Bayesian Estimation for Topp-Leone Distribution under Trimmed Samples

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ABSTRACT

Trimmed samples are widely utilized in several areas of statistical practice, especially when some sample values at either or both extremes might have been adulterated. In this article, the problem of estimating the parameter of Topp-Leone distribution based on trimmed samples under informative and uninformative has been addressed. The problem discussed using Bayesian approach to estimate the shape parameter of Topp-Leone distribution. Elicitation of hyperparameter through prior predictive approach is also discussed. Credible Intervals are also derived under different priors. A comparison is made using the Monte Carlo simulation. A real life data example has also been discussed.

KEYWORDS: Type II censored sample, Precautionary loss function (PLF), Weighted loss function (WLF), Entropy loss function (ELF), Squared-log error loss function (SLELF), Credible Intervals.

1. INTRODUCTION

Topp-Leone distribution is a continuous unimodal distribution with bounded support. It is a two-parametric family continuous distribution proposed by Topp and Leone (1955). Such a distribution is useful for modeling lifetime phenomena, different aspect of this class of distributions have been studied by Nadarajah and Kotz (2003). The random variable \(X\) with the range of values \((0,1)\) has one parametric Topp Leone distribution as \(X \sim TL(\theta,1)\). The cumulative distribution function (cdf) is
\[
F(x) = x^\theta (2-x)^\theta, \quad 0 < x < 1 \text{ and } \theta > 0.
\]

The corresponding probability density function (pdf) is:
\[
f(x) = \theta (2-2x)(2x-x^2)^{\theta-1} \quad 0 < x < 1 \text{ and } \theta > 0.
\]

The Topp Leone distribution does not seem to be very familiar to the statisticians and has not been investigated in much detail under the Bayesian paradigm. The purpose of this study is to obtain the estimates for the parameter assuming different asymmetric loss functions.

This distribution has attracted recent attention some key references are Ghitany et al. (2005), Van Dorp and Kotz(2004), Zhouaet el. (2006), Kotz and Seier (2007),Nadarajah (2009), and Genç (2012). As well as having finite support, the T-L distribution has a “J-shaped” density function and a hazard function that is “bathtub-shaped”. The latter characteristic is especially important in reliability applications in a wide range of fields, as is discussed recently by Reed (2011).

Censoring is of supreme importance in reliability studies. It has many types each of whom can be used in analysis of different kinds of data representing various real life circumstances. The situation may arise when complete information regarding all the units in the sample cannot be obtained. For example, the measuring instrument to be used, may not be capable of measuring the items above or below a particular point or the measurement of units above or below a certain point may not be of interest. For illustration, suppose it is desired to estimate the average life of electric bulb produced in a certain factory. The simple method would be to take a certain number of bulbs at random and burn them out to get the required number of bulbs for analysis. Instead of wasting the bulbs it might be decided to stop the experiment when a fixed number have burnt out. The random sample hence obtained would be a censored sample Type II. Wingo (1993) has considered Maximum likelihood estimation of Burr XII distribution under type II censoring. Howlader and Hossian(1995) investigate the Bayesian estimation and

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prediction for Rayleigh based on type II censored data. Singh et al. (2005) have discussed estimation of the parameter for exponentiated-Weibull family under type-II censoring scheme. Parakash(2009) studied the Bayesian shrinkage approach in Weibull under Type II censored data. Gholizadeh et al. (2011) have studied the Kumaraswamy distribution under progressively type II censored data. Mostafa (2012) presents an approach involving objective Bayesian reference analysis to the frailty model with cluster survival time and sources of heterogeneity that are not captured by covariates. In this study, the idea behind Bayesian technique to estimate the parameters of interest is adopted using Gibbs sampling. Rezaei and Tahmasbi (2012) studied a new two-parameter lifetime distribution with increasing failure rate for maximum survival time in exponential truncate Poisson distribution. Various properties of the proposed distribution are discussed and the estimation of the parameters attained by the EM algorithm.

The Topp-Leone distribution does not seem to be very familiar to the statisticians and has not been investigated so far in much detail under the Bayesian framework. Being first attempt of its kind is the prime feature and scientific contribution of this study. Bayes estimates for the unknown parameters are obtained assuming different asymmetric loss functions based on type II censored sample. The complete sample results are found to be a special case of the censored sample results.

The rest of paper is organized as follows. In section 2, the posterior distributions have been derived under non-informative and informative priors. Estimation of parameter along with the posterior risk has been discussed in section 3. Method of Elicitation of the hyper-parameters via prior predictive approach has been discussed in section 4. Credible intervals have been derived in Section 5. Simulation studies are conducted and results are illustrated based on real data set with graphical comparisons have been performed in sections 6 and 7 respectively. The conclusions regarding the study have been presented in section 8.

2. Likelihood Function and Posterior Distributions

In the failure censoring scheme, the \( n \) experimental units are placed under observation in a typical life test and the number of uncensored observations \( r \) is predetermined. The data (collected) consist of observations \( x_{(1)}, x_{(2)}, x_{(3)}, \ldots, x_{(r)} \) are the ordered lifetimes of these life testing items, this means that we have no information about survival item \((n-r)\) except that their lifetimes are greater than \(x_{(r)}\). The experiment is terminated when the \(r^{th}\) item fails and remaining \((n-r)\) items are regarded as censored data. The likelihood function for \(x_{(1)}, x_{(2)}, x_{(3)}, \ldots, x_{(r)}\) failed observations, as given by Cohen (1965) is:

\[
L(\theta | x) = \frac{n!}{(n-r)!} \prod_{i=1}^{r} f(x_{(i)}, \theta) \left[ 1 - F(x_{(r)}, \theta) \right]^{n-r},
\]

\[
L(\theta | x) = \frac{n!}{(n-r)!} \prod_{i=1}^{r} \left\{ \theta \left[ 2 - 2x_{(i)} \right] \left( 2x_{(i)} - x_{(i)}^2 \right)^{\theta-1} \left[ 1 - \left( 2x_{(i)} - x_{(i)}^2 \right)^{\theta} \right] \right\}^{n-r},
\]

After some algebra, the likelihood function reduces to:

\[
L(x, \theta) \propto \sum_{k=0}^{n-r} (-1)^k \binom{n-r}{k} \theta^r \exp \left[ \theta \left( \sum_{i=1}^{r} \ln \left( 2x_{(i)} - x_{(i)}^2 \right) + k \ln \left( 2x_{(r)} - x_{(r)}^2 \right) \right) \right],
\]

\[
L(x, \theta) \propto \sum_{k=0}^{n-r} (-1)^k \binom{n-r}{k} \theta^r \exp \left\{ -\theta \xi_r \right\},
\]

where \(\xi_r = -\left( \sum_{i=1}^{r} \ln \left( 2x_{(i)} - x_{(i)}^2 \right) + k \ln \left( 2x_{(r)} - x_{(r)}^2 \right) \right)\).

2.1 Posterior Distribution using Uniform Prior

Uniform prior reflects the lack of prior information and the Bayesian methodology can still work. Uniform prior may be proper or improper. Even if Uniform prior is improper, we can still have a proper posterior. Equation (4) presents an improper prior while the posterior given in equation (5) is proper one having total area under the curve equals to unity. The uniform prior for \(\theta\) is defined as:

\[
p(\theta) \propto k, \quad \theta > 0.
\]

In making use of Eq. (3) and (4) the posterior distribution given data is:
\[ p(\theta | \mathbf{x}) = \frac{\sum_{k=0}^{n-r} (-1)^k \binom{n-r}{k} \theta^k \exp\{-\theta \bar{\xi}_{x}\} \Gamma(r+1)}{\sum_{k=0}^{n-r} \left(\sum_{k=0}^{n-r} (-1)^k \binom{n-r}{k} \Gamma(r+1) \left(\bar{\xi}_{x}\right)^{r+1}\right)} \quad \theta > 0. \]  

(5)

\[ \int_0^\infty p(\theta | \mathbf{x}) d\theta = \int_0^\infty \frac{\sum_{k=0}^{n-r} (-1)^k \binom{n-r}{k} \theta^k \exp\{-\theta \bar{\xi}_{x}\} \Gamma(r+1)}{\sum_{k=0}^{n-r} \left(\sum_{k=0}^{n-r} (-1)^k \binom{n-r}{k} \Gamma(r+1) \left(\bar{\xi}_{x}\right)^{r+1}\right)} d\theta = 1 \]

Since the posterior density is a proper density function, and is recognized as the mixture of gamma density functions.

### 2.2 Posterior Distribution using Jeffreys Prior

Jeffreys prior is perhaps the most widely used non-informative prior in Bayesian analysis. The only requirement is a likelihood function from which the prior is then derived using Jeffreys’ rule, which is to take the prior distribution to be the determinant of the square root of the Fisher information matrix.

\[ p(\theta) \propto \frac{1}{\theta}, \quad \theta > 0. \]  

(6)

In making use of Eq. (3) and (6) the posterior distribution given data is:

\[ p(\theta | \mathbf{x}) = \frac{\sum_{k=0}^{n-r} (-1)^k \binom{n-r}{k} \theta^k \exp\{-\theta \bar{\xi}_{x}\} \Gamma(r+1) \left(\bar{\xi}_{x}\right)^{r+1}}{\sum_{k=0}^{n-r} \left(\sum_{k=0}^{n-r} (-1)^k \binom{n-r}{k} \Gamma(r+1) \left(\bar{\xi}_{x}\right)^{r+1}\right)} \quad 0 < x < 1 \text{ and } \theta > 0. \]  

(7)

The posterior density function is recognized as the mixture of gamma density functions.

### 2.3 Posterior Distribution using Exponential Prior

The use of prior information is equivalent to adding a number of observations to a given sample size, and therefore leads to a reduction of the posterior risk of the Bayes estimates. Bansal (2007) discussed a method to evaluate the significance of a prior information in terms of the number of additional observation supposed to be added to a given sample size. Exponential and gamma prior have been used for Bayesian analysis.

The informative prior for the parameter \( \theta \) is assumed to be exponential distribution:

\[ p(\theta) = \frac{w e^{-w \theta}}{\Gamma(w)}, \quad w, \theta > 0. \]  

(8)

The posterior distribution given data using Eq. (3) and (8) is:

\[ p(\theta | \mathbf{x}) = \frac{\sum_{k=0}^{n-r} (-1)^k \binom{n-r}{k} \theta^k \exp\{-\theta (w + \bar{\xi}_{x})\} \Gamma(r+1) \left(w + \bar{\xi}_{x}\right)^{r+1}}{\sum_{k=0}^{n-r} \left(\sum_{k=0}^{n-r} (-1)^k \binom{n-r}{k} \Gamma(r+1) \left(w + \bar{\xi}_{x}\right)^{r+1}\right)} \quad 0 < x < 1 \text{ and } \theta > 0. \]  

(9)

The posterior density function is recognized as the mixture of gamma density functions.

### 2.4 Posterior Distribution using Gamma Prior

The informative prior for the parameter \( \theta \) is assumed to be gamma distribution:

\[ p(\theta) = \frac{b^a}{\Gamma(a)} \theta^{a-1} e^{-b \theta}, \quad \theta > 0. \]  

(10)

The posterior distribution given data using Eq. (3) and (10) is:
The posterior density function is recognized as the mixture of gamma density functions.

3. Bayes Estimators and Posterior Risks under different Loss Functions

This section enlightens on the derivation of the Bayes Estimators and corresponding posterior risks under different loss functions. The comparison is made for non-informative as well as informative priors. If the Bayes decision is choice of Bayes estimator then Bayes estimator which minimizes risk function is the best decision. The Bayes estimators are computed under precautionary loss function (PLF), weighted loss function (WLF), entropy loss function (ELF) and squared-log error loss function (SLELF).

3.1. Precautionary Loss Function (PLF)

Norstrom(1996) introduced an alternative asymmetric precautionary loss function and also presented a general class of precautionary loss functions as a special case. A very useful and asymmetric precautionary loss function is

\[ L_{PLF}(\hat{\theta}, \theta) = \frac{(\theta - \hat{\theta})^2}{\theta} \]

The Bayes estimator and the posterior risk under Precautionary loss function are given below:

\[ \hat{\theta}_{PLF} = \sqrt{E_{\hat{\theta}X}(\theta^2)}, \quad \rho(\hat{\theta}_{PLF}) = 2\left\{ E_{\hat{\theta}X}(\theta^2) - E_{\hat{\theta}X}(\theta) \right\}. \]

The Bayes estimator and the posterior risk under uniform prior are:

\[
\hat{\theta} = \frac{\sum_{k=0}^{n-r} (-1)^k \binom{n-r}{k} \frac{\Gamma(r+3)}{(\xi_{ir})^{r+3}}}{\sum_{k=0}^{n-r} (-1)^k \binom{n-r}{k} \frac{\Gamma(r+1)}{(\xi_{ir})^{r+1}}}, \\
\rho(\hat{\theta}) = 2\left[ \frac{\sum_{k=0}^{n-r} (-1)^k \binom{n-r}{k} \frac{\Gamma(r+3)}{(\xi_{ir})^{r+3}}}{\sum_{k=0}^{n-r} (-1)^k \binom{n-r}{k} \frac{\Gamma(r+1)}{(\xi_{ir})^{r+1}}} - \frac{\sum_{k=0}^{n-r} (-1)^k \binom{n-r}{k} \frac{\Gamma(r+1)}{(\xi_{ir})^{r+2}}}{\sum_{k=0}^{n-r} (-1)^k \binom{n-r}{k} \frac{\Gamma(r+2)}{(\xi_{ir})^{r+2}}} \right].
\]

The Bayes estimator and the posterior risk under Jeffreys prior are:

\[
\hat{\theta} = \frac{\sum_{k=0}^{n-r} (-1)^k \binom{n-r}{k} \frac{\Gamma(r+2)}{(\xi_{ir})^{r+2}}}{\sum_{k=0}^{n-r} (-1)^k \binom{n-r}{k} \frac{\Gamma(r)}{(\xi_{ir})^{r}}}, \\
\rho(\hat{\theta}) = 2\left[ \frac{\sum_{k=0}^{n-r} (-1)^k \binom{n-r}{k} \frac{\Gamma(r+2)}{(\xi_{ir})^{r+2}}}{\sum_{k=0}^{n-r} (-1)^k \binom{n-r}{k} \frac{\Gamma(r)}{(\xi_{ir})^{r}}} - \frac{\sum_{k=0}^{n-r} (-1)^k \binom{n-r}{k} \frac{\Gamma(r+1)}{(\xi_{ir})^{r+1}}}{\sum_{k=0}^{n-r} (-1)^k \binom{n-r}{k} \frac{\Gamma(r)}{(\xi_{ir})^{r}}} \right].
\]

The Bayes estimator and the posterior risk under exponential prior are:
\begin{align*}
\hat{\theta} &= \frac{\sum_{k=0}^{n-r} (-1)^k \binom{n-r}{k} \Gamma(r+3)}{\sum_{k=0}^{n-r} (-1)^k \binom{n-r}{k} (w + \xi_{ir})^{-r+1}}, \\
\rho(\hat{\theta}) &= 2 \left[ \frac{\sum_{k=0}^{n-r} (-1)^k \binom{n-r}{k} \Gamma(r+3)}{\sum_{k=0}^{n-r} (-1)^k \binom{n-r}{k} (w + \xi_{ir})^{-r+1}} \right] - \left[ \frac{\sum_{k=0}^{n-r} (-1)^k \binom{n-r}{k} \Gamma(r+2)}{\sum_{k=0}^{n-r} (-1)^k \binom{n-r}{k} (w + \xi_{ir})^{-r+2}} \right].
\end{align*}

The Bayes estimator and the posterior risk under gamma prior are:

\begin{align*}
\hat{\theta} &= \frac{\sum_{k=0}^{n-r} (-1)^k \binom{n-r}{k} \Gamma(r+a+2)}{\sum_{k=0}^{n-r} (-1)^k \binom{n-r}{k} (b + \xi_{ir})^{-r+a+2}}, \\
\rho(\hat{\theta}) &= 2 \left[ \frac{\sum_{k=0}^{n-r} (-1)^k \binom{n-r}{k} \Gamma(r+a+2)}{\sum_{k=0}^{n-r} (-1)^k \binom{n-r}{k} (b + \xi_{ir})^{-r+a+2}} \right] - \left[ \frac{\sum_{k=0}^{n-r} (-1)^k \binom{n-r}{k} \Gamma(r+a+1)}{\sum_{k=0}^{n-r} (-1)^k \binom{n-r}{k} (b + \xi_{ir})^{-r+a+1}} \right].
\end{align*}

The Bayes estimator and the posterior risk under exponential prior are:

\begin{align*}
L(\theta, \hat{\theta}) &= \frac{(\theta - \hat{\theta})^2}{\theta}, \\
\hat{\theta}_{\text{WLF}} &= \frac{1}{E_{\theta | \mathbf{x}}(\theta^{-1} | \mathbf{x})}, \quad \rho(\hat{\theta}_{\text{WLF}}) = E_{\theta | \mathbf{x}}(\theta | \mathbf{x}) - \frac{1}{E_{\theta | \mathbf{x}}(\theta^{-1} | \mathbf{x})}.
\end{align*}

**3.2. Weighted Loss Function (WLF)**

As compare to other loss function, weighted loss function may be mathematically, more convenient to obtain Bayes estimates. The weighted loss function is defined as

\[ L(\theta, \hat{\theta}) = \frac{(\theta - \hat{\theta})^2}{\theta} \]

The Bayes estimator and the posterior risk under weighted loss function are given below:

\begin{align*}
\hat{\theta}_{\text{WLF}} &= \frac{1}{E_{\theta | \mathbf{x}}(\theta^{-1} | \mathbf{x})}, \quad \rho(\hat{\theta}_{\text{WLF}}) = E_{\theta | \mathbf{x}}(\theta | \mathbf{x}) - \frac{1}{E_{\theta | \mathbf{x}}(\theta^{-1} | \mathbf{x})}.
\end{align*}

The Bayes estimator and the posterior risk under uniform prior are:

\begin{align*}
\hat{\theta} &= \frac{\sum_{k=0}^{n-r} (-1)^k \binom{n-r}{k} \Gamma(r+1)}{\sum_{k=0}^{n-r} (-1)^k \binom{n-r}{k} (\xi_{ir})^{-r+1}}, \\
\rho(\hat{\theta}) &= 2 \left[ \frac{\sum_{k=0}^{n-r} (-1)^k \binom{n-r}{k} \Gamma(r+2)}{\sum_{k=0}^{n-r} (-1)^k \binom{n-r}{k} (\xi_{ir})^{-r+2}} \right] - \left[ \frac{\sum_{k=0}^{n-r} (-1)^k \binom{n-r}{k} \Gamma(r+1)}{\sum_{k=0}^{n-r} (-1)^k \binom{n-r}{k} (\xi_{ir})^{-r+1}} \right].
\end{align*}
The Bayes estimator and the posterior risk under Jeffreys prior are:

\[
\hat{\theta} = \frac{\sum_{k=0}^{n-r} (-1)^k \binom{n-r}{k} \frac{\Gamma(r)}{(\xi_{ir})^r}}{\sum_{k=0}^{n-r} (-1)^k \binom{n-r}{k} \frac{\Gamma(r-1)}{(\xi_{ir})^{r-1}}}, \\
\rho(\hat{\theta}) = \left( \frac{\sum_{k=0}^{n-r} (-1)^k \binom{n-r}{k} \frac{\Gamma(r+1)}{(\xi_{ir})^{r+1}}}{\sum_{k=0}^{n-r} (-1)^k \binom{n-r}{k} \frac{\Gamma(r)}{(\xi_{ir})^r}} \right)^2 - \left( \frac{\sum_{k=0}^{n-r} (-1)^k \binom{n-r}{k} \frac{\Gamma(r-1)}{(\xi_{ir})^{r-1}}}{\sum_{k=0}^{n-r} (-1)^k \binom{n-r}{k} \frac{\Gamma(r)}{(\xi_{ir})^r}} \right)^2.
\]

The Bayes estimator and the posterior risk under exponential prior are:

\[
\hat{\theta} = \frac{\sum_{k=0}^{n-r} (-1)^k \binom{n-r}{k} \frac{\Gamma(r+1)}{(w+\xi_{ir})^{r+1}}}{\sum_{k=0}^{n-r} (-1)^k \binom{n-r}{k} \frac{\Gamma(r)}{(w+\xi_{ir})^r}}, \\
\rho(\hat{\theta}) = \left( \frac{\sum_{k=0}^{n-r} (-1)^k \binom{n-r}{k} \frac{\Gamma(r+2)}{(w+\xi_{ir})^{r+2}}}{\sum_{k=0}^{n-r} (-1)^k \binom{n-r}{k} \frac{\Gamma(r+1)}{(w+\xi_{ir})^{r+1}}} \right)^2 - \left( \frac{\sum_{k=0}^{n-r} (-1)^k \binom{n-r}{k} \frac{\Gamma(r)}{(w+\xi_{ir})^r}}{\sum_{k=0}^{n-r} (-1)^k \binom{n-r}{k} \frac{\Gamma(r+1)}{(w+\xi_{ir})^{r+1}}} \right)^2.
\]

The Bayes estimator and the posterior risk under gamma prior are:

\[
\hat{\theta} = \frac{\sum_{k=0}^{n-r} (-1)^k \binom{n-r}{k} \frac{\Gamma(r+a)}{(b+\xi_{ir})^{r+a}}}{\sum_{k=0}^{n-r} (-1)^k \binom{n-r}{k} \frac{\Gamma(r+a-1)}{(b+\xi_{ir})^{r+a-1}}}, \\
\rho(\hat{\theta}) = \left( \frac{\sum_{k=0}^{n-r} (-1)^k \binom{n-r}{k} \frac{\Gamma(r+a+1)}{(b+\xi_{ir})^{r+a+1}}}{\sum_{k=0}^{n-r} (-1)^k \binom{n-r}{k} \frac{\Gamma(r+a)}{(b+\xi_{ir})^{r+a}}} \right)^2 - \left( \frac{\sum_{k=0}^{n-r} (-1)^k \binom{n-r}{k} \frac{\Gamma(r+a-1)}{(b+\xi_{ir})^{r+a-1}}}{\sum_{k=0}^{n-r} (-1)^k \binom{n-r}{k} \frac{\Gamma(r+a)}{(b+\xi_{ir})^{r+a}}} \right)^2.
\]

3.3. Entropy Loss Function (ELF)

The linex loss function is suitable for the estimation of the location parameter but not for the estimation of the scale parameter and other parametric functions. Calabrìa and Pulcini(1994) suggested the general entropy loss function for estimation these quantities. They proposed another alternative to the modified linex loss function named general entropy loss function and defined it as

\[
\mathcal{L}_{\text{GELF}}(\hat{\theta}, \theta) = b \left( \frac{\hat{\theta}}{\theta} \right)^p - p \ln \left( \frac{\hat{\theta}}{\theta} \right) - 1, \quad b > 0, \quad p \neq 0,
\]

which has a minimum at \( \hat{\theta} = \theta \). Without loss of generality, we assume that \( b = 1 \). This loss is a generalization of the entropy loss function used by several authors taking the shape parameter \( p = 1 \). The Bayes estimator of \( \theta \) and corresponding risks under the entropy loss are:
\[ \hat{\theta}_{\text{GELF}} = \left\{ E_{\phi_k} \left( \theta^{-1} | \mathbf{x} \right) \right\}^{-1} \text{ and } \rho(\hat{\theta}) = \ln \left\{ E \left( \theta^{-1} | \mathbf{x} \right) \right\} + E \left\{ \ln \left( \theta | \mathbf{x} \right) \right\}. \]

The Bayes estimator and the posterior risk under uniform prior are:

\[
\hat{\theta} = \frac{\sum_{k=0}^{n-r} (-1)^k \binom{n-r}{k} \frac{\Gamma(r+1)}{\xi_{ir}^{r+1}}}{\sum_{k=0}^{n-r} (-1)^k \binom{n-r}{k} \frac{\Gamma(r)}{\xi_{ir}^{r}}}, \\
\rho(\hat{\theta}) = \ln \frac{\sum_{k=0}^{n-r} (-1)^k \binom{n-r}{k} \frac{\Gamma(r)}{\xi_{ir}^{r}}}{\sum_{k=0}^{n-r} (-1)^k \binom{n-r}{k} \frac{\Gamma(r+1)}{\xi_{ir}^{r+1}}} + \psi(r+1) - \ln \frac{\sum_{k=0}^{n-r} (-1)^k \binom{n-r}{k} \frac{1}{\xi_{ir}^{r+1}}}{\sum_{k=0}^{n-r} (-1)^k \binom{n-r}{k} \frac{1}{\xi_{ir}^{r+2}}}. 
\]

The Bayes estimator and the posterior risk under Jeffreys prior are:

\[
\hat{\theta} = \frac{\sum_{k=0}^{n-r} (-1)^k \binom{n-r}{k} \frac{\Gamma(r)}{(\xi_{ir})^{r+1}}}{\sum_{k=0}^{n-r} (-1)^k \binom{n-r}{k} \frac{\Gamma(r-1)}{(\xi_{ir})^{r-1}}}, \\
\rho(\hat{\theta}) = \ln \frac{\sum_{k=0}^{n-r} (-1)^k \binom{n-r}{k} \frac{\Gamma(r-1)}{(\xi_{ir})^{r-1}}}{\sum_{k=0}^{n-r} (-1)^k \binom{n-r}{k} \frac{\Gamma(r)}{(\xi_{ir})^{r}}} + \psi(r) - \ln \frac{\sum_{k=0}^{n-r} (-1)^k \binom{n-r}{k} \frac{1}{(\xi_{ir})^{r+1}}}{\sum_{k=0}^{n-r} (-1)^k \binom{n-r}{k} \frac{1}{(\xi_{ir})^{r+2}}}. 
\]

The Bayes estimator and the posterior risk under exponential prior are:

\[
\hat{\theta} = \frac{\sum_{k=0}^{n-r} (-1)^k \binom{n-r}{k} \frac{\Gamma(r+1)}{(w+\xi_{ir})^{r+1}}}{\sum_{k=0}^{n-r} (-1)^k \binom{n-r}{k} \frac{\Gamma(r)}{(w+\xi_{ir})^{r}}}, \\
\rho(\hat{\theta}) = \ln \frac{\sum_{k=0}^{n-r} (-1)^k \binom{n-r}{k} \frac{\Gamma(r)}{(w+\xi_{ir})^{r}}}{\sum_{k=0}^{n-r} (-1)^k \binom{n-r}{k} \frac{\Gamma(r+1)}{(w+\xi_{ir})^{r+1}}} + \psi(r) - \ln \frac{\sum_{k=0}^{n-r} (-1)^k \binom{n-r}{k} \frac{1}{(w+\xi_{ir})^{r+1}}}{\sum_{k=0}^{n-r} (-1)^k \binom{n-r}{k} \frac{1}{(w+\xi_{ir})^{r+2}}}. 
\]

The Bayes estimator and the posterior risk under gamma prior are:

\[
\hat{\theta} = \frac{\sum_{k=0}^{n-r} (-1)^k \binom{n-r}{k} \frac{\Gamma(r+a)}{(b+\xi_{ir})^{r+a}}}{\sum_{k=0}^{n-r} (-1)^k \binom{n-r}{k} \frac{\Gamma(r+a-1)}{(b+\xi_{ir})^{r+a-1}}}. 
\]
$$\rho(\hat{\theta}) = \ln \left( \frac{\sum_{k=0}^{n-r} (-1)^k \binom{n-r}{k} \Gamma(r+a-1)}{\sum_{k=0}^{n-r} (-1)^k \binom{n-r}{k} \Gamma(r+a) + \zeta_{ir}^{r+a-1}} \right) + \varphi(r+a) - \ln \left( \frac{\sum_{k=0}^{n-r} (-1)^k \binom{n-r}{k} 1}{\sum_{k=0}^{n-r} (-1)^k \binom{n-r}{k} 1 + \zeta_{ir}^{r+a+1}} \right).$$

Where $\varphi(x) = \frac{\Gamma'(x)}{\Gamma(x)}$ is the digamma function.

### 3.4. Squared-Log Error Loss Function (SLELF)

The squared-log error loss function is of the form: $L(\theta, \hat{\theta}) = (\ln \hat{\theta} - \ln \theta)^2$, which is balanced with $\lim \ L(\theta, \hat{\theta}) \to \infty$ as $\hat{\theta} \to 0$ or $\infty$. A balanced loss function takes both error of estimation and goodness of fit into account but the unbalanced loss function only considers error of estimation. This loss function is convex for $\frac{\hat{\theta}}{\theta} \leq e$ and concave otherwise, but its risk function has a unique minimum with respect to $\hat{\theta}$. The Bayes estimator and posterior risk for the parameter $\theta$ under the squared-log error loss function may be given as:

$$\hat{\theta}_{SLELF} = \text{Exp} \left\{ E(\ln \theta | \mathbf{x}) \right\}, \quad \rho(\hat{\theta}_{SLELF}) = E \left\{ (\ln \hat{\theta} - \ln \theta)^2 \right\} - E \left\{ (\ln \theta | \mathbf{x}) \right\}^2,$$

where $\varphi(x) = \frac{\Gamma'(x)}{\Gamma(x)}$ is the digamma function and $\varphi'(x) = \frac{d^2}{dx^2} \{\log \Gamma(x)\} = \frac{d}{dx} \left\{ \frac{\Gamma'(x)}{\Gamma(x)} \right\}$ is the tri-gamma function.

The Bayes estimator and the posterior risk under uniform prior are:

$$\hat{\theta} = \frac{\sum_{k=0}^{n-r} (-1)^k \binom{n-r}{k} \Gamma(r+1) \exp(\varphi(r+1))}{\sum_{k=0}^{n-r} (-1)^k \binom{n-r}{k} \Gamma(r+1) + \xi_{ir}^{r+1}}, \quad \rho(\hat{\theta}) = \left( \frac{\sum_{k=0}^{n-r} (-1)^k \binom{n-r}{k} \Gamma(r+1) \varphi'(r+1)}{\sum_{k=0}^{n-r} (-1)^k \binom{n-r}{k} \Gamma(r+1) + \xi_{ir}^{r+1}} \right).$$

The Bayes estimator and the posterior risk under Jeffreys prior are:

$$\hat{\theta} = \frac{\sum_{k=0}^{n-r} (-1)^k \binom{n-r}{k} \Gamma(r) \exp(\varphi(r))}{\sum_{k=0}^{n-r} (-1)^k \binom{n-r}{k} \Gamma(r) + \xi_{ir}^{r+1}}, \quad \rho(\hat{\theta}) = \left( \frac{\sum_{k=0}^{n-r} (-1)^k \binom{n-r}{k} \Gamma(r) \varphi'(r)}{\sum_{k=0}^{n-r} (-1)^k \binom{n-r}{k} \Gamma(r) + \xi_{ir}^{r+1}} \right).$$

The Bayes estimator and the posterior risk under exponential prior are:

$$\hat{\theta} = \frac{\sum_{k=0}^{n-r} (-1)^k \binom{n-r}{k} \Gamma(r+1) \left( \frac{\exp(\varphi(r+1))}{w + \xi_{ir}^{r+1}} \right)}{\sum_{k=0}^{n-r} (-1)^k \binom{n-r}{k} \Gamma(r+1) \left( w + \xi_{ir}^{r+1} \right)}, \quad \rho(\hat{\theta}) = \left( \frac{\sum_{k=0}^{n-r} (-1)^k \binom{n-r}{k} \Gamma(r+1) \varphi'(r+1)}{\sum_{k=0}^{n-r} (-1)^k \binom{n-r}{k} \Gamma(r+1) \left( w + \xi_{ir}^{r+1} \right) + \xi_{ir}^{r+1}} \right).$$

The Bayes estimator and the posterior risk under gamma prior are:
Elicitation is the process of talking out the expert knowledge about some unknown quantity of interest, or the probability of some future event, which can be used to supplement any numerical data we may have. If the expert in question does not have a statistical background, as often happens, translating their beliefs into a statistical form suitable for the use in our analyses can be a challenging task as described in Dey (2007).

Prior elicitation is an organized and systematic approach to represent an expert’s opinion as a well-defined, coherent prior. In Bayesian analysis, specification and elicitation of the prior distribution is a common difficulty. The Bayesian approach allows the use of objective data or subjective opinion in specifying a prior distribution. Elicitation is the process of extracting experts’ knowledge about some parameter of interest, or the probability of some future event and also the quantification of this prior information accurately, which then supplements the given data. In any statistical analysis there will typically be some form of background knowledge available in addition to data at hand. Berger (1985) gives a description of numerous different methods for the elicitation of prior distribution. For different sampling models, different methods for specification of opinions have been developed. There are various methods of elicitation available in literature (reader desires more detail see Grimshaw et al. (2001), Kadane (1980), O’Hagan et al. (2006), Kadane et al. (1996), Jenkinson (2005) and Leon et al. (2003). Here we use the method based on the prior predictive distribution, which is developed by Aslam (2003).

4.1. Elicitation of Hyperparameter

Prior information should be made available in probabilistic terms. First the functional form of the family of distributions of the unknown parameter is decided. Secondly, to be more accurate, particular and specific we need to know the values of parameter(s) of the prior probability distribution. Estimating the numerical values of the parameters with the help of data available combined with prior information is called prior elicitation. Bayesian analysis elicitation of opinion is a crucial step. It helps to make it easy for us to understand what the experts believe in and what their opinions are. In statistical inference the characteristics of a certain predictive distribution proposed by an expert determine the hyperparameters of a prior distribution.

In this article, we focus on a probability elicitation method known as prior predictive elicitation. Predictive elicitation is a method for estimating hyperparameters of prior distributions by inverting corresponding prior predictive distributions. Elicitation of hyperparameter from the prior \( p(\theta) \) is conceptually difficult task because we first have to identify prior distribution and then its hyperparameters. The prior predictive distribution is used for the elicitation of the hyperparameters which is compared with the experts’ judgment about this distribution and then the hyperparameters are chosen in such a way so as to make the judgment agree closely as possible with the given distribution. According to Aslam (2003), the method of assessment is to compare the predictive distribution with experts’ assessment about this distribution and then to choose the hyperparameters that make the assessment agree closely with the member of the family. He discusses three important methods to elicit the hyperparameters: (i) Via the Prior Predictive Probabilities (ii) Via Elicitation of the Confidence Levels (iii) Via the Predictive Mode and Confidence Level.

4.2. Prior Predictive Distribution

The prior predictive distribution is:

\[
p(y) = \sum_{k=0}^{n-r} (-1)^k \binom{n-r}{k} \frac{\Gamma(r+a)}{(b+\xi_{ir})^{r+a}} \left[ \frac{\exp\left(\psi\left(r+a\right)\right)}{b+\xi_{ir}} \right] .
\]

\[
p(\hat{\theta}) = \sum_{k=0}^{n-r} (-1)^k \binom{n-r}{k} \frac{\Gamma(r+a)}{(b+\xi_{ir})^{r+a}} \left[ \frac{\exp\left(\psi\left(r+a\right)\right)}{b+\xi_{ir}} \right] .
\]
The predictive distribution under exponential prior is:
\[ p(y) = \int_0^\infty \theta (2-2y)(2y-y^2)^{\theta-1} \exp(-\theta) \, d\theta \]

After some simplification it reduces as
\[ p(y) = \frac{w(2-2y)}{(2y-y^2)\left[w-\ln(2y-y^2)\right]^{2}} , \quad 0 < y < 1. \quad (12) \]

The predictive distribution under gamma prior is:
\[ p(y) = \frac{ab^\alpha(2-2y)}{(2y-y^2)\left[b-\ln(2y-y^2)\right]^{\alpha+1}} , \quad 0 < y < 1. \quad (13) \]

By using the method of elicitation defined by Aslam (2003), we obtain the following hyperparameters \( w=1.09895, a =1.42153 \) and \( b=0.23541 \). Predictive Intervals for a mixture distribution can be seen in Saleem et al. (2010).

5. Credible Interval
The Bayesian credible intervals for type II censored data under uniform and Jeffreys prior, as discussed by Saleem and Aslam (2009), are presented in the following. The credible interval for type II censored data under uniform prior is:
\[ \frac{\chi^2[(r+1)\theta]}{2} \sum_{k=0}^{n-r}(1)\left(\frac{n-r}{k}\right)1^{(r+\theta)} < \theta_{Uniform} < \frac{\chi^2[(r+1)\theta]}{2} \sum_{k=0}^{n-r}(1)\left(\frac{n-r}{k}\right)1^{(r+\theta)} \]

\[ \frac{\chi^2[(r+1)\theta]}{2} \sum_{k=0}^{n-r}(1)\left(\frac{n-r}{k}\right)1^{(r+\theta)} < \theta_{Jeffreys} < \frac{\chi^2[(r+1)\theta]}{2} \sum_{k=0}^{n-r}(1)\left(\frac{n-r}{k}\right)1^{(r+\theta)} \]

\[ \frac{\chi^2[(r+1)\theta]}{2} \sum_{k=0}^{n-r}(1)\left(\frac{n-r}{k}\right)1^{(r+\theta)} < \theta_{Exponential} < \frac{\chi^2[(r+1)\theta]}{2} \sum_{k=0}^{n-r}(1)\left(\frac{n-r}{k}\right)1^{(r+\theta)} \]

\[ \frac{\chi^2[(r+1)\theta]}{2} \sum_{k=0}^{n-r}(1)\left(\frac{n-r}{k}\right)1^{(r+\theta)} < \theta_{Gamma} < \frac{\chi^2[(r+1)\theta]}{2} \sum_{k=0}^{n-r}(1)\left(\frac{n-r}{k}\right)1^{(r+\theta)} \]

6. Simulation Study
This section shows how simulation can be helpful and illuminating way to approach problems in Bayesian analysis. Bayesian problems of updating estimates can be handled easily and straight forwardly with simulation. In order to assess the statistical performances of these estimates, we conducted a simulation study. The risks using generated random samples of different sizes are computed for each estimator. The behavior of the different estimators under different censoring schemes and prior distribution has been examined. The term different censoring scheme means different values of \( n \) and \( r \). We considered five censoring schemes. Here, the inverse transformation method of simulation is used to compare the performance of different estimators. The study has been carried out for different values of \( (n,r) \) using \( \theta \in (3 \text{ and } 7) \). The estimation has been done under 10% and 20% right censored samples. Sample size is varied to observe the effect of small and large samples on the estimators. Changes in the estimators and their risks have been determined when changing the loss function and the prior distribution of \( \theta \)
while keeping the sample size fixed. All these results are based on 5,000 repetitions. In the Tables, the estimators for the parameter and the risk, are averaged over the total number of repetitions. Mathemtica 8.0 has been used to carry out the results. The results are summarized in the following Tables.

**Table 1**: Bayes Estimates and the corresponding Posterior Risks (given in parentheses) under PLF.

<table>
<thead>
<tr>
<th>n, r</th>
<th>Uniform</th>
<th>Jeffreys</th>
<th>Exponential</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$\theta = 3$</td>
<td>$\theta = 7$</td>
<td>$\theta = 3$</td>
</tr>
<tr>
<td>20, 16</td>
<td>3.37698</td>
<td>7.93585</td>
<td>3.21376</td>
</tr>
<tr>
<td>20, 18</td>
<td>3.35781</td>
<td>7.87548</td>
<td>3.19124</td>
</tr>
<tr>
<td>50, 40</td>
<td>3.15446</td>
<td>0.362237</td>
<td>0.153856</td>
</tr>
<tr>
<td>50, 45</td>
<td>0.061419</td>
<td>0.144979</td>
<td>0.061246</td>
</tr>
<tr>
<td>80, 72</td>
<td>3.10825</td>
<td>7.19567</td>
<td>3.06033</td>
</tr>
<tr>
<td></td>
<td>(0.037962)</td>
<td>(0.085511)</td>
<td>(0.037721)</td>
</tr>
</tbody>
</table>

**Table 2**: Bayes Estimates and the corresponding Posterior Risks (given in parentheses) under WLF.

<table>
<thead>
<tr>
<th>n, r</th>
<th>Uniform</th>
<th>Jeffreys</th>
<th>Exponential</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$\theta = 3$</td>
<td>$\theta = 7$</td>
<td>$\theta = 3$</td>
</tr>
<tr>
<td>20, 16</td>
<td>3.15454</td>
<td>7.43666</td>
<td>3.02616</td>
</tr>
<tr>
<td>20, 18</td>
<td>0.09474</td>
<td>7.24377</td>
<td>3.02171</td>
</tr>
<tr>
<td>50, 40</td>
<td>3.06782</td>
<td>7.14102</td>
<td>3.01320</td>
</tr>
<tr>
<td>50, 45</td>
<td>0.06161</td>
<td>0.143677</td>
<td>0.061584</td>
</tr>
<tr>
<td>80, 72</td>
<td>3.01820</td>
<td>7.07418</td>
<td>3.00090</td>
</tr>
<tr>
<td></td>
<td>(0.037860)</td>
<td>(0.088604)</td>
<td>(0.037953)</td>
</tr>
</tbody>
</table>

**Table 3**: Bayes Estimates and the corresponding Posterior Risks (given in parentheses) under ELF.

<table>
<thead>
<tr>
<th>n, r</th>
<th>Uniform</th>
<th>Jeffreys</th>
<th>Exponential</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$\theta = 3$</td>
<td>$\theta = 7$</td>
<td>$\theta = 3$</td>
</tr>
<tr>
<td>20, 16</td>
<td>3.17552</td>
<td>7.47985</td>
<td>2.97263</td>
</tr>
<tr>
<td>20, 18</td>
<td>3.160248</td>
<td>7.38475</td>
<td>2.97670</td>
</tr>
<tr>
<td>50, 40</td>
<td>3.04952</td>
<td>7.21902</td>
<td>2.98127</td>
</tr>
<tr>
<td>50, 45</td>
<td>3.04883</td>
<td>7.18131</td>
<td>2.99046</td>
</tr>
<tr>
<td>80, 72</td>
<td>3.04221</td>
<td>7.13121</td>
<td>3.0231</td>
</tr>
<tr>
<td></td>
<td>(0.005651)</td>
<td>(0.00568)</td>
<td>(0.005624)</td>
</tr>
</tbody>
</table>

**Table 4**: Bayes Estimates and the corresponding Posterior Risks (given in parentheses) under SLELF.

<table>
<thead>
<tr>
<th>n, r</th>
<th>Uniform</th>
<th>Jeffreys</th>
<th>Exponential</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$\theta = 3$</td>
<td>$\theta = 7$</td>
<td>$\theta = 3$</td>
</tr>
<tr>
<td>20, 16</td>
<td>3.25975</td>
<td>7.58267</td>
<td>3.07936</td>
</tr>
<tr>
<td>20, 18</td>
<td>3.24590</td>
<td>7.56833</td>
<td>3.07687</td>
</tr>
<tr>
<td>50, 40</td>
<td>3.09782</td>
<td>7.20198</td>
<td>3.02393</td>
</tr>
<tr>
<td>50, 45</td>
<td>3.09728</td>
<td>7.18520</td>
<td>3.01821</td>
</tr>
<tr>
<td>80, 72</td>
<td>3.06786</td>
<td>7.14553</td>
<td>3.01232</td>
</tr>
<tr>
<td></td>
<td>(0.013793)</td>
<td>(0.013793)</td>
<td>(0.013985)</td>
</tr>
</tbody>
</table>

**Table 5**: Bayes Estimates and the corresponding Posterior Risks (given in parentheses) under Gamma.

<table>
<thead>
<tr>
<th>n, r</th>
<th>PLF</th>
<th>WLF</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$\theta = 3$</td>
<td>$\theta = 7$</td>
</tr>
<tr>
<td>20, 16</td>
<td>3.36633</td>
<td>7.35212</td>
</tr>
<tr>
<td>20, 18</td>
<td>3.32031</td>
<td>7.32779</td>
</tr>
<tr>
<td>50, 40</td>
<td>3.112317</td>
<td>7.183333</td>
</tr>
<tr>
<td>50, 45</td>
<td>3.09708</td>
<td>7.16590</td>
</tr>
<tr>
<td>80, 72</td>
<td>3.07892</td>
<td>7.12631</td>
</tr>
</tbody>
</table>
Table 6: Bayes Estimates and the corresponding Posterior Risks (given in parentheses) under Gamma.

<table>
<thead>
<tr>
<th>n, r</th>
<th>$\theta = 3$</th>
<th>$\theta = 7$</th>
<th>$\theta = 3$</th>
<th>$\theta = 7$</th>
</tr>
</thead>
<tbody>
<tr>
<td>20, 16</td>
<td>3.09703(0.018902)</td>
<td>6.86949(0.018895)</td>
<td>3.17202(0.059079)</td>
<td>7.10785(0.059079)</td>
</tr>
<tr>
<td>20, 18</td>
<td>3.08370(0.021854)</td>
<td>6.87221(0.021852)</td>
<td>3.16578(0.052838)</td>
<td>7.08138(0.052838)</td>
</tr>
<tr>
<td>50, 40</td>
<td>3.03958(0.007534)</td>
<td>6.96440(0.007534)</td>
<td>3.04819(0.024438)</td>
<td>7.05957(0.024438)</td>
</tr>
<tr>
<td>50, 45</td>
<td>3.03520(0.008332)</td>
<td>6.98228(0.008332)</td>
<td>3.04799(0.021775)</td>
<td>7.03137(0.021775)</td>
</tr>
<tr>
<td>80, 72</td>
<td>3.02775(0.005548)</td>
<td>6.99473(0.005568)</td>
<td>3.03994(0.013713)</td>
<td>7.00997(0.013713)</td>
</tr>
</tbody>
</table>

Table 7: The lower (LL), the upper (UL) and the width of the 95% Credible Intervals under Uniform

<table>
<thead>
<tr>
<th>n, r</th>
<th>$\theta = 3$ Width</th>
<th>$\theta = 7$ Width</th>
</tr>
</thead>
<tbody>
<tr>
<td>20, 16</td>
<td>1.84192 4.83265</td>
<td>3.99073 6.25580</td>
</tr>
<tr>
<td>20, 18</td>
<td>1.91760 4.76879</td>
<td>3.85119 4.44355</td>
</tr>
<tr>
<td>50, 40</td>
<td>2.20714 4.06061</td>
<td>1.87887 5.12870</td>
</tr>
<tr>
<td>50, 45</td>
<td>2.22676 3.91219</td>
<td>1.75453 5.20104</td>
</tr>
<tr>
<td>80, 72</td>
<td>2.40054 3.80391</td>
<td>1.40337 5.49552</td>
</tr>
</tbody>
</table>

Table 8: The lower (LL), the upper (UL) and the width of the 95% Credible Intervals under Jeffreys

<table>
<thead>
<tr>
<th>n, r</th>
<th>$\theta = 3$ Width</th>
<th>$\theta = 7$ Width</th>
</tr>
</thead>
<tbody>
<tr>
<td>20, 16</td>
<td>1.76676 4.76323</td>
<td>3.00247 10.78103</td>
</tr>
<tr>
<td>20, 18</td>
<td>1.80106 4.59530</td>
<td>2.79424 4.15423</td>
</tr>
<tr>
<td>50, 40</td>
<td>2.14858 4.10033</td>
<td>1.95175 5.00562</td>
</tr>
<tr>
<td>50, 45</td>
<td>2.17819 3.91982</td>
<td>1.74163 5.08010</td>
</tr>
<tr>
<td>80, 72</td>
<td>2.36210 3.75504</td>
<td>1.39294 5.41792</td>
</tr>
</tbody>
</table>

Table 9: The lower (LL), the upper (UL) and the width of the 95% Credible Intervals under Exponential

<table>
<thead>
<tr>
<th>n, r</th>
<th>$\theta = 3$ Width</th>
<th>$\theta = 7$ Width</th>
</tr>
</thead>
<tbody>
<tr>
<td>20, 16</td>
<td>1.61143 4.22794</td>
<td>2.66151 8.08881</td>
</tr>
<tr>
<td>20, 18</td>
<td>1.64622 4.09391</td>
<td>2.44769 7.95492</td>
</tr>
<tr>
<td>50, 40</td>
<td>2.07875 3.84832</td>
<td>1.76957 4.44392</td>
</tr>
<tr>
<td>50, 45</td>
<td>2.09268 3.74156</td>
<td>1.64888 4.45103</td>
</tr>
<tr>
<td>80, 72</td>
<td>2.30049 3.64538</td>
<td>1.34489 5.01802</td>
</tr>
</tbody>
</table>

Table 10: The lower (LL), the upper (UL) and the width of the 95% Credible Intervals under Gamma

<table>
<thead>
<tr>
<th>n, r</th>
<th>$\theta = 3$ Width</th>
<th>$\theta = 7$ Width</th>
</tr>
</thead>
<tbody>
<tr>
<td>20, 16</td>
<td>1.78199 4.67543</td>
<td>2.89344 10.44791</td>
</tr>
<tr>
<td>20, 18</td>
<td>1.83170 4.55517</td>
<td>2.72347 4.03679</td>
</tr>
<tr>
<td>50, 40</td>
<td>2.17809 4.03222</td>
<td>1.85413 4.25485</td>
</tr>
<tr>
<td>50, 45</td>
<td>2.20646 3.94499</td>
<td>1.73853 5.06864</td>
</tr>
<tr>
<td>80, 72</td>
<td>2.37507 3.76355</td>
<td>1.38848 5.41397</td>
</tr>
</tbody>
</table>

7. Comparison of Bayes Estimators and Posterior Risks for Real Life Data Set

In this section, we analyze a real data set and illustrate the analysis of the posterior distribution of shape parameter of Topp-Leone distribution assuming informative and non-informative priors. The data set is taken from http://cdec.water.ca.gov/reservoir_map.html. The capacity of the reservoir for the month of February from 1991 to 2010. The data were transformed to the interval [0, 1]. The sample characteristics required to evaluate the estimates of shape parameter of Topp-Leone are as follows:

$$n = 20, r = 18$$ and $$\sum_{i=1}^{18} \ln\left(2x_i + x_i^2\right) = -2.26131$$ and $$x_r = 0.843485$$.

7.1. Graphical Results of Posterior Distribution for Real Life Data Set

The below graphs reveal that posterior distributions under different informative and non-informative priors. Figure 1 depicts posterior densities under uniform and Jeffreys priors. It is obvious that both the priors yield the approximately identical posterior inferences. Figure 2 shows that the posterior densities under exponential and gamma priors are not identical. However, gamma prior may be a better a choice because of its two hyper parameters which ensure better fit.
Table 11: Bayes Estimates and the corresponding Posterior Risks under the real Data Set.

<table>
<thead>
<tr>
<th>Prior</th>
<th>PLF BEs</th>
<th>PRs</th>
<th>WLF BEs</th>
<th>PRs</th>
<th>ELF BEs</th>
<th>PRs</th>
<th>SLELF BEs</th>
<th>PRs</th>
</tr>
</thead>
<tbody>
<tr>
<td>Uniform</td>
<td>9.40622</td>
<td>0.43271</td>
<td>8.75208</td>
<td>0.43777</td>
<td>8.75208</td>
<td>0.02226</td>
<td>8.94912</td>
<td>0.05404</td>
</tr>
<tr>
<td>Jeffreys</td>
<td>8.96830</td>
<td>0.43244</td>
<td>8.31431</td>
<td>0.43778</td>
<td>8.31431</td>
<td>0.02328</td>
<td>8.51012</td>
<td>0.05713</td>
</tr>
<tr>
<td>Exponential</td>
<td>6.3510</td>
<td>0.29209</td>
<td>5.90942</td>
<td>0.29553</td>
<td>5.90942</td>
<td>0.02225</td>
<td>6.04239</td>
<td>0.05404</td>
</tr>
<tr>
<td>Gamma</td>
<td>6.9472</td>
<td>0.39234</td>
<td>8.10169</td>
<td>0.39686</td>
<td>8.10169</td>
<td>0.02186</td>
<td>8.28072</td>
<td>0.05284</td>
</tr>
</tbody>
</table>

Table 12: The lower (LL), the upper (UL) and the width of the 95% Credible Intervals under Jeffreys Prior

<table>
<thead>
<tr>
<th>Prior</th>
<th>LL</th>
<th>UL</th>
<th>Width</th>
</tr>
</thead>
<tbody>
<tr>
<td>Uniform</td>
<td>5.53290</td>
<td>13.75953</td>
<td>8.22663</td>
</tr>
<tr>
<td>Jeffreys</td>
<td>5.15984</td>
<td>13.16504</td>
<td>8.00520</td>
</tr>
</tbody>
</table>

8. Conclusions
The findings of the simulation study are pretty interesting. The parameter has been under-estimated for majority of the cases. The extent of under estimation is more severe under exponential prior based on PLF, ELF and SLELF. Similarly, the increased true parametric values impose a negative impact on the convergence of the estimates under exponential prior. However, it can be observed that by increasing the sample size, the convergence of the estimated values toward the true parametric values tend to increase for each case. On the other hand, the amounts of posterior risks, based on each prior and loss function tend to decrease by increasing the sample size. It indicates that the estimators are consistent. Also, it is observed that for the fixed n (sample size) as r (effective sample size) increases, the performances of the estimates become better in terms of posterior risks and the width of the credible interval decreases in all the considered cases. It is interesting to note that the posterior risks under SLELF are independent of the choice of true parametric values. In making the comparison of non-informative priors the uniform prior gives the better estimates as the corresponding risks are smaller under WLF, ELF and SLELF. The performance of the PLF and WLF is the best under exponential prior, while The Bayes risks for gamma prior are least under ELF and SLELF. Similarly, the risks corresponding to the estimates under ELF loss function are the least for each prior. It can also be observed that performance of estimates under informative priors is better than those under non-informative priors and the Bayes estimates may turn out to be most efficient provided that useful prior information and appropriate values of the hyper-parameters are available. This simply indicates the use of prior information that makes a different in terms of gain in precision.

The length of credible intervals appears to be inversely proportional to sample size and directly proportional to the true parametric value. Under non-informative priors the length of the intervals is smaller in case of Jeffreys prior, while for informative priors the shorter intervals are observed using exponential prior. The results obtained from the real life data further strengthened the findings of the simulation study. So, from the above discussion, it can be summarized that the use of gamma prior under the ELF may be preferred to estimate the parameter of the Topp-Leone distribution.

As none of the citations mentioned in this paper have considered Bayesian analysis of Topp-Leone distribution so far, a formal comparison of the results is not due. The study can further be extended by considering generalized versions of the distribution under variety of circumstances.

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REFERENCES


