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Generalized Distance and Fixed Point Theorems for Weakly Contractive Mappings

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ABSTRACT

In this paper we consider generalized contractive mapping concerning generalized distance. The existence theorems for fixed points of α-admissible maps in complete metric spaces are proved and some generalization of fixed point theorems are obtained using w-distance. Then as corollaries some fixed point theorems in ordered metric spaces are proved. Our results generalize, improve and simplify the previous results in the literature. [1]

KEYWORDS: Fixed point; Generalized distance; Generalized cotraction; Ordered metric space.

1 INTRODUCTION AND PRELIMINARY

The Banach fixed point theorem for contraction mappings has been generalized and extended in many directions [1-8,20-22].

Recently, Samet, Lakzian [8], Samet et al. [15] introduced new types of generalized contractive mappings and established fixed point theorems for such mappings in complete metric spaces. Nieto and Rodriguez-Lopez [9,10], Ran and Reurins [14], Petrusel and Rus [13] presented some new results for contractions in partially ordered metric spaces.

Kada, Suzuki and Takahashi [7,18] in 1996 introduced the concept of w-distance on a metric space and prove some fixed point theorems. The study of fixed point theorem concerning generalized distance followed in other papers, see [7,19-21]. In this paper, Using concept of w-distance, we generalize contractions and prove some fixed point theorems in ordered metric spaces. Also, we introduce α-admissible maps and we study generalized contractions and prove various fixed point theorens for generalized contractive mappings by using the concept of w-distance in complete metric spaces. Finally, as corollaries we stablish some fixed point theorems for such mappings in ordered metric spaces. Our results generalize and improve some results in [2-8,12-17].

Definition 1.1 *([7,16,21]) Let X be a metric space with metric d. Then a function $p: X \times X \to [0,\infty)$ is called a w-distance on X if the following are satisfied:*
(i) $p(x,z) \leq p(x,y) + p(y,z)$ for any $x,y,z \in X$;
(ii) for any $x \in X$, $p(x,.): X \to [0,\infty)$ is lower semi-continuous;
(iii) for any $\varepsilon > 0$, there exists $\delta > 0$ such that $p(x,z) \leq \delta$ and $p(z,y) \leq \delta$ imply $d(x,y) \leq \varepsilon$.
Let recall that a real-valued function f defined on a metric space X is said to be lower semi-continuous at a point x_0 in X if either $\liminf_{x_n \to x_0} f(x_n) = \infty$ or $f(x_0) \leq \liminf_{x_n \to x_0} f(x_n)$, whenever $x_n \in X$ for each $n \in N$ and $x_n \to x_0$.

Lemma 1.2 *([7,20]) Let X be a metric space with metric d and p be a w-distance on X. Let $\{x_n\}$ and $\{y_n\}$ be sequences in X, let $\{\alpha_n\}$ and $\{\beta_n\}$ be sequences in $[0,\infty)$ converging to zero, and let $x,y,z \in X$. Then the following hold:*
(i) if $p(x_n,y) \leq \alpha_n$ and $p(x_n,z) \leq \beta_n$ for any $n \in N$, then $y = z$. In particular, if $p(x.y) = 0$ and $p(x,z) = 0$, then $y = z$;
(ii) if $p(x_n,y_n) \leq \alpha_n$ and $p(x_n,z) \leq \beta_n$ for any $n \in N$, then $d(y_n,z) \to 0$;
(iii) if $p(x_n,x_m) \leq \alpha_n$ for any $n,m \in N$ with $m > n$, then $\{x_n\}$ is a Cauchy sequence;
(iv) if $p(y,x_n) \leq \alpha_n$ for any $n \in N$, then $\{x_n\}$ is a Cauchy sequence.

Definition 1.3 *([15]) Let $T: X \to X$ and $\alpha: X \times X \to [0,\infty)$. We say that T is α-admissible if*

$$\alpha(x,y) \geq 1 \Rightarrow \alpha(Tx,Ty) \geq 1 \qquad for\ x,y \in X.$$

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2 MAIN RESULTS

First we introduce the following notations:
(i) We denote by Ψ the set of functions $\psi: [0,\infty) \to [0,\infty)$ satisfying the following hypotheses:
(h_1) ψ is continuous and nondecreasing,
(h_2) $\psi(t) = 0$ if and only if $t = 0$.
(ii) We denote by Φ the set of functions $\varphi: [0,\infty) \to [0,\infty)$ satisfying the following hypotheses:
(c_1) φ is continuous,
(c_2) $\varphi(t) = 0$ if and only if $t = 0$.

Since in every metric space (X,d), d is a w-distance, so our theorems are generalization of theorems in [8,15].
The following theorem is a generalization of theorem (2.1) in [15].

Theorem 2.1 *Let (X,d) be a complete metric space and $T: X \to X$ be an operator. Let p be a w-distance on (X,d) and suppose that,*
(i) T is α-admissible,
(ii) there exists $x_0 \in X$ such that $\alpha(x_0, T(x_0)) \geq 1$,
(iii) T is orbitally continuous,
(iv) T be a self-mapping satisfying

$$\alpha(x,y)\psi(p(T(x),T(y))) \leq \psi(p(x,y)) - \varphi(p(x,y)) \tag{1}$$

for $\psi \in \Psi$ and $\varphi \in \Phi$.
Then, T has a fixed point.
Proof: Let $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$. Define the sequence $\{x_n\}$ in X by

$$x_{n+1} = Tx_n, \qquad \text{for all } n \in N.$$

If $x_n = x_{n+1}$ for some $n \in N$, then $x^* = x_n$ is a fixed point for T. Assume that $x_n \neq x_{n+1}$ for all $n \in N$. Since T is α-admissible, we have

$$\alpha(x_0, x_1) = \alpha(x_0, Tx_0) \geq 1 \Rightarrow \alpha(Tx_0, Tx_1) = \alpha(x_1, x_2) \geq 1.$$

By induction, we get

$$\alpha(x_n, x_{n+1}) \geq 1, \qquad \text{for all } n \in N. \tag{2}$$

Applying the inequality (1) with $x = x_{n-1}$ and $y = x_n$ and using (2), we obtain

$$\begin{aligned}\psi(p(x_n, x_{n+1})) = \psi(p(Tx_{n-1}, Tx_n)) &\leq \alpha(x_{n-1}, x_n)\psi(p(Tx_{n-1}, Tx_n)) \\ &\leq \psi(p(x_{n-1}, x_n)) - \varphi(p(x_{n-1}, x_n)) \\ &\leq \psi(p(x_{n-1}, x_n)). \end{aligned} \tag{3}$$

Using the monotone property of the ψ-function, we get

$$p(x_n, x_{n+1}) \leq p(x_{n-1}, x_n).$$

It follows that $\{p(x_n, x_{n+1})\}$ is monotone decreasing and consequently, there exists $r \geq 0$ such that
$p(x_n, x_{n+1}) \to r$ as $n \to \infty$.
Letting $n \to \infty$ in (3) and using the continuity of ψ and φ, we obtain

$$\psi(r) \leq \psi(r) - \varphi(r),$$

which implies that $\varphi(r) = 0$ and then $r = 0$. So

$$p(x_n, x_{n+1}) \to 0 \quad \text{as } n \to \infty. \tag{4}$$

Next we show that $\{x_n\}$ is a Cauchy sequence. If otherwise, there exist an $\varepsilon > 0$ for which we can find two sequences of positive integers $\{n(k)\}$ and $\{m(k)\}$ such that for all positive integers k such that $n(k) > m(k) > k$, $p(x_{m(k)}, x_{n(k)}) \geq \varepsilon$ and $p(x_{m(k)}, x_{n(k)-1}) < \varepsilon$
Now,

$$\varepsilon \leq p(x_{m(k)}, x_{n(k)}) \leq p(x_{m(k)}, x_{n(k)-1}) + p(x_{n(k)-1}, x_{n(k)}).$$

That is, $\varepsilon \leq p(x_{m(k)}, x_{n(k)}) \leq \varepsilon + p(x_{n(k)-1}, x_{n(k)})$.
Taking the limit as $k \to \infty$ in the above inequality and using (4), we have

$$\lim_{k\to\infty} p(x_{m(k)}, x_{n(k)}) = \varepsilon. \tag{5}$$

Also,

$$p(x_{m(k)-1}, x_{n(k)-1}) \leq p(x_{m(k)-1}, x_{m(k)}) + p(x_{m(k)}, x_{n(k)-1}).$$

Taking the limit as $k \to \infty$ in the above inequality and using (4), we have

$$\lim_{k\to\infty} p(x_{m(k)-1}, x_{n(k)-1}) = \varepsilon. \tag{6}$$

For $x = x_{m(k)-1}$, $y = y_{n(k)-1}$, we have

$$\begin{aligned}\alpha(x_{m(k)-1}, x_{n(k)-1})\psi(p(x_{m(k)}, x_{n(k)})) &= \alpha(x_{m(k)-1}, x_{n(k)-1})\psi(p(T(x_{m(k)-1}), T(x_{n(k)-1}))) \\ &\leq \psi(p(x_{m(k)-1}, x_{n(k)-1})) + \varphi(p(x_{m(k)-1}, x_{n(k)-1})),\end{aligned}$$

Now, letting $k \to \infty$, using (5),(6) and the continuity of ψ and φ, we obtain $\psi(\varepsilon) \leq \psi(\varepsilon) - \varphi(\varepsilon)$, which implies that $\varepsilon = 0$, a contradiction with $\varepsilon > 0$.

Hence $\{x_n\}$ is a Cauchy sequence. from the completeness of X, there exists a $x^* \in X$ such that $x_n \to x^*$ as $n \to \infty$.

From the orbitally continuity of T, it follows that $x_{n+1} = Tx_n \to Tx^*$ as $n \to \infty$.

Now, we show that x^* is a fixed point. We have,

$\psi(p(x_{n+1}, x_{n+2})) \leq \alpha(x_n, x_{n+1})\psi(p(x_{n+1}, x_{n+2})) \leq$
$\psi(p(x_n, x_{n+1})) - \varphi(p(x_n, x_{n+1}))$,

Letting $n \to \infty$, we get

$$\psi(p(x^*, x^*)) \leq \psi(p(x^*, x^*)) - \varphi(p(x^*, x^*)). \qquad (7)$$

Similarly,

$$\psi(p(x_{n+1}, Tx_{n+1})) \leq \alpha(x_n, x_{n+1})\psi(p(x_{n+1}, Tx_{n+1})) \leq \psi(p(x_n, Tx_n)) - \varphi(p(x_n, Tx_n)).$$

Letting $n \to \infty$ and using the orbitally continuity of T, we obtain

$$\psi(p(x^*, Tx^*)) \leq \psi(p(x^*, Tx^*)) - \varphi(p(x^*, Tx^*)). \qquad (8)$$

Using (7) and (8), we get $p(x^*, x^*) = 0$ and $p(x^*, Tx^*) = 0$, so by lemma (1.2), we conclude that $x^* = Tx^*$. This completes the proof.

The following theorem is a generalization of theorem (2.2) in [15].

Theorem 2.2 *Let (X, d) be a complete metric space and $T: X \to X$ be an operator. Let p be a w-distance on (X, d) and suppose that,*

(i) T is α-admissible,

(ii) there exists $x_0 \in X$ such that $\alpha(x_0, T(x_0)) \geq 1$.

(iii) if $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}) \geq 1$ for all n and $x_n \to x$ as $n \to \infty$, then $\alpha(x_n, x) \geq 1$ for all n,

(iv) T be a self-mapping satisfying

$$\alpha(x, y)\psi(p(T(x), T(y))) \leq \psi(d(x, y)) - \varphi(x, y)$$

for $\psi \in \Psi$ and $\varphi \in \Phi$.

Then T has a fixed point.

Proof: Following the proof of Theorem (2.1), we know that $\{x_n\}$ is a Cauchy sequence in the complete metric space (X, d). Then, there exists $x^* \in X$ such that $x_n \to x^*$ as $n \to \infty$. On the other hand, from (ii) and (iii) we have

$$\alpha(x_n, x^*) \geq 1 \quad \text{for all} \quad n \in N. \qquad (9)$$

Now, from (ii) and (iv) we get

$$\psi(p(x_{n+1}, x_{n+2})) \leq \alpha(x_n, x_{n+1})\psi(p(x_n, x_{n+1})) \leq \psi(p(x_n, x_{n+1})) - \varphi(p(x_n, x_{n+1})).$$

Letting $n \to \infty$ and using the continuity of ψ and φ, we obtain $p(x^*, x^*) = 0$.

Also, by (iv) and (9) we have

$$\psi(p(x_{n+1}, Tx^*)) \leq \alpha(x_n, x^*)\psi(p(x_n, x^*)) \leq \psi(p(x_n, x^*)) - \varphi(p(x_n,^* x)).$$

letting $n \to \infty$, using the continuity of ψ and φ and the fact $p(x^*, x^*) = 0$, we have $p(x^*, Tx^*) = 0$. So by lemma (1.2) we have $Tx^* = x^*$, that is, x^* is a fixed point of T.

Corollary 2.3 *Let $(X, d, \leq)$ be an ordered metric space and $T: X \to X$ be a continuous and nondecreasing mapping w.r.t $\leq$. Let p be a w-distance on (X, d) and suppose that,*

(i) T be a self-mapping satisfying

$\psi(p(T(x), T(y))) \leq \psi(p(x, y)) - \varphi(p(x, y))$

for all $x, y \in X$ with $x \leq y$, $\psi \in \Psi$ and $\varphi \in \Phi$,

(ii) there exists $x_0 \in X$ such that $x_0 \leq Tx_0$,

(iii) T is orbitally continuous,

(iv) the metric d is complete.

Then, T has a fixed point.

Proof: Define the mapping $\alpha: X \times X \to [0, \infty)$ by

$$\alpha(x, y) = \begin{cases} 1 & if\ x \leq y, \\ 0 & otherwise. \end{cases}$$

It is enough to show that T is α-admissible.

Let $x, y \in X$ such that $\alpha(x, y) \geq 1$. By the definition of α, this implies that $x \leq y$. Since T is a nondecreasing mapping w.r.t $\leq$, we have $Tx \leq Ty$, which gives us that $\alpha(Tx, Ty) = 1$. Then T is α-admissible. From (ii), there exists $x_0 \in X$ such that $x_0 \leq Tx_0$. This implies that $\alpha(x_0, Tx_0) = 1$. Therefore, all the hypotheses of Theorem (2.1) are satisfied, and so T has a fixed point.

Corollary 2.4 *Let $(X, d, \leq)$ be an ordered metric space and $T: X \to X$ be a continuous and nondecreasing mapping w.r.t $\leq$. Let p be a w-distance on (X, d) and suppose that,*

(i) T be a self-mapping satisfying
$\psi(p(T(x), T(y))) \leq \psi(p(x, y)) - \varphi(p(x, y))$
for all $x, y \in X$ with $x \leq y$, $\psi \in \Psi$ and $\varphi \in \Phi$,
(ii) there exists $x_0 \in X$ such that $x_0 \leq Tx_0$,
(iii) if $\{x_n\}$ is a nondecreasing sequence in X such that $x_n \to x \in X$ as $n \to \infty$, then $x_n \leq x$ for all n,
(iv) the metric d is complete.
Then, T has a fixed point.

Proof: Define the mapping $\alpha: X \times X \to [0, \infty)$ by

$$\alpha(x, y) = \begin{cases} 1 & if\ x \leq y, \\ 0 & otherwise. \end{cases}$$

The reader can show easily that T is α-admissible, Now, let $\{x_n\}$ be a sequence in X such that $\alpha(x_n, x_{n+1}) \geq 1$ for all n and $x_n \to x \in X$ as $n \to \infty$. By the definition of α, we have $x_n \leq x_{n+1}$ for all n. From (iii), this implies that $x_n \leq x$ for all n, which gives us that $\alpha(x_n, x) = 1$ for all n. Thus all the hypotheses of Theorem (2.2) are satisfied, and so T has a fixed point.

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