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Fibonacci Length of Certain Coxeter's Families of Group Presentations

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ABSTRACT

In this paper, we consider some certain Coxeter's families of group presentations and compute their Fibonacci length in finite cases. First we study the Fibonacci length of finitely presented 4- parametric groups.

$$< a,b \mid a^{l} = b^{m}, (ab)^{n} = (ab^{-1})^{k} >$$

for positive integers l, m, n and k. They are indeed extensions of $(l, m|n, k)$-groups of M. Edjvet and R.M. Thomas considered for their finiteness property in 1997. We prove that there are some subclasses of these groups which are non-isomorphic groups of the same Fibonacci length. More interesting result is that, these lengths are independent of one of the involved parameters of the groups, and also the lengths involve the Wall number $\kappa(n)$. Moreover, the Fibonacci lengths of two homomorphic images of the groups have been calculated and compared with those of the groups.

Then, we consider the groups presented that by $(p, q, m; n) = < a,b \mid a^{p} = b^{q} = (ab)^{m} = [a,b]^{n} = 1 >$ where m, n, p and q are positive integers, and give some formulas for their Fibonacci lengths.

KEYWORDS: Groups, Fibonacci lengths, Special Automorphisms.
AMS Subject Classification: 20F05, 20B05, 20F28.

INTRODUCTION

The 4-paraeter groups

$$(1) \qquad G = (l, m \,|n, k) = < a,b \mid a^{l} = b^{m} = (ab)^{n} = (ab^{-1})^{k} = 1 >$$

were studied in 1997 by M. Edjvet and R.M. Thomas (see [12]) where, $l, m, n, k \geq 1$, $l \leq m$ *and* $n \leq k$. The study of these groups is a continuation of the study of the Coxeter groups of [5, 6] and the groups

$$\bar{G} = < a,b \mid a^{l} = b^{m}, (ab)^{n} = (ab^{-1})^{k} = 1 >$$

and

$$\bar{\bar{G}} = < a,b \mid a^{l} = b^{m}, (ab)^{n} = (ab^{-1})^{k} >$$

considered by H. Doostie, R. Gholamie and R.M. Thomas in 2003 (see [9]). It is shown in [9] that the group G is finite only in the following cases:

$(2,2 \,|n,k)$,	$n \geq 2$;	$(l,m \,|2,2)$,	$l \geq 4$;
$(3,3 \mid 3,k)$,	$k \geq 3$;	$(7,8 \,|2,3)$;	
$(3,3 \mid 4,4)$;		$(3,4 \,|3,4)$;	
$(3,3 \,|4,5)$;		$(3,4 \,|3,5)$;	
$(3,5 \mid 3,4)$;		$(4,m \,|2,3)$,	$m = 6,7,8,9$;
$(4,5 \mid 2,k)$,	$k = 3,4,5$;	$(5,5 \,|2,4)$;	
$(6,7 \,|2,3)$;		$(7,7 \,|2,3)$;);	
$(4,4 \,|2,k)$,	$k \geq 3$;	$(5,m \,|2,3)$,	$m \geq 3$;
$(2,3|n,k)$	$3 \leq gcd(n,k) \leq 5$;	$(2,3|n,k)$,	$3 \leq gcd(n,k) \leq 5$;
$(2,m|n,k)$,	$m \geq 3$, $gcd(n,k) \leq 2$;	$(2,m|n,k)$,	$m = 4,5, gcd(n,k) = 3$;
$(5,m|2,k))$,	$gcd(m,k) \leq 5$;	$(3,m \,|3,3)$,	$m \geq 4$.

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We consider all of the finite cases of $G, \bar{G}$ and $\bar{\bar{G}}$ and study the Fibonacci length of them. This computation compares the Fibonacci length of the extensions of *G* and gives us explicit formulas for lengths which are in certain cases independent of at least one of the parameters of the groups.

Also, in the last section, we consider the groups presented by

$$(2) \qquad (p, q, m; n) = < a, b \mid a^p = b^q = (ab)^m = [a, b]^n = 1 >,$$

where m, n, p and q are positive integers, and give some formulas for their Fibonacci lengths. These groups and the groups presented by

$$(3) \quad G^{m,n,p} =< a, b, c \mid a^m = b^n = c^p = (ab)^2 = (bc)^2 = (ca)^2 = (abc)^2 = 1 >$$

are called Coxeter's Families of groups presentations and have studied by many authors such as Edjvet, Juhasz and recently Havas and Holt (for example see [10, 11].

All of the groups $(m, n|l, k), (p, q, m; n)$ and $G^{m,n,p}$ (presented in (1), (2) and (3), respectively) have close relations with each other (from finiteness point of view and etc.) acording to the main works of Coxeter([5, 6]).

The periodic sequences of elements of finite algebraic structures have been studied by many authors, one may see [2, 5, 3, 4, 12, 9], for examples. Following the notations of the article [4] where, for a 2-generated non-abelian finite group $G =< a, b >$,

$$x_1 = a, x_2 = b, x_i = x_{i-2}x_{i-1}, \qquad i \geq 3$$

is called the Fibonacci sequence of *G* depending the generating set *{a,b}*. The least integer k (denoted by *LEN(G)*) such that $x_{k+1} = x_1$ and $x_{k+2} = x_2$ is called the Fibonacci Length of *G*, and the least integer m (denoted by *BLEN(G)*) such that $|x_{m+1}| = |x_1|$ and $|x_{m+2}| = |x_2|$, is called the basic Fibonacci length of *G*, where the $|x|$ denotes the order of the element x in the group *G*. Note, it is proved that *BLEN* divides *LEN* and the map $\theta : G \to G$ given by $a \to x_{m+1}$ and $b \to x_{m+2}$ is an automorphism of *G* with order $LEN/BLEN$ (for more details one may refer to [4]). We call this *θ*, the special automorphism of *G*. In 1990 Campbell, Doostie and Robertson [4] gained to compute the length of the groups D_{2n} and Q_{2^n}, and the simple groups of order less than 10^5. The Fibonacci length and the basic Fibonacci length of the groups $Aut(D_{2n})$ and $Aut(Q_{2^n})$ have been computed in 2000 by Doostie and Campbell (see [7]).

The Fibonacci sequence $\{f_n\}_{n=0}^{\infty}$ of numbers defined by $f_n = f_{n-2} + f_{n-1}$ for $n \geq 0$, and we seed the sequence with $f_0 = 0$ and $f_1 = 1$. Modulo some integer $n \geq 2$, it must ultimately become periodic as there are only n^2 different pairs of residues modulo n. Further, throughout this paper, we use $\kappa(n)$ to denote the fundamental period of the Fibonacci sequence modulo n, and call it the Wall number of n (see [14]).

We attempt here to calculate the Fibonacci lengths of certain remained cases of $\bar{G}$ *and* $\bar{\bar{G}}$ of the article [9].

2. THE GROUPS $(2, 2|n, K), (n \geq 2, k \geq 2, n \leq k)$

As a result of Proposition 2.1 of [9] we get the finite cases as follows :

$$\begin{cases} if \quad d = \gcd(n, k) & then \quad G \cong D_{2d,} \text{ the dihedral groups of order } 2d, \\ if \quad n \neq k & then \quad \bar{G} \cong D_{2|n-k|,} \\ if \quad n \neq k & then \quad \bar{\bar{G}}, \text{is finite of order } 4n|n-k|. \end{cases}$$

As a quick result of [4] we deduce that $LEN_{\{a,b\}}(G) = LEN_{\{a,b\}}\bar{G} = 6, BLEN_{\{a,b\}}(G) = 3$ and $\theta : G \to G$ will be the special automorphism and then,

$$\theta : \begin{cases} a \to bab, \\ b \to b. \end{cases}$$

To calculate the Fibonacci length of $\bar{\bar{G}}$ we consider the Fibonacci sequence $\{f_i\}$ of numbers as

$$f_1 = f_2 = 1, \qquad f_{i+1=} f_i + f_{i-1} \quad i \geq 2.$$

and define the Fibonacci sequence of elements of $\bar{\bar{G}}$ as usual by:

$$x_1 = a, x_2 = b, x_i = x_{i-2}x_{i-1}, \qquad i \geq 3.$$

Then, we have:

Lemma 2.1. *For every integers n and k where* $n \neq k$ *and* $n, k \geq 2$, *every element of sequence* $\{x_i\}$ *may be presented by*

$$x_{i=} \begin{cases} a^{f_i}, & if \ \ i \equiv 1 & (mod\ 6), \\ a^{-1+f_i}b, & if \ \ i \equiv 2,3, -1 & (mod\ 6), \\ a^{-3+f_i}bab, & if \ \ i \equiv -2 & (mod\ 6), \\ a^{-2+f_i}ba, & if \ \ i \equiv 0 & (mod\ 6). \end{cases}$$

Proof. For $i \leq 6$ we get the result at once. Then using an induction method (by considering six cases) we may get the result. For example let $i \equiv 1\ (mod\ 6)$ then, $i-1 \equiv 0\ (mod\ 6)$ and $i-2 \equiv -1\ (mod\ 6)$. So,

$$\begin{aligned} x_i &= x_{i-2}x_{i-1} \\ &= a^{-1+f_{i-2}}.b.a^{-2+f_{i-1}}.ba \quad \text{(by induction hypothesis)} \\ &= a^{-1+f_{i-2}}.a^{-2+f_{i-1}}.b^2a \quad (\text{for, } a^2 = b^2 \text{ and } f_{i-1} \text{ is even in this case}) \\ &= a^{-3+f_i}.a^2a \qquad (\text{for, } a^2 = b^2) \\ &= a^{f_i}. \end{aligned}$$

as required. □

Proposition 2.2. *For every integers n and k where $2 \leq n \leq k$, the Fibonacci length of the group $\bar{\bar{G}}$ is independent of k. $t = LEN_{\{a,b\}}\bar{\bar{G}}$ if and only if $t \equiv 0\ (mod\ 6)$ and $f_{t+3} \equiv 2\ (mod\ 4n)$.*

Proof. As a result of the proof of Proposition 2.1 of [9] we conclude the validity of the relation $a^{4n} = 1$ in the group $\bar{\bar{G}}$. So, $\bar{\bar{G}}$ has a presentation isomorphic to

$< a, b\ |a^{4n} = 1, a^2 = b^2, (ab)^n = (ab^{-1})^k >$.

Now, let $t = LEN_{\{a,b\}}\bar{\bar{G}}$. Then, t is the least positive integer such that the equations $x_{t+1} = x_1$ and $x_{t+2} = x_2$ hold in $\bar{\bar{G}}$. Equivalently we get:

$$t+1 \equiv 1\ (mod\ 6), a^{f_{t+1}} = a, a^{-1+f_{t+2}}.b = b$$

by considering the Lemma 2.1 and its possible cases. These equations yield in turn the numerical equations

$$\begin{cases} f_{t+1} \equiv 1 & (mod\ 4n) \\ f_{t+2} \equiv 1 & (mod\ 4n) \\ t \equiv 0 & (mod\ 6), \end{cases}$$

by considering the relation a^{4n}=1 of $\bar{\bar{G}}$. Consequently we get $f_{t+3} \equiv 2\ (mod\ 4n)$ and $t \equiv 0\ (mod\ 6)$, as required. □

Proposition 2.3. *For every integers n and k where $n \leq k$, $LEN_{\{a,b\}}\bar{\bar{G}} = \kappa(2^n.n)$. So, the Fibonacci length of the group $\bar{\bar{G}}$ is independent of k.*

Proof. It is similar to the proof of 2.2. □

3. THE GROUPS $(l, m|2, 2), (l \geq 4, m \geq 2)$

Let

$$\bar{\bar{G}} = < a, b\ |\ a^l = b^m, (ab)^2 = (ab^{-1})^2 >$$

by the Lemma 3.1 of [9], $|\bar{\bar{G}}| = 4l$ and its proof shows that the relation $a^{4m} = 1$ holds in $\bar{\bar{G}}$. Similar to the last session, every element of the Fibonacci sequence of elements of $\bar{\bar{G}}$ may be presented as follows

$$x_1 = a, x_2 = b,$$

$$x_k = \begin{cases} a^{f_{k-2}}, & \text{if } k \equiv 1, -2 \quad (mod 6), \\ a^{f_{k-2}}.b, & \text{if } k \equiv 2 \text{ or } 3 \quad (mod 6), \\ a^{f_{k-2}}.b^{-1}, & \text{if } k \equiv 0 \text{ or } -1 \quad (mod 6), \end{cases}$$

for every $k \geq 3$. The proof is easy by considering different cases for k and by using an induction method. Using this feat we get the following result:

Proposition 3.1. *(i) For non-abelian cases of G (at least one of l and m is even), LEN(G) =6, (ii) For every even value of l and odd values of m, $t = LEN_{\{a,b\}}\bar{G}$ if and only if $f_{t-2} \equiv 1\ (mod\ 4l)$ and $f_{t-1} \equiv 0\ (mod\ 4l)$, (iii) For every integers l and m where $l \geq 4$, the Fibonacci length of the group $\bar{\bar{G}}$ is independent of m, and $t = LEN_{\{a,b\}}\bar{\bar{G}} = \kappa(2^2.l)$.*

Proof. (i) It is sufficient to consider the Fibonacci sequences of elements $x_1 = a, x_2 = b, x_3 = ab, x_4 = bab = a^{-1}$, $x_5 = aba^{-1}, x_6 = ba^{-1}, x_7 = aba^{-1}ba^{-1} = a,\ x_8 = b$. So *LEN(G)* =6 for all values of l and *m* whenever at least one of them is even.

(ii) In $\bar{\bar{G}}$ and $\bar{G}$ the relation $(ab)^2 = (ab^{-1})^2$ yields $aba^{-1} = b^{-1}$ (for *m* odd) and then we get $a^tba^{-t} = b^{\pm 1}$ if *t* is even or odd, respectively. Then every element of the Fibonacci sequence x_i of $\bar{\bar{G}}$ and $\bar{G}$ will be presented as:

$$x_{k=}\begin{cases} a^{f_{k-3}}, & \text{if } k \equiv 1,4 & (mod\ 6), \\ a^{f_{k-3}}.b, & \text{if } k \equiv 2\ or\ 3 & (mod\ 6), \\ a^{f_{k-3}}.b^{-1}, & \text{if } k \equiv 0\ or -1 & (mod\ 6), \end{cases}$$

where $k \geq 7$ and $x_1 = a, x_2 = b$, $x_3 = ab, x_4 = a$, $x_5 = a^2b^{-1}$, $x_6 = a^3b^{-1}$. The proof is straight forward and follows by induction on k. Let $t = LEN(\bar{G})$. Then $x_{t+1} = x_1$ and $x_{t+2} = x_2$; i.e. $t \equiv 0\ (mod\ 6)$, $f_{t-2} \equiv 1\ (mod\ l)$ and $f_{t-1} \equiv 0\ (mod\ l)$as required. On the other hand, the relation b^{4m}=1 holds in $\bar{\bar{G}}$ (see [9]). So b^{4l}=1 holds in $\bar{\bar{G}}$.

(iii) The proof is similar to the proof of (ii). □

4. THE GROUPS $(\mathbf{2},\mathbf{3}|\boldsymbol{n},\boldsymbol{k})$, $(\mathbf{3} \leq \boldsymbol{gcd}(\boldsymbol{n},\boldsymbol{k}) \leq \mathbf{5})$

For the values $gcd(n,k) = 3,4,5$ we have the groups

$$G_1 = < a,b \mid a^2 = b^3 = (ab)^3 = 1 >,$$
$$G_2 = < a,b \mid a^2 = b^3 = (ab)^4 = 1 >,$$
$$G_3 = < a,b \mid a^2 = b^3 = (ab)^5 = 1 >,$$

respectively. Then we deduce:

Proposition 4.1. $LEN(G_1) = 16$, $LEN(G_2) = 18$, $LEN(G_3) = 50$. *Moreover, the special automorphisms are given by:*

$$\theta_1 : \begin{cases} a \to a, \\ b \to ba, \end{cases} \qquad \theta_2 : \begin{cases} a \to a, \\ b \to aba, \end{cases} \qquad \theta_3 : \begin{cases} a \to bab^{-1}, \\ b \to bab^{-1}ab^{-1}. \end{cases}$$

Proof. The Fibonacci sequence of elements for G_1 are

$$a, b, ab, bab, ab^{-1}ab, abab, ab^{-1}, b, a, ba, aba,$$
$$b^{-1}a, abab^{-1}a, ab^{-1}a, b^{-1}, ba, a, b$$

which gives that $LEN(G_1) = 16$. Considering the orders yields $N(G_1) = 8$, and θ_1 comes immediately. For G_2 we do as before and simplify the words. We get:

$$a, b, ab, bab, ab^{-1}ab, ab^{-1}abab, b^{-1}ab^{-1}, ab^{-1}a, ab$$
$$a, aba, ba, b^{-1}ab^{-1}, b^{-1}aba, bab^{-1}, bab, b^{-1}, ba, a, b.$$

So, $LEN(G_2) = 18$, $BLEN(G_2) = 9$, and θ_2 will be defined as required. To simplify the words of G_3 we have to use some extra relations of G_3. Using $a^2 = b^3 = (ab)^5 = 1$, we will get the identities :

$$(ab^2ab)^5 = 1,$$
$$(ab^2ab^2ab)^3 = 1,$$
$$(ab^2ab^2abab^2ab)^2 = 1,$$

in G_3. Now the long words of the Fibonacci sequence has to be simplified by hand calculations. So, we get:

$$a, b, ab, bab, ab^2ab, babab^2ab, ab^2ab^2abab^2ab, b^{-1}aba$$
$$babab^{-1}, bab, bab^{-1}, bab^{-1}, bab^{-1}ab^{-1}, ab^{-1}, b^{-1}aba,$$
$$ababa, b^{-1}abb^{-1}aba, abab^{-1}abab^{-1}a, ab^{-1}ab^{-1}a, ab^{-1}ab^{-1}ababab^{-1}a,$$
$$b^{-1}aba, abab^{-1}ababab^{-1}a, b^{-1}aba, abab^{-1}ababab^{-1}ab^{-1}aba, abab^{-1}ab^{-1}abab^{-1}a$$
$$b^{-1}ab^{-1}, ab^{-1}ab^{-1}a, ab, ab^{-1}a, abab^{-1}a, b^{-1}a, abab^{-1}ab^{-1}a, ab^{-1}ab^{-1}a,$$
$$abab^{-1}abab^{-1}a, ababab^{-1}a, ab^{-1}ab, b^{-1}a, ab^{-1}, b, a, ba, aba, b^{-1}a, abab^{-1}a,$$
$$ab^{-1}a, ababa, ba, b^{-1}ab^{-1}, bab^{-1}ab^{-1}, bab^{-1}, bab, b^{-1}, ba, a, b.$$

Now, by simple calculations we can show that all of the consecutive pairs before the 51st term is not equal to the generators set, i.e. $\{a,b\}$, except the first and second terms. Therefore, $LEN(G_3) = 50$, $BLEN(G_3) = 10$ and

$$\theta_3 = \begin{cases} a \to bab^{-1}, \\ b \to b(ab^{-1})^2. \end{cases}$$

This completes the proof. □

5. THE GROUPS $(\mathbf{2},\boldsymbol{m}|\boldsymbol{n},\boldsymbol{k})$, $(\boldsymbol{m} \geq \mathbf{3}, \boldsymbol{gcd}(\boldsymbol{n},\boldsymbol{k}) \leq \mathbf{2})$

If $gcd(n,k) = 1$ then G is abelian and there is nothing to calculate, and if $gcd(n,k) = 2$, then

$$G = < a,b \mid a^2 = b^m = 1, (ab)^2 = 1 > \cong D_{2m}$$

So, $LEN = 6$, $BLEN = 3$.

6. THE GROUPS $(\mathbf{2},\boldsymbol{m}|\boldsymbol{n},\boldsymbol{k})$, $(\boldsymbol{gcd}(\boldsymbol{n},\boldsymbol{k}) = \mathbf{3}, \boldsymbol{m} = \mathbf{4},\mathbf{5})$

For $m = 4$, G is equal to:

$G = (2,4|n,k) = < a,b \mid a^2 = b^4 = (ab)^3 > \cong < a,c \mid a^2 = c^3 = (ac)^4 = 1 >.$

So, as we showed in section 4, $LEN(G) = 18$. For $m = 5$, we have $G \cong < a, c \mid a^2 = c^3 = (ac)^5 = 1 >$, and so $LEN(G) = 50$.

7. SPECIAL AND REMAINING CASES

In the sixteen finite cases for G, not all of the groups $\bar{G}$ and $\bar{\bar{G}}$are finite (see [12,9]). In the following table we have collected the computer results for the orders and Fibonacci lengths (the codes were written in GAP ([13]).

$(l, m \mid n, k)$	$\lvert G \rvert$	$\lvert \bar{G} \rvert$	$\bar{\bar{G}}$	$LEN(G)$	$LEN(\bar{G})$	$LEN(\bar{\bar{G}})$
(7,8|2,3)	10752	∞	∞	256	-	-
(3,3|4,4)	168	1008	8064	10	120	120
(3,4|3,4)	168	∞	∞	14	-	-
(3,3|4,5)	1080	3240	51840	22	88	264
(3,4|3,5)	1080	∞	∞	20	-	-
(3,5|3,4)	1080	1080	34560	20	20	240
(4,6|2,3)	120	3840	∞	30	30	-
(4,7|2,3)	168	168	4368	128	128	2688
(4,8|2,3)	1152	∞	∞	36	-	-
(4,9|2,3)	2448	2448	∞	480	480	-
(4,5|2,3)	1	5	15	-	-	-
(4,5|2,4)	160	320	4480	30	30	240
(4,5|2,5)	360	360	28080	128	128	896
(5,5|2,4)	360	3600	43200	80	240	240
(6,7|2,3)	1092	1092	∞	392	392	-
(7,7|2,3)	1092	7644	∞	30	240	-

The remaining cases for $\bar{G}$ and $\bar{\bar{G}}$ which have not been studied for their finiteness property, are as follows:

$LEN(\bar{\bar{G}})$:

$(4,4 \mid 2, k)$, k= 4,5;

$(2,3 \mid n, k)$, $gcd(n, k) = 5$ and $\frac{n+k}{5} \equiv \pm 1 \ (mod\ 3)$;

$(2,3 \mid n, k) and\ (2,5 \mid n, k)$, $gcd(n, k) = 3,4$;

$(2,4 \mid n, k)$, $gcd(n, k) = 3$;

$(3, m \mid n, k)$; $gcd(m, k) = 1, \dots, 5$;

$(2, m \mid n, k)$, $m \geq 3$, $gcd(m, k) = 1,2$;

$(5, m \mid 2,3)$, $m \geq 3$.

$LEN(\bar{G})$:

$(3, m \mid 2, k)$, $gcd(m, k) = 1, \dots, 5$;

$(5, m \mid 2,3)$, $m \geq 3$;

$(2,3 \mid n, k)$, $gcd(n, k) = 5$ and $\frac{n+k}{5} \equiv \pm 1 \ (mod\ 3)$;

$(2,5 \mid n, k)$ $gcd(n, k) = 3$.

8. CONJECTURES

Using some procedures in GAP ([13]), related to the computations of Fibonacci length, we give the following conjectures on the groups of $(3,3 \mid 3, k)$, $k \geq 3$*:*

(i) For every $k \geq 3$, $LEN(G) = \frac{8k}{gcd(k,3)}$ and,

$$LEN(\bar{G}) = \begin{cases} 24k, & if\ k = 6 + 2^{t+1}, \\ 8k, & \text{Otherwise.} \end{cases}$$

(ii) Let $\bar{\bar{G}} = U_k$. Then, $LEN(U_{2^t}) = LEN(U_{3.2^t}) = 3.2^{t+3}$.

9. THE GROUPS $(p, q, m; n)$

For positive integer m, n, p and q, let G the group defined by the presentation

$G = (p, q, m; n) = < a, b \mid a^p = b^q, (ab)^m = [a, b]^n = 1 >$.
In any of m, n, p or q equals *1* then clearly $(p, q, m; n)$ is finite abelian, so for the reminder of this paper we assume that each of m, n, p and q is at least *2*. These groups were studied by Edjevet in [10], and the main result of that paper is the following:
Theorem 9.1. *If* $2 \leq p \leq q \leq m$, $2 \leq n$, $(p, n) \neq (3,2)$, *and* $(p, q, m; n)$ *is not* $(2, 3, 13; 4)$ *then the group* $(p, q, m; n)$ *is finite if and only if it is one of the following:*

$$
\begin{array}{ll}
(2,2,m;n), & (2 \leq m, 2 \leq n); \\
(2,3,m;n), & (2 \leq m, 2 \leq n \leq 3); \\
(2,3,m;n), & (3 \leq m \leq 6, 4 \leq n); \\
(2,3,7;n), & (4 \leq m \leq 8); \\
(2,3,m;n), & (8 \leq m \leq 9, 4 \leq n \leq 5); \\
(2,3,m;4), & (10 \leq m \leq 11); \\
(2,4,m;2), & (4 \leq m); \\
(2,4,4;n), & (3 \leq n); \\
(2,4,5;n), & (3 \leq n \leq 4); \\
(2,4,7;3); & \\
(2,5,m;2), & (5 \leq m \leq 9); \\
(2,6,7;2); & \\
(3,3,3;n), & (n \geq 3); \\
(3,3,4;3). &
\end{array}
$$

The groups $(p, q, m; n)$ and $(m, n|l, k)$ has close relations with each other and the groups $G^{m,n,p}$ defined by the presentation

$$< a, b, c \mid a^2 = b^2 = (ab)^2 = c^2 = (ac)^m = (bc)^n = (abc)^p = 1 >$$

according to the papers of the Edjevet, and Juhasz (see [10,11]).

Now, in this section we calculate the Fibonacci lengths of some finite cases of $(p, q, m; n)$.

Proposition 9.2. *Let* $G = (2,2, m; n)$. *Then the following statements are hold:*

(i) *if (m,n)=1, then LEN(G) = 3,*

(ii) *if* $(m, n) \neq 1$ *and m and n have a common prime factor, then*

$$LEN(G) = 6, \quad BLEN(G) = 3,$$

(iii) *if* $m = 2^t.r$, *where* $2 \nmid r$ *but* $2|n$, *and m and n have no common prime factor, then in case* $t = 1$, $LEN(G) = 3$, *but in case* > 1 $LEN(G) = 6$, $BLEN(G) = 3$.

Proof. (i) If $(m, n) = 1$, since

$$(2, 2, m; n) = < a, b \mid a^2 = b^2, (ab)^m = [a, b]^n = 1 >$$

and so $a^{-1} = a$ and $b^{-1} = b$, hence

$$[a, b]^n = (ab)^{2n} = 1.$$

Therefore, if $(m, 2n) = 1$, then $a = b^{-1} = b$ then $|G| = 2$, but if $(m, 2n) = 2$, then $(ab)^2 = 1$ and so the Fibonacci series of *G* is as follows:
$a, b, ab, bab = a, aab = b$.

(ii) Since for some prime $p, p|(m, 2n)$, then the Fibonacci series of *G* is as follows:
$a, b, ab, bab, abbab = b, babb = ba, bba = a, baa = b$.

Thus, $LEN(G)| 6$. On the other hand, since $(ab)^m$=1, $(ab)^{2n}$=1, and $p|(m, 2n)$, hence $(ab)^2 \neq 1$ and so $bab \neq a^{-1} = a$. Therefore, $LEN(G) = 6$.

(iii) If $t > 1$ then $2^2 \mid (m, 2n)$ and so $|ab| \geq 4$. Hence $|ab| \neq 2$ and so $bab \neq a$.
Therefore in this case we have $LEN(G) = 6$.

But in case of $t = 1$, we have $(m, 2n) = 2k$, where $2 \nmid$ k and so, $|ab| = 2$. Thus $bab = a$ and we have $LEN(G) = 3$. □

Proposition 9.3. *Let* $G = (2,3,3; n)$. *Then in case of n is even we have* $LEN(G) = 16$, *and in case of n is odd we have* $LEN(G) = 8$. *In both cases* $BLEN(G) = 4$.

Proof. By the relations of G, we have $b^2 = b$ and $a = a^{-1}$ and so the Fibonacci series of G is as follows:

$$a, b, ab, bab, abbab = [a,b], bab[a,b] = b^{-1}a, ab^{-1}, b, a, ba, aba, b^{-1}a, abab^{-1}a, ab^{-1},$$
$$abab^{-1}a^2b^{-1}a = b^2, ab^{-1}ab^{-1}, b^{-1}ab^{-1}ab^{-1} = aab^{-1}ab^{-1}a = b.$$

So, $LEN(G)|16$.

Now, since $[a,b]N = 1$ then $(ab^{-1}ab)^n = 1$. On the other hand, since $(ab)^3 = 1$ thus $bab^{-1} = ab^{-1}ab$. Therefore,
$(bab^{-1})^n = b^{-1}a^nb = 1$.
If n is odd then $bab^{-1} = 1$ and so $ba = b$ and in this case we have $LEN(G) = 8$.
But if n is even then we have $LEN(G) = 16$. □

Proposition 9.4. *Let* $G = (2,4,m;n)$. *Then in case of finiteness of G (as in the theorem 9.1) we have*

$$LEN(G) = \begin{cases} n(m+2), & if\ m = 4k\ (k \in Z), n \neq 2, \\ 3, & if\ m = 2k+1, n = 2, \\ 12, & if\ m = 4k, n = 2, \\ 24, & if\ m = 4k+2, n = 2. \end{cases}$$

Proof. It is similar to the proofs of the above propositions. □

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REFERENCES

[1] Aydin H., Dikici R. and Smith. C.G., (1993), Wall and Winson revisited, In Applications of Fibonacci numbers, eds. G. A. Bergum, et al., Vol.5, pp. 61-68.

[2] Aydin H. and Smith C.G., (1994), Finite P-quotients of some cyclically presented groups, J. Londan Math. Soc., 49, pp. 83-92.

[3] Brinson O.J., (1992), Complete Fibonacci sequences in finite fields, Fibonacci Quarterly, 30, pp. 295-304.

[4] Campbell C.M., Doostie H. and Robertson E.F., (1990), Fibonacci length of generating pairs in groups, In Applications of Fibonacci numbers, eds. G. A. Bergum, et al., Vol.3, pp. 27-35.

[5] Coxeter H.S.M., (1939), The abstract groups $G^{m,n,p}$, Trans. Amer. Math. Soc., 45, pp. 73-150.

[6] Coxeter H.S.M. and Moser W.O.J., (1984), Generators and Relation for Discrete Groups, 4th Edition, Springer-Verlag.

[7] Doostie H. and Campbell C.M., (2000), Fibonacci length of automorphism groups Involving Tribonacci numbers, Vietnam J. of Math., 28, pp. 57-65.

[8] Doostie H. and Golamie R., (2000), Computing of the Fibonacci lengths of finite groups, Internat. J. of Appl. math., 4(2), pp. 149-156.

[9] Doostie H., Gholamie R. and Thomas R.M., (2003), Certain Extension of $(l,m|n,k)$-groups, Southeast Asian Bulletin of Math, 27, pp. 21-34.

[10] Edjevet M., (1994), On certain quotients of the triangle groups, J. Algebra, 169, pp. 367-391.

[11] Edjevet M. and Juhnsz A., (2008), The groups $G^{m,n,p}$, J. Algebra, 169, pp. 248-266.

[12] Edjevet M. and Thomas R.M., (1997), The $(l,m|n,k)$-groups, J. Pure and Applied Algebra, 114, pp. 175-208.

[13] The GAP Group, (2006), GAP – Groups, Algorithms and Programming, Version 4.4 Aachen, St Andrews, (http://www.gap-system.org).

[14] Wall D.D., (1960), Fibonacci series modulo m, Amer. Math. Monthly, 67, pp. 525-532 .

[15] Wilcox H.J., (1986), Fibonacci sequences of period n in groups, Fibonacci Quart., 24, pp. 356-361.