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Identification of Linear Partial Difference Equations with Constant Coefficients

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ABSTRACT

In this paper, an algorithm based on the structural model is developed for parameter estimation of a linear two dimensional partial difference equation. Identification of two dimensional partial difference equations paves the way for identifying equations with higher dimensions. The proposed algorithm is based on two main parts, instrumental variable method and pre-filtering technique. Based on the analysis results, 2DPIV algorithm is presented. Finally, the performance of the proposed algorithm is evaluated by a simulation example.

KEYWORDS: Parameter Estimation, Partial Difference Equations, Structural Model, Instrumental Variable, Transfer Function.

I. INTRODUCTION

The parameter estimation of two-dimensional partial difference equations is an important problem that has applications in many branches of science such control, environmental science, image processing, aerospace, etc. 0, [2], [3]. In recent years, there has been increasing interest in the use of transfer function (TF) models in identification problems. TF modeling is appropriate for modeling of linear systems. TF models have been widely used for identification and parameter estimation problems.

There exist a number of approaches for the parameter estimation of partial difference equations [4]. During the last decades, several methods have been proposed for parameter estimation of ordinary difference equations, but parameter partial difference equation identification has not been studied as much as one dimensional system identification [5]. The authors In [4] have been proposed some methods to determine the model order of two dimensional ARMA that can be described as a linear partial difference equation. Hangbini *et al*. present a two dimensional system identification scheme that makes use of amplitude estimation [6]. There have been some other algorithms reported for the estimation of parameters in two dimensional systems, such as linear prediction based methods [7]–[8], least-squares (LS) methods [9]–[11], and maximum-likelihood (ML) methods [10], [12]–[3].

Instrumental Variable approaches to system identification have been developed since 1970s [13]. The IV approach to least squares parameter estimation has its foundations in classical statistical estimation theory, where it represents one approach to the problem of estimating structural model parameters [14]. In the structural model, the basic relationship between the parameters is in the normal estimation equation form but the elements of variable vector are not exactly known and can only be observed in error. The first application of the IV method in the process identification was by Joseph *et al*.[15]. Although they did not refer to it by name, Joseph *et al.* suggested an IV procedure for identifying the parameters of a process described by an ordinary difference equation model. For identification of partial difference equations, the IV method has not been used so far.

In this paper, SRIV algorithm [16] is evaluated and extended to the two dimensional extended algorithm for two dimensional partial difference equation. Young *et al.* proposed SRIV method to estimate the parameters of system model in one dimensional case [16]. The pre-filering technique in our algorithm, presents a consistent estimation that can be minimum variance for white noise. The order of partial difference equation is assumed to be known and the structure is fixed. Note that the paper does not attempt to determine the order of partial difference equation and the identification problem is to estimate the unknown parameter of system. This problem is known as parameter estimation.

The paper is organized as follows. Section II presents the modeling and problem formulation needed by the proposed two dimensional pre-filtered IV algorithms that is presented in section III. In section III, the general one dimensional IV method is extended to two dimensional IV method and then an algorithm to estimate the parameters of two dimensional partial difference equation, using the proposed two dimensional IV method is presented. Simulation results to support the efficiency of the proposed algorithm are presented in section IV. Finally, we draw some conclusions.

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II. Problem Formulation

In this section a two dimensional partial difference equation is formulated and then, described in transfer function (TF) form. It is assumed that the partial difference equation under estimation is a linear equation with constant coefficients. The input r and the output x are related by the following linear two dimensional partial difference equation:

$$\sum_{i=0}^{n_1}\sum_{j=0}^{n_2}\alpha_{ij}x(k-i,l-j) = \sum_{i=0}^{m_1}\sum_{j=0}^{m_2}\beta_{ij}r(k-i,l-j) \quad (1)$$

where the constant coefficients α_{ij}'s and β_{ij}'s in (1), are the parameters to be estimated. n_1, n_2 are the highest difference order of x with respect to the variables k and l, respectively. Also m_1, m_2 are the highest difference order of r with respect to the variables k and l, respectively. The set $\{(n_1,n_2),(m_1,m_2)\}$ is called as the order of equation. Here, we suppose $n_1 \geq max(m_1,m_2)$ and $n_1 \geq max(m_1,m_2)$. In TF terms, the above linear two dimensional partial difference equation can be written as the following TF:

$$x(z_1^{-1},z_2^{-1}) = \frac{N(z_1^{-1},z_2^{-1})}{D(z_1^{-1},z_2^{-1})}r(z_1^{-1},z_2^{-1}) \quad (2)$$

where

$$D(z_1^{-1},z_2^{-1}) = 1 + \alpha_{01}z_2^{-1} + \cdots + \alpha_{0n_2}z_2^{-n_2} + \alpha_{10}z_1^{-1} + \cdots + \alpha_{n_1n_2}z_1^{-n_1}z_2^{-n_2}$$

$$N(z_1^{-1},z_2^{-1}) = \beta_{00} + \beta_{01}z_2^{-1} + \cdots + \beta_{0m_2}z_2^{-m_2} + \beta_{10}z_1^{-1} + \cdots + \beta_{m_1m_2}z_1^{-m_1}z_2^{-m_2}$$

Operator z_k^{-1} is partial difference operator, i.e. $z_1^{-i}z_2^{-j}f(z_1^{-1},z_2^{-1})$ is the two dimensional z-transform of $f(k-i,l-j)$.

The output of system, x, is corrupted by additive measurement noise e. We consider the white noise, in this paper. The measurement y of x, can be written as the following observation equation:

$$y(k,l) = x(k,l) + e(k,l) \quad (3)$$

where $e(k,l)$ is the two dimensional white noise with zero mean and finite and constant variance.

By considering (2), the observation equation (3) can be written as the following two dimensional TF Model:

$$y(z_1^{-1},z_2^{-1}) = \frac{N(z_1^{-1},z_2^{-1})}{D(z_1^{-1},z_2^{-1})}r(z_1^{-1},z_2^{-1}) + e(z_1^{-1},z_2^{-1}) \quad (4)$$

We consider the unknown parameter vector $\boldsymbol{\theta}$, in terms of the parameters in the TF polynomials $D(z_1^{-1},z_2^{-1})$ and $N(z_1^{-1},z_2^{-1})$ as:

$$\boldsymbol{\theta} = \left[\alpha_{01} \dots \alpha_{0n_2}\ \alpha_{10} \dots \alpha_{n_1n_2}\ \beta_{00}\ \beta_{01}\ \dots \beta_{0m_2}\ \beta_{10}\ \beta_{m_1m_2}\right]^{\boldsymbol{T}} \quad (5)$$

The partial difference equation (1) can be formulated as the following vector form:

$$\mathrm{x}(\mathrm{k},\mathrm{l}) = \boldsymbol{\varphi}^{\mathrm{T}}(\mathrm{k},\mathrm{l})\boldsymbol{\theta} \quad (6)$$

such that

$$\boldsymbol{\varphi}^{T}(k,l) = [-x(k,l-1)\ \dots\ -x(k-n_1,l-n_2)\ r(k,l)\ \dots\ r(k-m_1,l-m_2)\,] \quad (7)$$

The number of measured samples are $N_1 \times N_2$, i.e. $(k,l) = (iT_1, jT_2)$ such that $(1 \leq i \leq N_1, 1 \leq j \leq N_2)$ and $i,j \in \mathbb{N}$.

Now, the parameter estimation problem posed by the TF model (4) is to estimate unknown parameter vector $\boldsymbol{\theta}$ based on $N_1 \times N_2$ uniformly sampled input-output data set $\{r(iT_1,jT_2); y(iT_1,jT_2)\}$ such that $(1 \leq i \leq N_1, 1 \leq j \leq N_2)$ and $i,j \in \mathbb{N}$. It should be noted that the order of the partial difference equation, i.e. set $\{(n_1,n_2),(m_1,m_2)\}$ is known a priori.

III. Parameter Estimation

According to the modeling and formulations stated above, in this section the parameter estimation problem is analyzed. At first, the data are pre-filtered by a suitable filter and then, by developing a two dimensional IV algorithm, the unknown parameters are yielded.

A. Data Pre-filtering

In this sub-section, the prediction error minimization (PEM) approach is used. Minimization of a least squares criterion function in ε, provides the basis for optimal parameter estimation. Under the Gaussian normality assumptions on $e(k_1,k_2)$, a suitable error function can be given by,

$$\varepsilon(z_1^{-1},z_2^{-1}) = \left(y(z_1^{-1},z_2^{-1}) - \frac{N(z_1^{-1},z_2^{-1})}{D(z_1^{-1},z_2^{-1})}r(z_1^{-1},z_2^{-1})\right) \quad (8)$$

which can be written as

$$\varepsilon(z_1^{-1},z_2^{-1}) = \frac{1}{D(z_1^{-1},z_2^{-1})}\left(D(z_1^{-1},z_2^{-1})y(z_1^{-1},z_2^{-1}) - N(z_1^{-1},z_2^{-1})r(z_1^{-1},z_2^{-1})\right) \quad (9)$$

The filter p can be defined as

$$p(z_1^{-1}, z_2^{-1}) = \frac{1}{D(z_1^{-1}, z_2^{-1})} \tag{10}$$

This filter is inverse of characteristic equation of system i.e. $1/D(z_1^{-1}, z_2^{-1})$. This filter has a physical meaning. Indeed, it attenuates all signals outside the pass band of system, including high frequency noise. By inserting filter p into brackets in (9) and pre-filtering y and r by p, (9) takes the following form:

$$\varepsilon(z_1^{-1}, z_2^{-1}) = \left(D(z_1^{-1}, z_2^{-1})y_p(z_1^{-1}, z_2^{-1}) - N(z_1^{-1}, z_2^{-1})r_p(z_1^{-1}, z_2^{-1})\right) \tag{11}$$

The subscripts p denote that the variable has been pre-filtered by filter p. As a result, the estimation equation can be written as:

$$y_p(k,l) = \boldsymbol{\varphi}_p^T(k,l)\boldsymbol{\theta} + e(k,l) \tag{12}$$

where

$$\boldsymbol{\varphi}_p^T(k,l) = \left[-y_p(k,l-1)\ \dots\ -y_p(k-n_1,l-n_2)\ r_p(k,l)\ \dots\ r_p(k-m_1,l-m_2)\right] \tag{13}$$

B. Two Dimensional IV Approach

Now, we present the two dimensional IV estimation method. This method is extended based on the traditional one dimensional IV approach. Given the estimation equation (12), the two dimensional IV normal estimation equations are obtained as:

$$\left(\frac{1}{N_1N_2}\sum_{i=1}^{N_1}\sum_{j=1}^{N_2}\widehat{\boldsymbol{\phi}}_p(k,l)\boldsymbol{\varphi}_p^T(k,l)\right)\boldsymbol{\theta} - \left(\frac{1}{N_1N_2}\sum_{i=1}^{N_1}\sum_{j=1}^{N_2}\widehat{\boldsymbol{\phi}}_p(k,l)y_p(k,l)\right) = 0 \tag{14}$$

It should be noted again that $(k,l) = (iT_1, jT_2)$.

Optimization with respect to $\boldsymbol{\theta}$ yields a two dimensional IV solution $\widehat{\boldsymbol{\theta}}$:

$$\widehat{\boldsymbol{\theta}} = argmin_{\boldsymbol{\theta}} \left\| \left(\frac{1}{N_1N_2}\sum_{i=1}^{N_1}\sum_{j=1}^{N_2}\widehat{\boldsymbol{\phi}}_p(k,l)\boldsymbol{\varphi}_p^T(k,l)\right)\boldsymbol{\theta} - \left(\frac{1}{N_1N_2}\sum_{i=1}^{N_1}\sum_{j=1}^{N_2}\widehat{\boldsymbol{\phi}}_p(k,l)y_p(k,l)\right)\right\|$$

$$\widehat{\boldsymbol{\theta}} = \left[\sum_{i=1}^{N_1}\sum_{j=1}^{N_2}\widehat{\boldsymbol{\phi}}_p(k,l)\boldsymbol{\varphi}_p^T(k,l)\right]^{-1}\left[\sum_{i=1}^{N_1}\sum_{j=1}^{N_2}\widehat{\boldsymbol{\phi}}_p(k,l)y_p(k,l)\right] \tag{15}$$

In these equations, $\widehat{\boldsymbol{\varphi}}_p$ is the IV vector defined as follows:

$$\widehat{\boldsymbol{\varphi}}_p^T(k_1,k_2) = \left[-\hat{x}_p(k,l-1)\ \dots\ -\hat{x}_p(k-n_1,l-n_2)\ r_p(k,l)\ \dots\ r_p(k-m_1,l-m_2)\right] \tag{16}$$

such that

$$\hat{x}(z_1^{-1}, z_2^{-1}) = \frac{\hat{N}(z_1^{-1}, z_2^{-1})}{\hat{D}(z_1^{-1}, z_2^{-1})} r(z_1^{-1}, z_2^{-1}) \tag{17}$$

Till now, two techniques pre-filtering and IV method have been given which are the bases for the main algorithm.

C. Two Dimensional Pre-filtered IV (2DPIV) Algorithm

In this section, we summarize the results as pre-filtered IV (PIV) algorithm to estimate the parameters of a partial difference equation.

2DPIV Algorithm:

Step 1. Make an initial estimation by using a discrete two dimensional estimation algorithm or based on prior knowledge of system.

Step 2. Generate the IV variables from (17) with the aid of estimation in last iteration (for first iteration, use initial estimation in Step 1).

Step 3. Pre-filter the input and output signals according to (16).

Step 4. Estimate the system parameters according to sub-section B.

Step 5. Check the convergence condition. If it is not satisfied, go to Step 2.

End

Let us conclude this section with some remarks:

Remark.1: The convergence condition in Step 5 can be described as: The covariance matrix **P** associated with estimation θ, converges to zero, as the number of samples goes infinity. The covariance matrix **P** is defined as follow:

$$\boldsymbol{P}(k,l) = \left[\sum_{\mathrm{i}=1}^{\mathrm{N}_1}\sum_{\mathrm{j}=1}^{\mathrm{N}_2}\widehat{\boldsymbol{\phi}}_p(k,l)\boldsymbol{\varphi}_p^T(k,l)\right]^{-1} \tag{18}$$

Remark.2: Applying IV method and pre-filtering technique in multi dimensional parameter estimation is a new idea and have not been fully investigated.

Remark.3: A necessary condition for identifiably of system is stability of the TF model, i.e. the roots of $D(z_1^{-1}, z_2^{-1})$ should lie inside the unit circle. This condition must be observed in Step 1 of the 2DPIV algorithm, too.

IV. Simulation Results

In the following example, the 2DPIV algorithm is applied to 2500 samples of simulated data generated by the following TF model:

$$x(k,l) + 0.6x(k,l-j) = 3r(k,l)$$

The input signal r is zero mean white noise with variance 1. Also e is zero mean white noise with variance 0.005. Here, $N_1 = 50$ and $N_2 = 50$. The observation y is shown in figure 1.

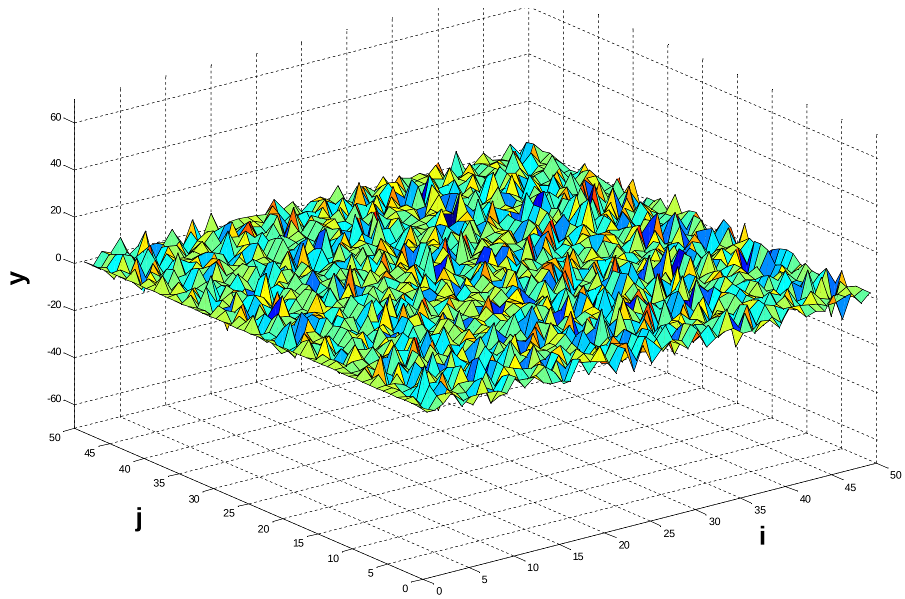

figure 1. Observation of system

In above partial difference equation, $\alpha_{01} = 0.6, \beta_{00} = 3$. Therefore the system parameters vector (5) are $\theta = [\alpha_{01}\ \ \beta_{00}]^T = [0.6 \quad 3]^T$. By applying 2DPIV algorithm and considering initial estimation as $\theta = [1\ 5]^T$, the estimation of system parameters vector after 2 iterations has been obtained as $\hat{\theta} = \left[a_{0,1}\ \ b_{0,0}\right]^T = [0.6002 \quad 2.9981]^T$. The rational error between $\hat{\theta}$ and its real value θ is about $[0.03\% \quad 0.06\%]^T$.

The error between the output x and its estimation $\hat{x}$ is shown in figure 2. The maximum error is 0.0073. It is obvious that the estimation error is too low.

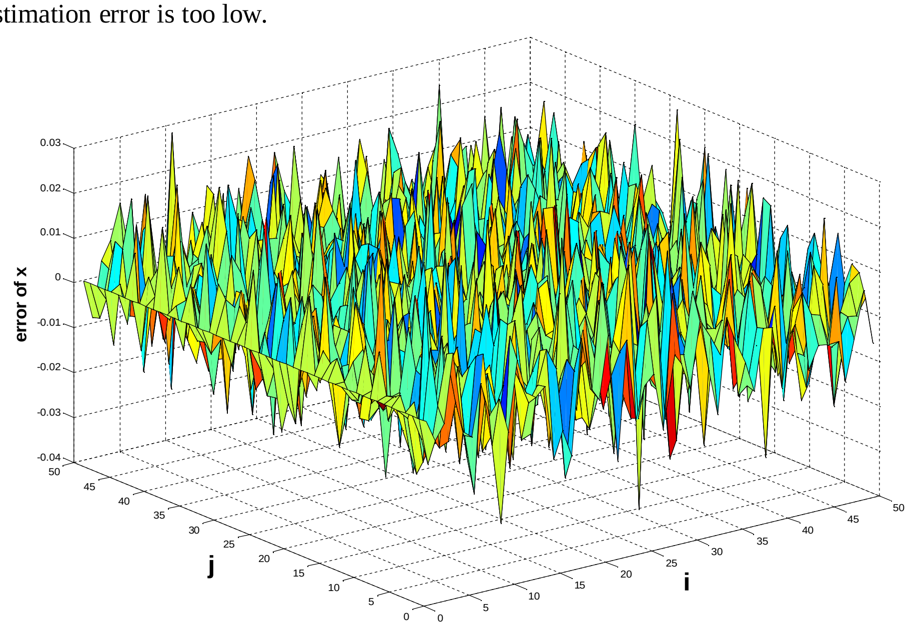

figure 2. Difference between output x and its estimation $\hat{x}$

Now we consider the output of system with two dimensional step input, i.e. $r(k,l) = 1$. Figure 3 shows the estimated output of system and the error between the real and estimated step response of system is shown in figure 4.

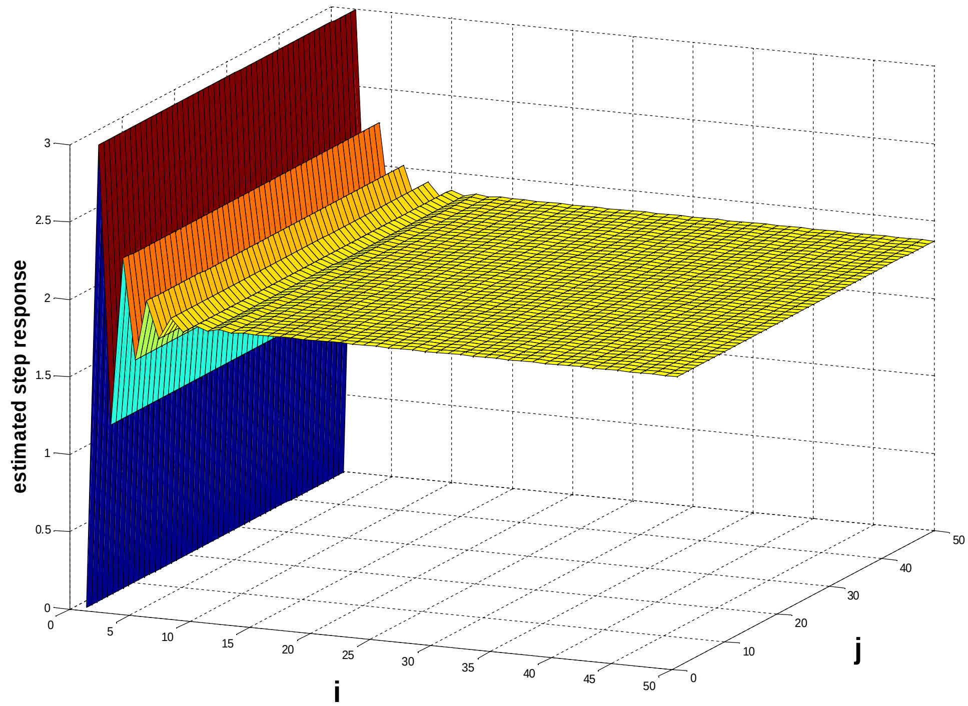

figure 3. Estimated output $\hat{x}$

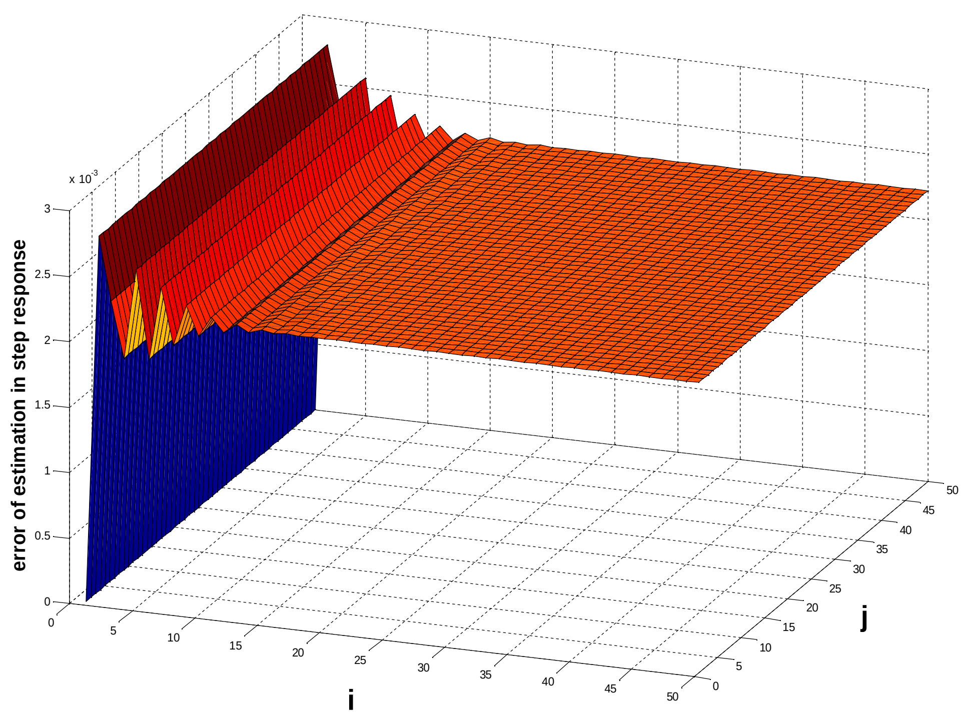

figure 4. Difference between the real and estimated step response of system

From figure 4 we can see that the amount of error is very low.

V. Conclusion

In this paper, pre-filtered IV (PIV) algorithm was presented to estimate the parameters of two dimensional partial difference equations. The algorithm was given based on two techniques: pre-filtering and IV method. An example was put worth to demonstrate the effectiveness of the proposed approach. There are many new topics in identification the partial difference equation that can be considered as future works such as model order determination and consideration the colored noise in analysis.

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