfrac{L}{N}$, where, *M* denotes the final time level t_M and *N*+1 is the number of nodes. In order to check the accuracy of the proposed method, the maximum norm errors and L_2 norm errors between numerical and exact solutions are given by the following definitions

Maximum norm error: $$\|e_M\|_\infty = \max_{0\le i\le N} \left| u(x_i,t_M) - U_i^M \right|$$

L_2 norm error: $$\|e_M\|_2 = \frac{1}{N}\left(\sum_{i=0}^{N} \left| u(x_i,t_M) - U_i^M \right|^2 \right)^{\frac{1}{2}}$$

The accuracy of the proposed method is tested, for different values of parameters *h*, *k*, *m* and *b*.

Some important non-dimensional parameters in numerical analysis are defined as follows:

Courant number: $C_r = m\dfrac{k}{h}$

Diffusion number: $S = b\dfrac{k}{h^2}$

Grid *Peclet* number: $P_e = \dfrac{C_r}{S} = \dfrac{m}{b}h$

When the *Peclet number* is high, the convection term dominates and when the *Peclet* number is low, the diffusion term dominates.

Example 1

The following convection-diffusion integro-differential equation is considered

$$u_t(x,t) + m u_x(x,t) - b u_{xx}(x,t) = \int_0^t (t-s)^{-\alpha} u(x,s)\,ds + f(x,t), \quad x \in [0,1], \quad \alpha = \frac{1}{2}, \quad t > 0$$

with m = 0.05, b = 0.4 and the following initial condition

$$u(x,0) = \sin \pi x, \quad 0 \le x \le 1$$

and boundary conditions

$$u(0,t) = u(1,t) = 0, \quad 0 \le t \le T$$

The exact solution of the problem is

$$u(x,t) = (t+1)^2 \sin \pi x$$

The numerical solutions for two different grid sizes N = 100 and N = 50 with k = 0.0001, at different time levels M, are presented in Table 2. The numerical solutions for two different grid sizes N = 100 and N = 50 with k = 0.00001, at different time levels M, are presented in Table 3. P_e = 0.00125 and P_e = 0.0025 for h = 0.01 and h = 0.02, respectively. Here, *Peclet* number P_e is low, which indicates that the diffusion term dominates. In Tables 2 and 3, the time increment k, the space increment $h = \dfrac{1}{N}$ and time level M are varied to test the accuracy of the proposed method, which indicate that the proposed method is substantially efficient. It can be observed from the Tables 2 and 3, that the proposed method approximates the exact solution very efficiently.

In order to indicate the effects of the proposed method for larger time level M, the exact solution and the approximate solution are plotted using N=100, M=1000 and k =0.00001 as shown in Fig. 1. It is clear from the Fig. 1 that the numerical solution is highly consistent with the exact solution, which indicates that cubic B-spline collocation method is very effective.

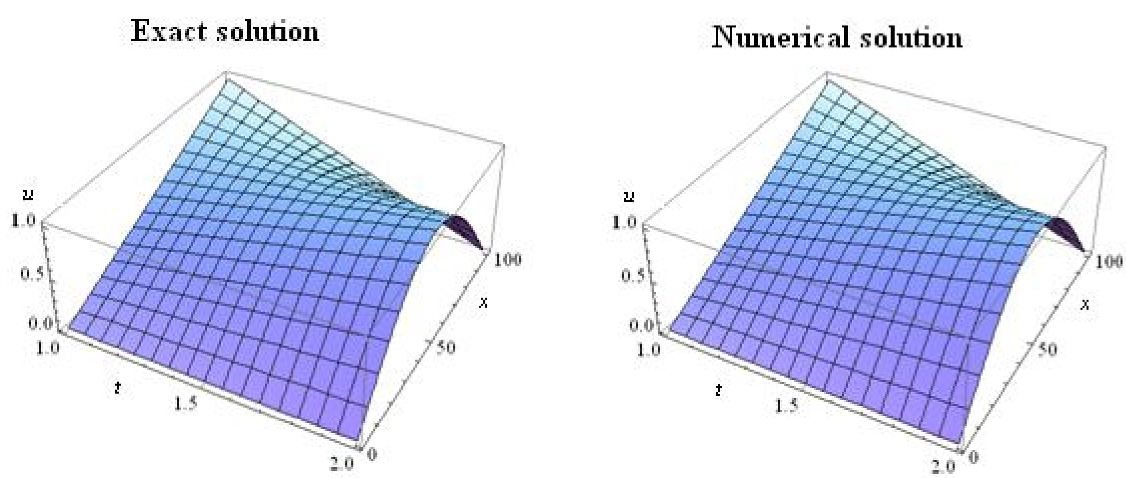

Figure 1: The results at N=100, M=1000 and k=0.00001 for Example 1.

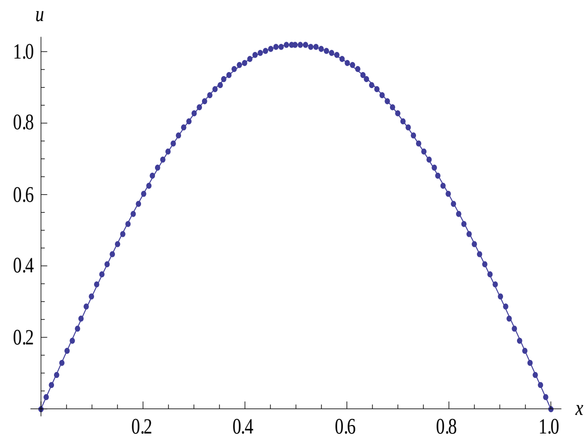

Figure 2: The exact and numerical solutions at M=1000.
Dotted line: numerical solution, Solid line: the exact solution

Table 2: $\|e\|_\infty$ and $\|e\|_2$ for k = 0.0001

h	M	P_e	$\|e_M\|_\infty$	$\|e_M\|_2$
0.01	10	0.0012	9.4351 E-06	1.3253 E-07
	50	5	1.1446 E-05	3.3175E-07
	100		1.1943 E-05	5.7312 E-07
	500		3.5394 E-05	2.4398E-06
0.02	10	0.0025	2.6417 E-05	5.6276 E-07
	50		3.9890 E-05	1.0863 E-06
	100		4.4030 E-05	1.3976 E-06
	500		5.3632 E-05	2.8654 E-06

Table 3: $\|e\|_\infty$ and $\|e\|_2$ for k = 0.00001

h	M	P_e	$\|e_M\|_\infty$	$\|e_M\|_2$
0.01	10	0.00125	4.3771 E-06	4.4106 E-08
	50		8.2580 E-06	9.5928E-08
	100		9.5573 E-06	1.2493 E-07
	500		1.1579 E-05	2.2516 E-07
0.02	10	0.0025	6.7929 E-06	1.3756 E-07
	50		2.004 E-05	4.1125 E-07
	100		2.6950 E-05	5.8715 E-07
	500		4.0214 E-05	1.2581 E-06

Example 2

The following convection-diffusion integro-differential equation is considered

$$u_t(x,t)+m u_x(x,t)-b u_{xx}(x,t)=\int_0^t (t-s)^{-\alpha} u(x,s)\,ds+f(x,t),\quad x\in[0,1]\quad,\alpha=\frac{1}{4}\quad,t>0$$

with m = 0.5, b = 0.001 and the following initial condition

$$u(x,0)=2\sin^2\pi x,\quad 0\le x\le 1$$

and boundary conditions

$$u(0,t)=u(1,t)=0,\quad 0\le t\le T$$

The exact solution of the problem is

$$u(x,t)=2(t^2+t+1)\sin^2\pi x$$

The numerical solutions for two different grid sizes N = 50 and N = 10 with k = 0.0001, at different time levels M, are presented in Table 4. P_e = 10 and P_e = 50 for h = 0.02 and h = 0.1, respectively. Here, *Peclet* number P_e is high, which indicates that the convection term dominates. The numerical solutions for two different grid sizes N = 100 and N = 50 with k = 0.00001, at different time levels M, are presented in Table 5. P_e = 5 and P_e = 10 for h = 0.01 and h = 0.02 respectively. Here, *Peclet* number P_e is high, which indicates that the convection term dominates. In Tables 4 and 5, the time increment k, the space increment $h=\dfrac{1}{N}$ and time level M are varied to test the accuracy of the proposed method, which indicate that the proposed method is substantially efficient. It can be observed from the Tables 4 and 5, that the proposed method approximates the exact solution very efficiently.

In order to indicate the effects of the proposed method for larger time level M, the exact solution and the approximate solution are plotted using N=100, M=1000 and k =0.00001 as shown in Fig. 3. It is clear from the Fig. 3 that the numerical solution is highly consistent with the exact solution, which indicates that the proposed method is very effective.

In Fig. 4, the exact solution is represented by solid line and the numerical solution is represented by dotted line at M=1000 time level.

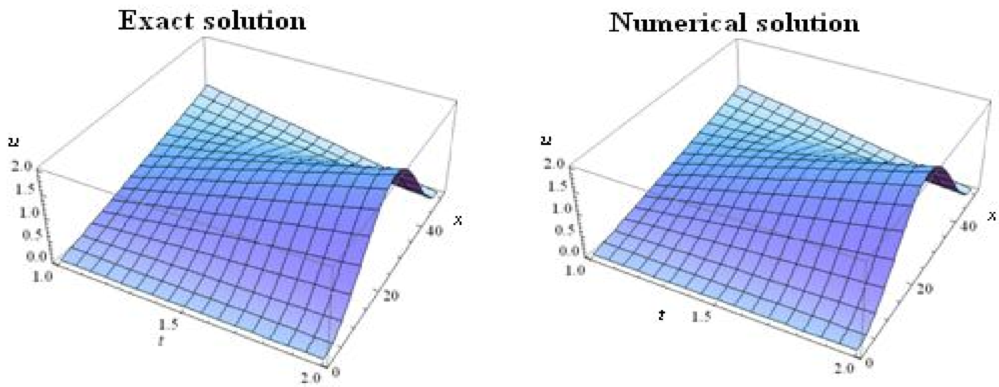

Figure 3: The results at N=100, M=1000 and k=0.00001 for Example 2.

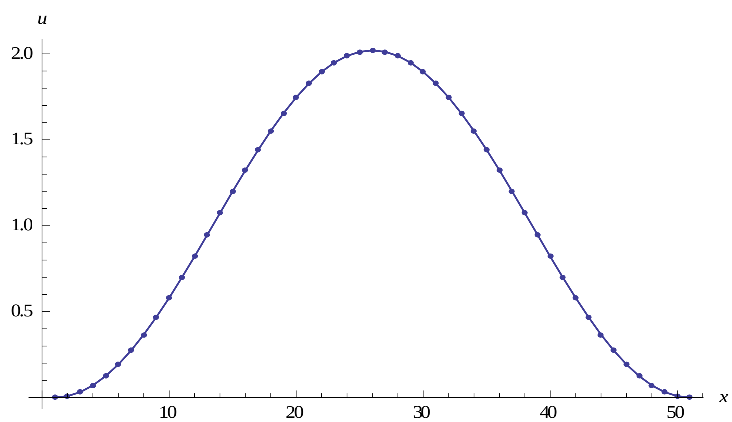

Figure 4: The exact and numerical solutions at M=1000.
Dotted line: numerical solution, Solid line: the exact solution

Table 4: $\|e\|_\infty$ and $\|e\|_2$ for k = 0.0001

h	M	P_e	$\|e_M\|_\infty$	$\|e_M\|_2$
0.02	10	10	7.0636 E-07	5.8752 E-08
	50		3.5371 E-06	2.9412 E-07
	100		7.0872 E-06	5.8915 E-07
	500		3.5937 E-05	2.9826 E-06
0.1	10	50	3.5325 E-06	7.5091 E-07
	50		1.7733 E-05	3.7616 E-06
	100		3.5640 E-05	7.5412 E-06
	500		1.8429 E-04	3.8465 E-05

Table 5: $\|e\|_\infty$ and $\|e\|_2$ for k = 0.00001

h	M	P_e	$\|e_M\|_\infty$	$\|e_M\|_2$
0.01	10	5	6.6157 E-09	3.9406 E-10
	50		3.3084 E-08	1.9705 E-09
	100		6.6182 E-08	3.9417 E-09
	500		3.3144 E-07	1.9732 E-08
0.02	10	10	7.1775 E-09	5.8970 E-10
	50		3.5196 E-08	2.9488 E-09
	100		7.1813 E-08	5.8985 E-09
	500		3.5988 E-07	2.9526 E-08

Example 3

The following convection-diffusion integro-differential equation is considered

$$u_t(x,t)+mu_x(x,t)-bu_{xx}(x,t)=\int_0^t (t-s)^{-\alpha} u(x,s)\,ds+f(x,t), \quad x\in[0,4\pi], \quad \alpha=\frac{1}{4}, \quad t>0$$

with m = 0.1, b = 0.1 and the following initial condition

$$u(x,0) = 2\sin^2 x, \quad 0 \le x \le 4\pi$$

and boundary conditions

$$u(0,t) = u(4\pi,t) = 0, \quad 0 \le t \le T$$

The exact solution of the problem is

$$u(x,t) = 2(t+1)\sin^2 x$$

The numerical solutions for two different grid sizes N = 10 and N = 50 with k = 0.0001 at different time levels M, are presented in Table 6. The numerical solutions for two different grid sizes N = 10 and N = 50 with k = 0.00001 at different time levels M, are presented in Table 7.

P_e = 1.25663 and P_e = 0.25132 for $h = \frac{4\pi}{10}$ and $h = \frac{4\pi}{50}$, respectively. Here, *Peclet* number P_e corresponds to $h = \frac{4\pi}{10}$ is high, which indicates that the convection term dominates. P_e corresponds to $h = \frac{4\pi}{50}$ is low, which indicates that the diffusion term dominates. In Tables 6 and 7, the time increment k, the space increment $h = \frac{1}{N}$ and time level M are varied to test the accuracy of the proposed method, which indicate that the proposed method is substantially efficient. It can be observed from the Tables 6 and 7, that the proposed method approximates the exact solution very efficiently.

In order to indicate the effects of the proposed method for larger time level M, the exact solution and the approximate solution are plotted using N=100, M=100 and k =0.00001 as shown in Fig. 5. It is clear from the Fig. 5 that the numerical solution is highly consistent with the exact solution, which indicates that the proposed method is highly effective.

In Fig. 6, the exact solution is represented by solid line and the numerical solution is represented by dotted line at M=100 time level.

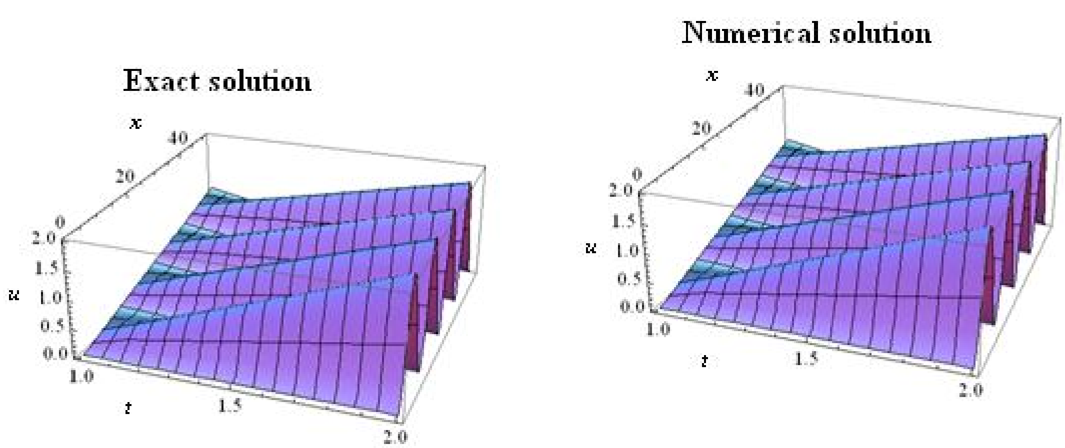

Figure 5: The results at N=50, k=0.00001 and M=100 for Example 3.

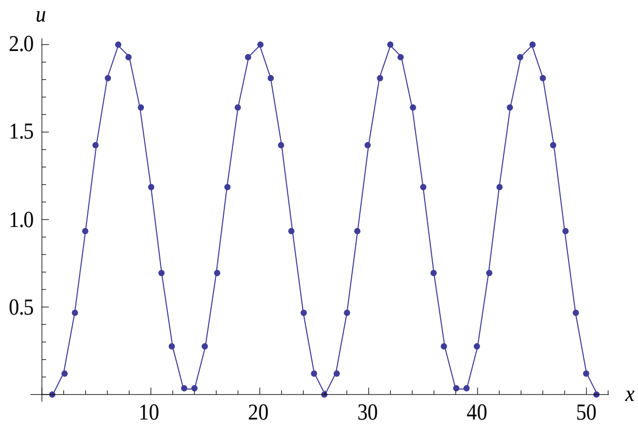

Figure 6: The exact and numerical solutions at M=100.
Dotted line: numerical solution, Solid line: the exact solution

Table 6: $\|e\|_\infty$ and $\|e\|_2$ for k = 0.0001

N	M	P_e	$\|e_M\|_\infty$	$\|e_M\|_2$
10	10 50 100	1.25663	1.8012 E-04 8.1725 E-04 1.5025 E-03	3.6509 E-05 1.6629 E-04 3.0190 E-04
50	10 50 100	0.25132	6.3946 E-06 1.7846 E-04 6.8299 E-04	7.5264 E-07 1.4880 E-05 5.6108 E-05

Table 7: $\|e\|_\infty$ and $\|e\|_2$ for k = 0.00001

N	M	P_e	$\|e_M\|_\infty$	$\|e_M\|_2$
10	10 50 100	1.25663	1.9055 E-05 9.3283 E-05 1.8239 E-04	3.7849 E-06 1.8669 E-05 3.6800 E-05
50	10 50 100	0.25132	9.4661 E-07 4.2270 E-06 7.1358 E-06	7.2971 E-08 2.8923 E-07 6.3521 E-07

Example 4

The following convection-diffusion integro-differential equation is considered

$$u_t(x,t)+mu_x(x,t)-bu_{xx}(x,t)=\int_0^t (t-s)^{-\alpha}\,u(x,s)\,ds+f(x,t),\quad x\in[0,1],\quad \alpha=\frac{1}{3},\quad t>0$$

with m = 0.005, b = 0.5 and the following initial condition

$$u(x,0)=1-\cos 2\pi\, x+2\pi^2\, x(1-x),\quad 0\le x\le 1$$

and boundary conditions

$$u(0,t)=u(1,t)=0,\quad 0\le t\le T$$

The exact solution of the problem is

$$u(x,t)=(t+1)^2\,(1-\cos 2\pi\, x+2\pi^2\, x(1-x))$$

The numerical solutions for two different grid sizes N = 50 and N = 100 with k = 0.0001 at different time levels M, are presented in Table 8. The numerical solutions for two different grid sizes N = 50 and N = 100 with k = 0.00001 at different time levels M, are presented in Table 9. P_e = 0.0002 and P_e = 0.0001 for h = 0.02 and h = 0.01, respectively. Here, *Peclet* number P_e is low, which indicates that the diffusion term dominates. In Tables 8 and 9, the time increment k, the space increment h and time level M are varied to test the accuracy of the proposed method, which indicate that the proposed method is substantially efficient. It can be observed from the Tables 8 and 9, that the proposed method approximates the exact solution very efficiently.

In order to indicate the effects of the proposed method for larger time level M, the exact solution and the approximate solution are plotted using N=100, M=500 and k =0.00001 as shown in Fig. 7. It is clear from the Fig. 7 that the numerical solution is highly consistent with the exact solution, which indicates that cubic B-spline collocation method is very effective.

In Fig. 8, the exact solution is represented by solid line and the numerical solution is represented by dotted line at M=500 time level.

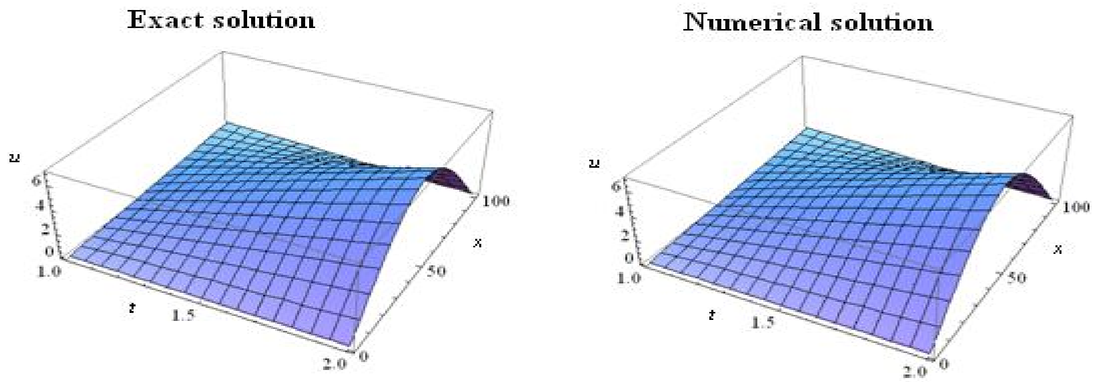

Figure 7: The results at N=100, M=500 and k=0.00001 for Example 4.

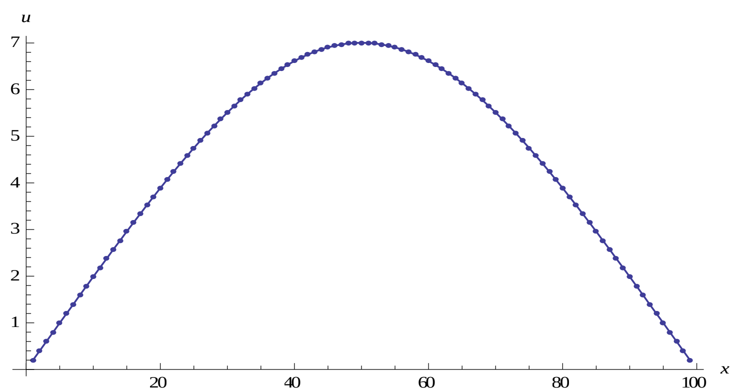

Figure 8: The exact and numerical solutions at M=500.
Dotted line: numerical solution, Solid line: the exact solution

Table 8: $\|e\|_\infty$ and $\|e\|_2$ for k = 0.0001

h	M	P_e	$\|e_M\|_\infty$	$\|e_M\|_2$
0.02	10	0.0002	2.4282 E-05	2.0070 E-06
	50		1.0090 E-04	8.8800 E-06
	100		1.7172 E-04	1.5778 E-05
	500		4.6998 E-04	4.1795 E-05
0.01	10	0.0001	6.6290 E-06	5.0793 E-07
	50		3.1392 E-05	2.4155 E-06
	100		5.9245 E-05	4.6368 E-06
	500		2.2874 E-04	1.9337 E-05

Table 9: $\|e\|_\infty$ and $\|e\|_2$ for k = 0.00001

h	M	P_e	$\|e_M\|_\infty$	$\|e_M\|_2$
0.02	10	0.0002	2.9281 E-06	2.4829 E-07
	50		1.2548 E-05	1.1989 E-06
	100		2.4765 E-05	2.3503 E-06
	500		1.1947 E-04	1.0652 E-05
0.01	10	0.0001	1.9020 E-06	4.5422 E-08
	50		3.0941 E-06	2.0137 E-07
	100		5.9627 E-06	3.9130 E-07
	500		2.6198 E-05	1.7531 E-06

Example 5

The following convection-diffusion integro-differential equation is considered

$$u_t(x,t)+mu_x(x,t)-bu_{xx}(x,t)=\int_0^t (t-s)^{-\alpha} u(x,s)\,ds+f(x,t), \qquad x\in[0,1], \qquad \alpha=\frac{1}{3}, \qquad t>0$$

with m = 0.5, b = 0.005 and the following initial condition

$$u(x,0)=\cos\pi x, \quad 0\le x\le 1$$

And boundary conditions

$$u(0,t)=(t+1), \qquad u(1,t)=-(t+1), \qquad t\ge 0$$

The exact solution of the problem is

$$u(x,t) = (t+1)\cos \pi x$$

The numerical solutions at N = 100 with k = 0.0001 and k = 0.00001 at different time levels M are presented in Table 10 and 11, respectively. In Tables 10 and 11, the time increment k and time level M are varied to test the accuracy of the proposed method, which indicates that the proposed method is substantially efficient. P_e = 1 for h = 0.01. Here, *Peclet* number P_e is high, which indicate that the convection term dominates. From Figures 9 and 10, it can be observed that the numerical solution is highly consistent with the exact solution, which indicates that the proposed method is very effective.

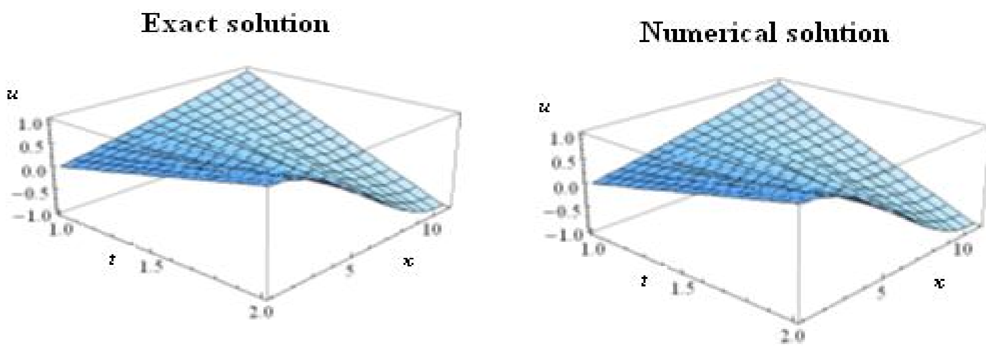

Figure 9: The results at N=100, k=0.00001 and M=100 for Example 5

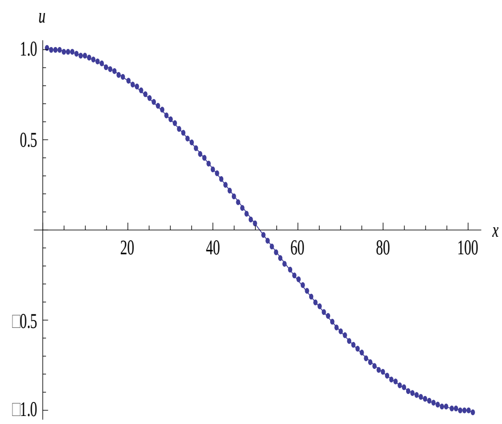

Figure 10: The exact and numerical solutions at M=100.
Dotted line: numerical solution, Solid line: the exact solution

Table 10: $\|e\|_2$ for N=100

h	M	P_e	k=0.0001 $\|e_M\|_2$	k=0.00001 $\|e_M\|_2$
0.01	10	1.0	1.1709 E-06	1.4690 E-08
	50		2.0104 E-05	2.5430 E-07
	100		7.2658 E-05	9.2099 E-07
	500		1.5005 E-03	1.9062 E-05

Table 11: $\|e\|_\infty$ for N=100

h	M	P_e	k=0.0001 $\|e_M\|_\infty$	k=0.00001 $\|e_M\|_\infty$
0.0	10	1.	3.0131 E-05	3.7552 E-07
1	50	0	3.3049 E-04	5.0060 E-06
	100		1.0609 E-03	1.6837 E-05
	500		2.1712 E-02	2.8412 E-04

Table 12: The following table summarizes the differences of examples 1-5

Example	Exact solution	Interval	Initial condition	Boundary conditions	*Peclet* number P_e
1	$u(x,t) = (t+1)^2 \sin \pi x$	[0,1]	$u(x,0) = \sin \pi x$, $0 \le x \le 1$	$u(0,t) = u(1,t) = 0$, $0 \le t \le T$	Low
2	$u(x,t) = 2(t^2 + t + 1)\sin^2 \pi x$	[0,1]	$u(x,0) = 2\sin^2 \pi x$, $0 \le x \le 1$	$u(0,t) = u(1,t) = 0$, $0 \le t \le T$	High
3	$u(x,t) = 2(t+1)\sin^2 x$	$[0,4\pi]$	$u(x,0) = 2\sin^2 x$, $0 \le x \le 4\pi$	$u(0,t) = u(4\pi,t) = 0$, $0 \le t \le T$	High for $h = \frac{4\pi}{10}$ and low for $h = \frac{4\pi}{50}$
4	$u(x,t) = (t+1)^2$ $(1 - \cos 2\pi x + 2\pi^2 x(1-x))$	[0,1]	$u(x,0) =$ $(1 - \cos 2\pi x + 2\pi^2 x(1-x))$, $0 \le x \le 1$	$u(0,t) = u(1,t) = 0$, $0 \le t \le T$	Low
5	$u(x,t) = (t+1)\cos \pi x$	[0,1]	$u(x,t) = \cos \pi x$, $0 \le x \le 1$	$u(0,t) = (t+1)$, $u(1,t) = -(t+1)$, $0 \le t \le T$	High

7 Conclusion

In this paper, the convection-diffusion integro-differential equation with a weakly singular kernel was solved using a collocation method with cubic B-spline basis functions. The backward Euler scheme is used for time discretization and the cubic B-spline collocation method is used for space discretization. The proposed method efficiently worked to give accurate results for values of P_e upto 50. The performance of the proposed method for the considered problems was measured by calculating the maximum norm error and L_2 -norm error, presented in Tables 2-11. The parameters *h*, *k* and *M* are varied in order to test the accuracy of the proposed method. The proposed method is also valid and efficient for different values of α, $(0 < \alpha < 1)$. It is observed from the numerical examples, that the proposed method possesses a high degree of efficiency and accuracy. Moreover, the numerical results are in good agreement with the exact solutions. The numerical solution of time-fractional convection-diffusion equations using B-spline collocation method are in progress.

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