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Strongly Commuting Regular Rings

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ABSTRACT

In this paper, ring R satisfying in the condition $xy=(yx)^2a(yx)^2$ for all $x,y\in R$ with some $a\in R$ and is called strongly commuting regular. We observe the structure of a strongly commuting regular ring and we show that a ring with this property is π-regular and strongly regular and commuting regular. Also we study some significant results of this investigation which will be used for group rings and we briefly study the relationship between the idempotent in strongly commuting regular rings.
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KEYWORDS: strongly π-regular, π-regular, commuting regular, strongly computing regular, idempotent, group ring.

1. INTRODUCTION

R will represent an associative ring with identity. $N=N(R)$ the set of all nilpotent elements in R. $Z=Z(R)$, center of R. $J=J(R)$, Jacobson radical of R and $Id(R)$, is the set of all idempotent elements in R. $P(R)$, the prime radical. An element $a\in R$ is called nilpotent,if $a^n=0$,for some integer $n>1$. A ring R is said to be a reduced ring if $N=N(R)=\{0\}$, and a semiprime ring if for each $a\in R$, $aRa=0$ implies that $a=0$. In 1936, Von Neumann defined that an element x in R is regular if $x=xyx$, for some $y\in R$,the ring R is regular if each element of R is regular. Some properties of regular rings have been studied byGoodearl[4] and Fisher and snider [6]. A ring R is called π-regular if for each $x\in R$ exists a positive integer n, depending on x, and $y\in R$ such that $x^n=x^nyx^n$, and is called strongly π-regular if for each $x\in R$ exists a positive integer n, and $y\in R$ such that $x^n=x^{n+1}y$. The strongly π-regular has roles in ring theory as we see in [1],[8]. In 2004 Safari Sabet and Yamini[2] defined that a ring is called commuting regular if for each $x,y\in R$ there exists $a\in R$ such that $xy=yxayx$, see [2]. Then some results on commuting regular rings have been studied in [3,5,7,9].

R is called locally finite if every finite subset in it generates a finite semigroup multiplicatively and is called abelian if every idempotent is central. An idempotent $e\neq 0$ is called local if eRe is a local ring and e is called primitive if the ring eRe has no nontrivial idempotents. In section 2 we define strongly commuting regular ring and give some basic results about them. In section 3 we study strongly commuting regular group rings. In section 4 we study the relationship between idempotents in this rings.

2. Strongly Commuting Regular Rings

Definition 2.1. A ring R is said to be strongly commuting regular if for each $x,y\in R$ there exists $a\in R$ such that $xy=(yx)^2a(yx)^2$.

Proposition 2.2. Let R be a strongly commuting regular ring, then R is an abelian ring.

Proof. Let e be an arbitrary element in $Id(R)$ and $x\in R$, since R is strongly commuting regular, therefore there exist $a\in R$ such that $ex=(xe)^2a(xe)^2$, $xe=(ex)^2a(ex)^2$ thus $exe=(ex)^2a(ex)^2=xe$, $exe=(xe)^2a(xe)^2=ex$ which implies that $ex=xe$, this means that R is abelian.

Lemma 2.3. Let x be an element in the Jacobson radical $J(R)$. If for $y\in R$, $yx=y$ then $y=0$.

Proof. There exists $t\in R$ such that $x+t-tx=x+t-xt=0$ therefore,

$$0=y(x+t-xt)=yx+yt-yxt=y+yt-yt=y.$$

Proposition 2.4. Let R be a strongly commuting regular ring, then $J(R)\subseteq N(R)$.

Proof. Let $x\in J(R)$ be an arbitrary element. Since R is strongly commuting regular, it is for some $a\in R$, $xy=(yx)^2a(yx)^2$. If $x=y$ then $x^2=x^4ax^4$, but in view of lemma 2.3, $x^2=0$. Therefore $J(R)\subseteq N(R)$.

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Lemma 2.5. Let R be a locally finite ring, $a\in R$, then a^t is an idempotent for some positive integer t.

Proof. Since R is locally finite, $a^m=a^{m+n}$ for some integers $m,n\geq 1$.Then inductively we have $a^m=a^ma^n=a^ma^{2n}=...=a^ma^{mn}=a^{m(n+1)}$
hence letting $s=n+1$ then we obtain $a^m=(a^m)^s$ with $s\geq 2$. But

$$a^{(s-1)m}=a^{(s-2)m}a^m=a^{(s-2)m}(a^m)^s=a^{2(s-1)m}=(a^{(s-1)m})^2$$

so a^t is an idempotent in R with $t=(s-1)m=mn$.

Proposition 2.6. Let R be a strongly commuting regular ring, then $N=N(R)$, is a nilpotent ideal. In fact $RN=NR=N^2=0$.

Proof. Let $x\in N$ then, there exists $a\in R$ such that $x^2=x^4ax^4$.Clearly, x^4ax^2, x^2ax^4 are idempotent elements because $(x^4ax^2)^2=x^4ax^2x^4ax^2=x^4ax^2$,$(x^2ax^4)^2=$
$x^2ax^4x^2ax^4=x^2ax^4$ therefore, it is in the center of R. This shows that

$$x^2=x^4ax^4=x^6ax^2$$
$$=x^{10}ax^2ax^2$$
$$=x^{14}(ax^2)^3a$$
$$=x^{18}(ax^2)^4=...$$

but x is a nilpotent element, so $x^2=0$. Now suppose that $y\in R$ is an arbitrary element. Strongly commuting regularity of R implies that;

$$xy=(yx)^2a(yx)^2$$
$$=(yx)yxa(yx)^2$$
$$=yxxaybxay(yx)^2$$
$$=yx^2aybxay(yx)^2=0.$$

For some a and b in R, this proves that $xy=0$. Therefore $NR=0$ and particular $N^2=0$. Similarly, we can show that $RN=0$. To complete the proof it suffices to note that for every $x,y\in N$, $(x+y)^2=x^2+xy+yx+y^2=0$.

Proposition 2.7. Let R be a strongly commuting regular ring, if R is semiprime, then R is reduced. *Proof.* It is obvious by proposition 2.6 that $NRN=0$, therefore $N=\{0\}$ since R is semiprime, so R is reduced.

Corollary 2.8. Let R be a strongly commuting regular ring, if R is semiprime then it is semiprimitive. *Proof.* by propositions 2.4,2.7.

Corollary 2.9. Let R be a strongly commuting regular ring, if R is reduced then it is semiprimitive. *Proof.* by propositions 2.7,2.8.

Lemma 2.10. Suppose that R a strongly commuting regular domain, then R is a division ring.

Proof. Since R is a strongly commuting regular, therefore for each $0\neq x\in R$, there exist $a\in R$ such that $x^2=x^4ax^4$. But $e=x^2ax^4$ is an idempotent element, this implise that $x^2=x^2e$ and R being a domain we get $x=xe$. For every $y\in R$, $xy=xey$. So $y=ey$. That is $e=1_R$. but $x^2ax^4=e$ implies that every x has an inverse.thus, R is a division ring.

Proposition 2.11. Let R be a strongly commuting regular domain.If R is reduced then it is a local ring. *Proof.* By lemma 2.10.

Theorem 2.12. Suppose that R is a strongly commuting regular ring then R is π-regular and in particular it is strongly π-regular and commuting regular.

Proof. By strongly commuting regularity of R, for each $x\in R$ there exists an element c in R such that $x^2=x^4cx^4$. with suppose $c'=x^2cx^2$,then we have $x^2=x^2c'x^2$ which implies that, R is π-regular. In particular, x^4cx^2 and x^2cx^4 as idempotent elements are in the center of R, so $x^2=c'x^6=x^6c'$; $c'=cx^2=x^2c\in R$, i.e, R is strongly π-regular. Also for each $x,y\in R$ there exists an element c in R such that $xy=(yx)^2c(yx)^2$ thus we have $xy=(yx)yxcyx(yx)$ with suppose $yxcyx=d\in R$ therefore we have $xy=(yx)d(yx)$. Then R is a commuting regular ring.

Proposition 2.13. Let R be a locally finite ring. If R is strongly commuting regular, then $N(R)=J(R)$.

Proof. let $0\neq a\in N(R)$ with $a^l=0$ for some $r\in R$, then $(ar)^s$ is a non zero idempotent for some positive integer s, by lemma 2.5, say $(ar)^s=ab$ with $b\in R$.But it is central since R is strongly commuting regular, and so ;

$0\neq(ar)^{s+1}=(ar)(ar)^s ar((ar)^s)^l(ar)(ab)^l(a(ab)r)(ab)^{l-1}=(aa(ab)br)(ab)^{l-2}=...=$
$(a^l b^{l-1}r)(ab)=0$, is a contradiction. Thus $ar\in N(R)$ for all $r\in R$, similarly $ra\in N(R)$ for all $r\in R$.

Consequently, aR and Ra are nil and so they are contained in $J(R)$, hence we have $N(R)\subseteq J(R)$. We also get $J(R)\subseteq N(R)$ since R is strongly commuting regular,then it is strongly π-regular by theorem 2.12,so by lemma 2.4(2) [3], showing $N(R)=J(R)$.

Lemma 2.14. If R is a strongly commuting regular, then each homomorfic image of R is a strongly commuting regular ring.

Proof. Let S be a ring, $\Omega : R\longrightarrow S$ be an epimorphic and $a,b\in S$, then there exist $c,d\in R$ such that $a=\Omega(c)$, $b=\Omega(d)$.Since R is a strongly commuting regular ring, there exists t such that $cd=(dc)^2t(dc)^2$ and so;

$ab=\Omega(c)\Omega(d)$
$=\Omega(cd)=\Omega((dc)^2t(dc)^2)$
$=\Omega((dc)^2)\Omega(t)\Omega((dc)^2)$
$=(\Omega(dc))^2\Omega(t)(\Omega(dc))^2$
$=(\Omega(d)\Omega(c))^2\Omega(t)(\Omega(d)\Omega(c))^2=(ba)^2c(ba)^2$.

Where $c=\Omega(t)$.

3.Strongly Commuting Regular Group Rings

Let R be a ring and G a group. we shall denote the group ring of G over R as RG, the augmentation ideal of RG is generated by $\{1-g \mid g\in G\}$. We shall use Δ to denote the augmentation ideal of RG, it is known that R is a homomorphic image of RG since $RG/\Delta\cong R$. If RG is strongly commuting regular, then R is strongly commuting regular.

Corollary 3.1. Let R be a ring and G a group. If RG is a strongly commuting regular ring, then R is a strongly commuting regular ring.

Lemma 3.2. Every factor ring of strongly commuting regular ring is strongly commuting regular .

Proof. Let R be strongly commuting regular and commuting regular , we have for each $x,y\in R$, there exists $z\in R$;

$$xy=(yx)^2z(yx)^2.$$

thus we have

$$((x+I)(y+I))=((y+I)(x+I))^2(z+I)((y+I)(x+I))^2$$

therefore $\overline{x}\overline{y}=(\overline{y}\overline{x})^2\overline{z}(\overline{y}\overline{x})^2$.It follows that R/I is strongly commuting regular.

Proposition 3.3. Let $R=R_1\times R_2$ be a ring.Then R is strongly commuting regular if and only if so is each R_i when $i=1,2$.

Proof. If R is strongly commuting regular then by lemma 3.16 we have each R_i is strongly commuting regular, when $i=1,2$.Coversely if R_i is strongly commuting regular. Then we have for each $x_1,x_2\in R_1$, $y_1,y_2\in R_2$; there exist $c_1\in R_1$, $c_2\in R_2$ such that $x_1x_2=(x_2x_1)^2c_1(x_2x_1)^2$, $y_1y_2=(y_2y_1)^2c_2(y_2y_1)^2$.let $z_1=(x_1,y_1)$, $z_2=(x_2,y_2)$ be belong to R. Then we have

$z_1z_2 = (x_1,y_1)(x_2,y_2)$
$= (x_1x_2,y_1y_2)$
$= ((x_2x_1)^2c_1(x_2x_1)^2, (y_2y_1)^2c_2(y_2y_1)^2)$
$= ((x_2x_1)^2,(y_2y_1)^2)(c_1,c_2)((x_2x_1)^2,(y_2y_1)^2)$

$=((x_2,y_2)(x_1,y_1))^2c((x_2,y_2)(x_1,y_1))^2$
$=(z_2z_1)^2c(z_2z_1)^2$

when $c=(c_1,c_2)\in R$.

Corollary 3.4. A direct product $R=\prod_{i\in I} R_i$ of rings $\{R_i\}_{i\in I}$ is strongly commuting regular if and only if so is each R_i when $i\in I$.

Theorem 3.5. Let R be a ring in which 2 is invertible and $G=\{1,g\}$ be a group.then RG is strongly commuting regular if and only if R is strongly commuting regular.

Proof. If RG is strongly commuting regular then by corollary 3.1, R is strongly commuting regular.Conversely, if R is strongly commuting regular and 2 is invertible in R then $RG\cong R\times R$ via the map $a+bg\leftrightarrow(a+b,a-b)$. hence RG is strongly commuting regular by proposition 3.3.

4. Idempotents in Strongly Commuting Regular Rings

For any idempotent e in a ring,we have the peirce decompositions:

$R=eRe\oplus eRf\oplus fRe\oplus fRf$

Where $f=1-e$ is the complementary idempotent to e. Two rings eRe and fRf be characterized by the equations:

$$eRe=\{rR : er=r=re\},\ fRf=\{rR : ef=r=rf\}$$

Remark 4.1. e is a central idempotent iff $fRe=eRf=0$.

Lemma 4.2. Let R be a strongly commuting regular ring. For every idempotent e in it,we have the peirce decomposition:$R=eRe\oplus fRf$

Proof. by proposition 2.2 and remark 4.1.

Proposition 4.3. Let $e=0$ be every idempotent in R Then R is a strongly commuting regular ring if and only if eRe and fRf are strongly commuting regular.

Proof. By proposition 3.3 and lemma 4.2.

Proposition 4.4. Let R be a strongly commuting regular domain. If R be reduced then any idempotent is local idempotent.

Proof. If R be a strongly commuting regular domain then eRe is a strongly com-muting regular domain. Since R is reduced so is eRe. Therefore eRe is a local ring by proposition 2.11, then by definition , e is an local idempotent.

Remark 4.5. A local idempotent is always a primitive idempotent.

Corollary 4.6. In studing the structure of idempotent,if R be a strongly commuting regular domain and reduced then we have following chart:

idempotent ⟶central idempotent⟶local idempotent ⟶primitive idempotent.

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