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# On Graphs Associated to Conjugacy Classes of Some Metacyclic 2-Groups

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**ABSTRACT**

We consider the graph Γ associated with the conjugacy classes of a group $G$. Its vertices are the non-central conjugacy class sizes of $G$, and two distinct vertices $u$ and $v$ are joined by an edge if and only if their class sizes have a non-trivial common divisor, i.e. gcd $(|u|,|v|) > 1$. In this article, we characterize certain properties of the graph Γ structured on some finite metacyclic 2- groups. More specifically, we show that the chromatic number and clique number of these graphs are the same.
**KEYWORDS:** Graph, Metacyclic group, Conjugacy class.

## INTRODUCTION

There are many possible ways for associating a graph with a group, for the purpose of investigating these algebraic structures using properties of the associated graph, see for example [1,3,8,10]. Referring to Bertram et al.[2] for a finite group$G$, the conjugacy class graph $\Gamma(G) = \Gamma$ is defined as the following: the vertex set is the set of conjugacy class sizes of $G$, and two distinct vertices $u$ and $v$ are connected by an edge if and only if their class lengths have a non-trivial common divisor i.e. $(|u|,|v|) > 1$. This graph has been widely studied. See, for instance [4] and [7].

The graphs of non-abelian$p$-groups are complete, i.e. there is an edge between any two vertices in $\Gamma$. Let $k$ be a positive integer. A $k$-vertex coloring of a graph $\Gamma$ is an assignment of $k$ color to the vertices of $\Gamma$ such that no two adjacent vertices have the same color. The vertex chromatic number $\chi(\Gamma)$ of a graph$\Gamma$, is the minimum $k$ for which $\Gamma$ has a $k$-vertex coloring.

A subset $\mathfrak{A}$ of the vertices of Γ is called a *clique* if the induced subgraph on $\mathfrak{A}$ is a complete graph. The maximum size of a clique in a graph Γ is called the *clique number* of Γ and denoted by ω(Γ ).

Suppose that $G$ is connected and if $u$ and $v$ are vertices in $G$, then $d(u, v)$ denotes the length of the shortest path between $u$ and $v$. The largest distance between all pairs of the vertices of $\Gamma$ is called the *diameter* of $\Gamma$ and is denoted by $d(\Gamma)$. A graph $\Gamma$ is connected if there is a path between each pair of the vertices of $\Gamma$. The length of the shortest cycle in a graph $\Gamma$ is called girth of $\Gamma$ and is denoted by girth($\Gamma$). The goal of this paper is to investigate certain properties of the conjugacy classes graph $\Gamma$,which are structured on some metacyclic 2- groups. In particular, we show that the chromatic number and clique number of the graph Γ are identical.

### 1. Some Basic Results

A group $G$is called *metacyclic* if it contains a normal cyclic subgroup whose quotient is also cyclic. If $G$ is a finite metacyclic $p$-group, then

$$G \simeq \langle a, b | a^{p^{\alpha}} = 1,\ b^{p^{\beta}} = a^{p^{\alpha - r}}, a^b = a^{\lambda} \rangle,$$

for some $\alpha, \beta, r$ and λ, we use the notation $\mathcal{G}(2, \alpha, r, \gamma)$ to denote the metacyclic 2-group of the form:

$$G \simeq \langle a, b | a^{2^{\alpha}} = 1,\ b^2 = a^{2^{\alpha - r}}, [b, a] = a^{2^{\alpha - \gamma} - 2} \rangle,$$

where $r, \gamma \in \{0,1\}$ and $\alpha$ is a positive integer such that$\alpha \geq 2$.

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The following theorem presents three classifications of the form$\mathcal{G}(2, \alpha, r, \gamma)$ of metacyclic 2-groups.

**Theorem 2. 1** (Beuerle, 2005) *Let $G$ be a metacyclic 2-group of type $\mathcal{G}(2, \alpha, r, \gamma)$. Then G is isomorphic to exactly one of the following:*

$$(1)\mathcal{G}(2, \alpha, 1, 1) = \langle a, b | a^{2^\alpha} = 1, b^2 = a^{2^{\alpha-1}}, [b, a] = a^{-2}\rangle \simeq Q_{2^{\alpha+1}},$$
$$(2)\ \mathcal{G}(2, \alpha, 0, 0) = \langle a, b | a^{2^\alpha} = b^2 = 1, [b, a] = a^{-2}\rangle \simeq D_{2^{\alpha+1}}, \quad (2.1)$$
$$(3)\mathcal{G}(2, \alpha, 0, 1) = \langle a, b | a^{2^\alpha} = b^2 = 1, [b, a] = a^{2^{\alpha-1}-2}\rangle \simeq SD_{2^{\alpha+1}},$$

*where $Q_{2^{\alpha+1}}$, $D_{2^{\alpha+1}}$ and $SD_{2^{\alpha+1}}$ are the generalized quaternion, dihedral and semi-dihedral groups of order $2^{\alpha+1}$, respectively.*

In this paper we focus on some specific metacyclic 2-groups which are given in Theorem 2.1. Our goal is to find the number of edges, diameter, girth,chromatic number and clique number of the generalized quaternion groups, dihedral groups, and semi-dihedral groups.

We denote by $k(G)$ the number of conjugacy classes of $G$. Also, $V(\Gamma)$ and $E(\Gamma)$ the set of vertices and edges of graph $\Gamma$, respectively. Furthermore, $d(v)$ shows the degree of graph $\Gamma$.
In this section we give some preliminaries which are useful to prove our results.

**Lemma 2.1** [5]*Let$G$ be a metacyclic* 2*-group of type $\mathcal{G}(2, \alpha, r, \gamma)$. Then*
(1) $|G| = 2^{\alpha+1}$,
(2) $Z(G) = \langle a^{2^{\alpha-1}}, b\rangle$,
(3) $|Z(G)| = 2$.

**Lemma 2.2** [9]*Let $G$ be a metacyclic* 2*-group of type $\mathcal{G}(2, \alpha, r, \gamma)$. Then $k(G) = 2^{\alpha-1} + 3$.*

Now, we describe certain properties of the conjugacy classes graph Γ,associated to the metacyclic 2-groups of Equation (2.1). More specifically, we show that the chromatic number and clique number of these graphs are identical.

## 2. Some properties of conjugacy class graphs

**Lemma 3.1** *Let $G$ be a metacyclic* 2*-group of type $\mathcal{G}(2, \alpha, r, \gamma)$. Then $\Gamma$ is complete.*

**Proof.** Using Lemma 2.1 the order of the group$G$is$2^{\alpha+1}$. Let $u = {x_1}^G$and $v = {x_2}^G$ be two arbitrary non-central conjugacy classes of$G$, where $x_1, x_2 \in G$.It is easy to see that gcd$(|{x_1}^G|, |{x_2}^G|) > 1$. It follows that $u$ and $v$are adjacent. Thus $\Gamma$ is complete. ■

The following lemma is used to prove Theorem 3.1 and the results followed.

**Lemma 3.2** *Considera group of type $\mathcal{G}(2, \alpha, r, \gamma)$. Then $|V(\Gamma)| = 2^{\alpha-1} + 1$.*

**Proof.**As mentioned in the Introduction section, the vertices of graph $\Gamma$ are the non-central conjugacy class sizes of $G$, from which it follows that the order of center of vertices of $\Gamma$ is $|V(\Gamma)| = k(G) - |Z(G)|$. Hence by applying Lemmas 2.1 and 2.2 we conclude that

$$|V(\Gamma)| = (2^{\alpha-1} + 3) - 2 = 2^{\alpha-1} + 1,$$

as required. ■

In the following, we obtain a formula for the cardinality of $E(\Gamma)$ in terms of $\alpha$.

**Theorem 3.1***Let $G$ be a metacyclic* 2*-group of type $\mathcal{G}(2, \alpha, r, \gamma)$. Then $|E(\Gamma)| = 2^{2\alpha-3} + 2^{\alpha-2}$.*

**Proof.**Lemma 3.1shows thatthe graph $\Gamma$ is complete. Hence, for any vertex of $\Gamma$ the degree of $v$is $d(v) = |V(\Gamma)| - 1 = 2^{\alpha-1}$ . According to Theorem 2.1 of [6], the degree of graph $\Gamma$ is

$$d(\Gamma) = \sum_{\xi=1}^{|V(\Gamma)|} d(v_\xi) = 2|E(\Gamma)|.$$

Hence we have the following identity:

$$d(\Gamma) = \sum_{\xi=1}^{|V(\Gamma)|} d(v_\xi) = \sum_{\xi=1}^{2^{\alpha-1}+1} 2^{\alpha-1} = 2|E(\Gamma)|.$$

It follows that $(2^{\alpha-1} + 1) \cdot 2^{\alpha-1} = 2|E(\Gamma)|$. Thus $|E(\Gamma)| = 2^{2\alpha-3} + 2^{\alpha-2}$. ■

The above theorem shows that the graphs associated to the groups $\mathcal{G}(2,2,1,1)$and$\mathcal{G}(2,2,0,0)$, quaternion and dihedral groups of order 8, respectively, are triangle.

**Theorem 3.2***Let $G$ be a metacyclic 2-group of type $\mathcal{G}(2, \alpha, r, \gamma)$. If $\chi(\Gamma)$ is the chromatic number and $\omega(\Gamma)$is the clique number of $\Gamma$, then $\omega(\Gamma) = \chi(\Gamma) = 2^{\alpha-1} + 1$.*

**Proof.** UsingLemma 3.1,$\Gamma$ is a complete graph. Suppose that $\mathfrak{A}$ is a subset of the vertices of $\Gamma$ in which the induced subgraph on $\mathfrak{A}$ is a complete graph, then the maximum size of $\mathfrak{A}$ is the order of $|V(\Gamma)|$, that is $k(G) - |Z(G)$. It follows that $\omega(\Gamma) = 2^{\alpha-1} + 1$. On the other hand, $|V(\Gamma)|$ is the smallest number of colors needed to color the vertices of the graph $\Gamma$ so that no two adjacent vertices have the same color. Thus we deduce that

$$\chi(\Gamma) = k(G) - |Z(G) = 2^{\alpha-1} + 1 = \omega(\Gamma),$$

and the proof is complete. ■

**Proposition 3.1***Let $G$ be a metacyclic 2-group of type $\mathcal{G}(2, \alpha, r, \gamma)$. Then $d(\Gamma) = 1$, also the graph $\Gamma$ is connected. In particular, the girth of $\Gamma$ is equal to 3.*

**Proof.**Since $\Gamma$ is complete the graph$\Gamma$ is connected. Let $u$ and $v$ be two distinct vertices of the graph $\Gamma$. Thus $u$ and $v$ are adjacent, hence $d(u, v) = 1$. Thus the largest distance between all pairs of the vertices is equal to 1, that is $d(\Gamma) = 1$. If $\{u, v\}$ is an arbitrary edge of $\Gamma$ then $\{u, v, uv\}$ is a triangle. Hence the girth of $\Gamma$ is 3. ■

**Definition 3.1** If $V(\Gamma)$ is a partition into nonempty subsets $V_1, V_2, \dots, V_{\mathfrak{N}}$ such that two vertices $u$ and $v$ are connected if and only if both $u$ and $v$ belong to the same set$V_i$. The subgraphs $\Gamma[V_1], \Gamma[V_2], \dots, \Gamma[V_{\mathfrak{N}}]$ are called the *components*of $\Gamma$. The notation $\mathfrak{N}(\Gamma)$ is used to the number of components of $\Gamma$.

**Proposition 3.2**Let $G \simeq \mathcal{G}(2, \alpha, r, \gamma)$ *be a metacyclic 2-group. Then for any $v \in V(\Gamma)$, $\mathfrak{N}(\Gamma - v) \leq 2^{\alpha-2}$.*

**Proof.**The graph$\Gamma$is complete, hence it is connected. Also, for any $v \in V(\Gamma)$:

$$d(v) = |V(\Gamma)| - 1 = k(G) - |z(G)| - 1 = 2^{\alpha-1}.$$

Since $d(v)$ is an even degree in $\Gamma$ a result of [3] implies that

$$\mathfrak{N}(\Gamma - v) \leq \frac{1}{2} d(v) = 2^{\alpha-2}. \quad ■$$

**Definition3.2** A *spanning*subgraph of $\Gamma$ is a subgraph $\bar{\Gamma}$with $V(\Gamma) = V(\bar{\Gamma})$. Also, a Hamiltonian cycle of $\Gamma$ is a cycle that contains every vertex of $\Gamma$.

**Proposition 3.3** *The conjugacy class graph of every non-abelian metacyclic 2-group $\mathcal{G}(2, \alpha, r, \gamma)$ is Hamiltonian.*

**Proof.** First note that the degree of any vertex $v$ in the conjugacy class graph $\Gamma$ of a non-abelian 2-group $\mathcal{G}(2,\alpha,r,\gamma)$ is equal to $|V(\Gamma)|-1=2^{\alpha-1}$. It follows that $(v)\geq\frac{1}{2}|V(\Gamma)|$ . Hence by Dirac's theorem [3, Theorem 4.3], $\Gamma$ is Hamiltonian. ■

**Proposition 3.4** *Let $G$ be a metacyclic 2-group of type $\mathcal{G}(2,\alpha,r,\gamma)$. If $\mathcal{S}$ is a proper subset of $V(\Gamma)$, then* $\mathfrak{N}(\Gamma-\mathcal{S})\leq|\mathcal{S}|$.

**Proof.** Since $\Gamma$ is complete, it contains a Hamiltonian cycle $\mathcal{C}$. Thus for every proper subset $\mathcal{S}$ of $V(\Gamma)$, $\mathfrak{N}(\mathcal{C}-\mathcal{S})\leq|\mathcal{S}|$. Also, $\mathcal{C}-\mathcal{S}$ is a spanning subgraph of $\Gamma-\mathcal{S}$ and so

$$\mathfrak{N}(\Gamma-\mathcal{S})\leq\mathfrak{N}(\mathcal{C}-\mathcal{S})\leq|\mathcal{S}|.$$

The proof is complete. ■

A graph $\Gamma$ is *planar* if it can be embedded in the plane so that no two edges intersect geometrically except at a vertex which both are incident.

**Proposition 3.5** *Every graph associated to the metacyclic 2-group $G\simeq\mathcal{G}(2,3,r,\gamma)$ is non-planar.*

**Proof.** Let $\Gamma$ be a graph associated to a metacyclic 2-group $G$. Then

$$|V(\Gamma)|=k(G)-|z(G)|=2^{\alpha-1}+1.$$

It follows that $|V(\Gamma)|=5$ if $\alpha=3$. Hence $\Gamma$ is a complete graph with 5 vertices. A result of Bondy and Murty [6, Theorem 9.1], implies that $\Gamma$ is non-planar. ■

Abdollahi et al. (2006) proved that the non-commuting graph $\Gamma$ is planar if and only if $G\simeq S_3$ or $D_8$ or $Q_8$. The following proposition shows the result when the conjugacy class graph $\Gamma$ is planar.

**Proposition 3.6** *Let $G$ be a non-abelian metacyclic 2-group of type $\mathcal{G}(2,\alpha,r,\gamma)$. Then the graph $\Gamma$is planar if and only if $G$ is isomorphic to $\mathcal{G}(2,2,1,1)$ or $\mathcal{G}(2,2,0,0)$.*

**Proof.** It is easy to see that the conjugacy class graphs of $\mathcal{G}(2,2,1,1)\simeq Q_8$ and $\mathcal{G}(2,2,0,0)\simeq D_8$ are planar. Now suppose that $\Gamma$ is planar. Then a result of [9] shows that for every vertex $v\in V(\Gamma)$, $d(v)=2^{\alpha-1}\leq 5$. We recall that in the presentation of metacyclic 2-groups, $\alpha\geq 2$. From the inequality above, we conclude that $\alpha=2$ or 3. Applying $\alpha=2$ to the groups of type $\mathcal{G}(2,\alpha,1,1)$ and $\mathcal{G}(2,\alpha,0,0)$ yields

$$\mathcal{G}(2,2,1,1)=\langle a,b|a^4=1,b^2=a^2,[b,a]=a^{-2}\rangle\simeq Q_8,$$

and

$$\mathcal{G}(2,2,0,0)=\langle a,b|a^4=b^2=1,[b,a]=a^{-2}\rangle\simeq D_8,$$

respectively. On the other hand, if we apply $\alpha=2$ to the group of type $G\simeq\mathcal{G}(2,\alpha,0,1)$ then we have

$$G\simeq\langle a,b|a^4=b^2=1,[b,a]=1\rangle.$$

It follows that $a$ commutes with $b$, a contradiction. Now suppose that $\alpha=3$. Then $|V(\Gamma)|=k(G)-|z(G)|=5$. By a result of Bondy [6] since the complete graph of order 5 is not planar, we get a contradiction. This completes the proof.

## 3. Conclusion

In this paper we have shown that the clique and chromatic number ofconjugacy classes graph , associated to the generalized quaternion, dihedral and semi-dihedral groups are identical. Also, the conjugacy class graph $\Gamma$ is planar if and only if the metacyclic 2-group $G$ is isomorphic to the quaternion and dihedral groups of order 8.

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