J. Basic. Appl. Sci. Res., 3(4)872-880, 2013
© 2013, TextRoad Publication

ISSN 2090-4304
Journal of Basic and Applied Scientific Research
www.textroad.com

Inverse Data Envelopment Analysis Model in the Present of Non-Discretionary and Discretionary Data to Preserve Relative Efficiency Values: The Case of Variable Returns to Scale

Neda Fathi and Mohamad Izadikhah

Department of Mathematics, Arak Branch, Islamic Azad University, Arak, Iran

ABSTRACT

This paper studies the proposed models by S.Lertworasirikul et al.(2011) in the present of non-discretionary and discretionary data. We consider the inverse BCC model for a resource allocation problem, where increases of some outputs and decreases of the other outputs of the considered DMU can be taken into account simultaneously. The inverse BCC problem is in the form of a multi objective non-linear programming model (MONLP), which is not easy to solve. We propose a linear programming model, which gives a Pareto-efficient solution to the inverse BCC problem. However, there exists at least an optimal solution to the proposed model if and only if the new output vector is in the set of current production possibility set.

KEYWORDS: Data envelopment analysis(DEA), Non-discretionary data, Discretionary data, Efficiency, Linear programming, Variable returns to scale. Inverse optimization, Resource allocation.

1. INTRODUCTION

Data envelopment analysis (DEA), is currently a popular technique for analysing technical efficiency and it has been used in a number of applications. DEA originally proposed by Charnes, Cooper, and Rhodes [6] is a non-parametric technique for measuring and evaluating the relative efficiencies of a set of entities, called decision making units (DMUs), with the common inputs and outputs.

An interesting problem is how to preserve the relative efficiency value of a considered DMU if the internal technical structure of the considered DMU slightly changes in a short term. In recent years, inverse optimization of DEA models has been studied. For DEA models, constraint parameters are input and output values of DMUs. Therefore, inverse problems of DEA models can be classified into two types depending on what parameters are changed and what parameters need to be varied to keep the optimal objective value unchanged. Resource allocation problem and investment analysis problem. The resource allocation problem of DEA is an inverse DEA problem of determining the best possible inputs for given outputs such that the current efficiency value of a considered DMU (DMU_o) with respect to other DMUs remains unchanged. The investment analysis problem of DEA is an inverse DEA problem of determining the best possible outputs for given inputs such that the current efficiency value of a considered DMU_o with respect to other DMUs remains unchanged.

Normally, the internal technical structure of a DMU should not change dramatically in a short term (Yan et.al. [18]). Therefore, the inverse DEA models can be used to such resource allocation and investment analysis problems. Wei et al. [17] proposed, for the first time, an inverse DEA model for input and output estimation. In their work, an inverse DEA model was discussed to answer the following question:

Among a group of DMUs, if we increase certain inputs of a particular unit and assume that the DMU maintains its current efficiency value with respect to other units, how much more outputs could the unit produce? Or, if the outputs need to be increased to a certain value and the efficiency of the unit remains unchanged, how much more inputs should be provided to the unit? In their developed inverse DEA model, the increases in input and output values were assumed to be nonnegative values, and the inverse DEA model was transformed into and solved as a multi-objective linear programming (MOLP) problem. Yan, et al. [18] discussed an inverse DEA problem with preference cone constraints to represent decision makers' preferences, which was useful in resource planning. Jahanshahloo et al. [11]. extended the inversed DEA problem and the developed solution method by Yan, et al. [18] to the case of determining outputs of the considered DMU when some or all of inputs were increased and the efficiency value of the considered DMU with respect to other DMUs needed to be improved by specified percentage of its current efficiency value. They proposed by using inverse data envelopment analysis model, a method to estimate output levels of a decision making unit is presented when some or all of its input entities are increased and its current efficiency level is improved. Jahanshahloo et al. [12] showed that the inversed DEA models could be

***Corresponding Author:** Neda Fathi, Department of Mathematics, Arak Branch, Islamic Azad University, Arak, Iran, edafathi112@yahoo.com

used to estimate inputs for a DMU when some or all outputs and the efficiency value of the DMU were increased or preserved, and also identified extra inputs when outputs were estimated using the proposed models by Yan et al. [18] and Jahanshahloo et al. [11]. Jahanshahloo et al [13] proposed a modified inverse DEA model for sensitivity analysis of efficiency classifications of efficient and inefficient DMUs in which important policies over inputs, outputs and DMUs were represented by preference cones.

Hadi-Vencheh and Foroughi [10] discussed an extended inverse DEA model where an increase of some inputs (outputs) and a decrease due to some of the other inputs (outputs) are taken into account at the same time. A proposed solving method was based on DEA and MOLP. In their paper, they also showed that the solution proposed by Wei et al. [17] did not guarantee the efficiency result for input estimating, i.e., it might fail in a special case. Actually, in the paper of Wei, et al. [17], only the increase of inputs (outputs) is considered whereas each DMU may concern the increase of some of inputs (outputs) and the decrease of the other inputs (outputs) simultaneously. Alinezhad et al [1] proposed a methodology that uses an interactive MOLP for solving the inverse DEA problems. Previous studies on inverse DEA problems mostly consider the efficiency value of the considered DMU without considering the effect of input/output changes on efficiency values of other DMUs. Recently, Lertworasirikul et al. [15] studied the inverse Data Envelopment Analysis (inverse DEA) for the case of variable returns to scale (inverse BCC). The developed inverse BCC model can preserve relative efficiency values of all decision making units (DMUs) in a new production possibility set composing of all current DMUs and a perturbed DMU with new input and output values. They considered the inverse BCC model for a resource allocation problem, where increases of some outputs and decreases of the other outputs of the considered DMU can be taken into account simultaneously. They proposed a linear programming model, which gives a Pareto-efficient solution to the inverse BCC problem.

In this paper, we discuss the proposed models by Lertworasirikul et al. [15] when some of inputs and outputs values are non-discretionary.

Often the assumption of homogeneous environments is violated and factors that describe the differences in the environments need to be include in the analysis .these factors, and other factors outside the control of the DMUs ,are frequently called non-discretionary factors instances from the DEA literature include snowfall or weather in evaluating the efficiency of maintenance units, the number of competitors in the branches of a restaurant china. In this paper, we discuss the proposed models by Lertworasirikul et al. [15] in the presence of such data.

In Section 2, we provide a brief review on DEA models. Section 3 states the inverse BCC model in the present of non-discretionary and discretionary data and presents our proposed model to determine the best possible values of inputs for the perturbed DMU to preserve relative efficiency values of all DMUs. In Section 4, a solution approach to the inverse BCC in the present of non-discretionary and discretionary data is presented. Finally, conclusions are given in Section 5.

2. The DEA model in the present of non-discretionary and discretionary data

Assume that there are n DMUs, where each DMU_i(i=1,…,n), uses m different inputs,x_{ji} (j=1,…,m), to produce r different outputs, y_{ki} (k=1,. . . ,r) . We denote y_{ki} (k=1,. . . ,r) by the level of the kth output from unit i(i=1,. . . ,n) and by x_{ji} (i=1,. . . ,n)the level of the jth input to the ith unit. Let i=o be the evaluated unit. Assume that input and output data are semi-positive and some of them are non-discretionary. So we have:

$$x_i = \begin{bmatrix} x_{1i} \\ x_{2i} \\ \vdots \\ x_{mi} \end{bmatrix} \geq 0, x_i \neq 0, y_i = \begin{bmatrix} y_{1i} \\ y_{2i} \\ \vdots \\ y_{ri} \end{bmatrix} \geq 0, y_i \neq 0$$

$$x_o = \begin{bmatrix} x_{1o} \\ x_{2o} \\ \vdots \\ x_{mo} \end{bmatrix} \geq 0, x_o \neq 0, y_i = \begin{bmatrix} y_{1o} \\ y_{2o} \\ \vdots \\ y_{ro} \end{bmatrix} \geq 0, y_o \neq 0$$

The primal and dual BCC models for obtain efficiency of DMU_o are in the following form.

$$\begin{aligned}
&(\boldsymbol{PBCC_o}) \text{ maximize } \left(-\textstyle\sum_{j\in\{NI\}} x_{jo}v_j + \sum_{k\in\{DO\}} u_k y_{ko} + \sum_{k\in\{NO\}} u_k y_{ko} - u_o\right) \\
&s.t: \quad \textstyle\sum_{j\in\{DI\}} v_j x_{jo} = 1 \\
&\qquad -\textstyle\sum_{j\in\{DI\}} v_j x_{ji} - \sum_{j\in\{NI\}} v_j x_{ji} + \sum_{k\in\{DO\}} u_k y_{ki} + \sum_{k\in\{NO\}} u_k y_{ki} - u_o \leq 0, \ for \ i = 1, \dots, n \\
&\qquad\qquad u_o \ is \ free, u_k, v_j \geq 0, j = 1, \dots, m, k = 1, \dots, r
\end{aligned} \qquad \textbf{(1)}$$

$$\begin{aligned}
&(\boldsymbol{DBCC_o}) \text{ minimize } \left(\theta - \varepsilon\left(\textstyle\sum_{j\in\{DI\}} s_j^- + \sum_{k\in\{DO\}} s_k^+\right)\right) \\
&s.t: \quad \textstyle\sum_{i=1}^n \lambda_i x_{ji} + s_j^- = \theta_o x_{jo}, \quad j \in \{DI\} \\
&\qquad \textstyle\sum_{i=1}^n \lambda_i x_{ji} + s_j^- = x_{jo}, \quad j \in \{NI\} \\
&\qquad \textstyle\sum_{i=1}^n \lambda_i y_{ki} - s_k^+ = y_{ko}, \quad k \in \{Do\}
\end{aligned}$$

$\sum_{i=1}^{n} \lambda_i y_{ki} - s_k^+ = y_{ko}$, $\quad k \in \{No\}$
$\sum_{i=1}^{n} \lambda_i = 1$
$\lambda_j \geq 0$, $s_k^+ \geq 0$, $s_j^- \geq 0$, θ *is free*, **(2)**

Where i = 1,. . . ,n, j = 1,. . . ,m, k=1,. . . , r, x_{jo} is the input j of the considered DMU (DMU_o), x_{ji} is the input j of DMU_i, y_{ko} is the output k of DMU_o, y_{ki} is the output k of DMU_i, u_k is the weight of output k, v_j is the weight of input j, $u_o$is scalar λ_i is the convex combination of DMU_i, θ_o is the objective function or the technical efficiency valueof DMU_o. The indices and notation will be used throughout this paper.

From the BCC models, DMU_owill be technically efficient if the maximal efficiency,θ_o^*, is equal to 1. If $\theta_o^* < 1$, it is possible to produce the given outputs using smaller input values, which may be obtained as a convex combination of inputs of other DMUs. Theλ_i, i = 1,. . . ,n obtained from the $DBCC_o$ model provide a reference set for inefficient DMUs. The convex combination of the reference set is the projected point on the production frontier of the inefficient DMUs. The set of feasible activities or all DMUs is called production possibility set. The BCC model has its production frontier spanned by the convex hull of the existing DMUs. From the production frontier of the BCC model, a DMU is inefficient if it is possible to reduce any input without increasing any other inputs and achieve the same value of outputs or it is possible to increase any output without reducing any other outputs and use the same values of inputs (Lertworasirikul, et.al.[14]).

In this paper we propose an inverse BCC model to preserve efficiency values of all DMUs relative to other DMUs based on the current production frontier. With the same production frontier, relative efficiency values of all DMUs can be maintained. To preserve the production frontier, the production possibility set is composed of all current DMUs and the considered DMU with the changes in its input and output values.

3. The inverse BCC model

Denote the considered DMU with current input and output values by $DMU_o$and the considered $DMU_o$with its input and output changes (perturbedDMU_o) by$DMU_{o'}$. The developed inverse BCC model for a resource allocation problem is introduced to answer the following question. For a group of current DMUs with their relative efficiency values of $\theta_1^*, \theta_2^*, \theta_3^*, \ldots, \theta_n^*$, suppose that the output values of are changed from y_o to $y_o + \Delta y_o \geq 0, \Delta y_o \neq 0$,we want to find the minimum $x_o + \Delta x_o$ where $x_o + \Delta x_o$ is a semi-positive vector such that $DMU_{o'}$ with new input and output values $(x_o + \Delta x_o, y_o + \Delta y_o)$ still has its relative efficiency value of θ_o^*, and all other DMUs still have their relative efficiency values of $\theta_1^*, \theta_2^*, \theta_3^*, \ldots, \theta_n^*$.

Note that the current production possibility set before the changes of input and output values of DMU_o is composed of n DMUs (DMU_i, i = 1,. . . ,n). However, after input and output values of DMU_o are changed, we consider the new production possibility set composing of n + 1 DMUs (DMU_i, i = 1,. . . ,n, and $DMU_{o'}$) and try to preserve the production frontier. For non–discretionary data we let: Δx_{jo}=0 , Δy_{ko}=0,(j$\in \{NI\}, k \in \{NO\}$)where{NI},{NO} refer to non–discretionary input and outputs indices. .

3.1. Primal form of the inverse BCC model

The mathematical models (primal and dual models) of the inverse BCC for a resource allocation problem when some data are non-discretionary are as follows.

($IBCC_o$) minimize Δx_o

$$
\begin{aligned}
\text{s.t. } & \sum_{j\in\{DI\}} v_j (x_{jo} + \Delta x_{jo}) = 1 \\
& -\sum_{j\in\{DI\}} v_j x_{ji} - \sum_{j\in\{NI\}} v_j x_{ji} + \sum_{k\in\{DO\}} u_k y_{ki} + \sum_{k\in\{NO\}} u_k y_{ki} - u_o \leq 0 , \ for\ \ i = 1, \ldots , n \\
& -\sum_{j\in\{DI\}} v_j (x_{jo} + \Delta x_{jo}) - \sum_{j\in\{NI\}} v_j (x_{jo} + \Delta x_{jo}) + \sum_{k\in\{DO\}} u_k^* y_{ki} + \sum_{k\in\{NO\}} u_k^* y_{ki} - u_o \leq 0 \\
& x_{jo} + \Delta x_{jo} \geq 0 \\
& u_o \ is\ free , u_k, v_j \geq 0, j = 1, \ldots , m, k = 1, \ldots , r \qquad \textbf{(3)}
\end{aligned}
$$

Where $x_{jo} + \Delta x_{jo} \neq 0$ for each and $\sum_{k\in\{DO\}} u_k^* y_{ko} + \sum_{k\in\{NO\}} u_k^* y_{ko} - \sum_{j\in\{NI\}} x_{jo} v_j^* - u_o = \sum_{k\in\{DO\}} u_k (y_o + \Delta y_o) + \sum_{k\in\{NO\}} u_k (y_o + \Delta y_o) - \sum_{j\in\{NI\}} x_{jo} v_j - u_o = \theta_o^*$ is the relative efficiency value of DMU_o before the changes in its output values. Using Δx_o from the $IBCC_o$ model, the relative efficiency values of all DMU_l for l = 1,2, . . . ,n from solving the following $IBCC_l$ model must be equal to θ_l^* where θ_l^* is the current relative efficiency value of DMU_lbefore DMU_ochanges its output values.

($IBCC_l$) maximize$(\sum_{k\in\{DO\}} u_k y_{kl} + \sum_{k\in\{NO\}} u_k y_{kl} - \sum_{j\in\{NI\}} x_{jo} v_j - u_o)$

$$
\begin{aligned}
\text{s.t. } & \sum_{j\in\{DI\}} v_j x_{jl} = 1 \\
& -\sum_{j\in\{DI\}} v_j x_{jl} - \sum_{j\in\{NI\}} v_j x_{jl} + \sum_{k\in\{DO\}} u_k y_{kl} + \sum_{k\in\{NO\}} u_k y_{kl} - u_o \leq 0 , \ for\ \ i = 1, \ldots , n
\end{aligned}
$$

$-\sum_{j\in\{DI\}} v_j(x_{jo}+\Delta x_{jo}) - \sum_{j\in\{NI\}} v_j(x_{jo}+\Delta x_{jo})\ \sum_{k\in\{DO\}} u_k(y_{ko}+\Delta y_{ko}) + \sum_{k\in\{NO\}} u_k(y_{ko}+\quad \Delta y_{ko}) - u_o \le 0$

$u_o\ is\ free\ , u_k, v_j \ge 0, j = 1,\dots,m, k = 1,\dots,r$ **(4)**

3-2. Dual form of the inverse BCC model

We have :

$(\mathbf{DIBCC_o})$ minimize Δx_o

s.t. $\sum_{i=1}^{n} \lambda_i x_{ji} + \lambda_{o'}(x_{jo}+\Delta x_{jo}) + s_j^- = \theta_o^*(x_{jo}+\Delta x_{jo}), \quad j\in\{DI\}$

$\sum_{i=1}^{n} \lambda_i x_{ji} + \lambda_{o'}(x_{jo}+\Delta x_{jo}) + s_j^- = (x_{jo}), \quad j\in\{NI\}$

$\sum_{i=1}^{n} \lambda_i y_{ki} + \lambda_{o'}(y_{ko}+\Delta y_{ko}) - s_k^+ = y_{ko} + \Delta y_{ko}, \quad k\in\{Do\}$

$\sum_{i=1}^{n} \lambda_i y_{ki} + \lambda_{o'}(y_{ko}+\Delta y_{ko}) - s_k^+ = y_{ko}\quad , \quad k\in\{No\}$

$\sum_{i=1}^{n} \lambda_i + \lambda_{o'} = 1$

$x_o + \Delta x_o \ge 0$

$\lambda_{o'}, \lambda_i \ge 0, i = 1,\dots,n.$ **(5)**

Where $x_o + \Delta x_o \ne 0$ and θ_o^* is the relative efficiency value of DMU_o before the changes in its output values. Using Δx_o from the DIBCC0 model, the relative efficiency value of DMU_l for l = 1,2,. . . ,n from solving the $DIBCC_l$ model must be equal to θ_l^*.

$(\mathbf{DIBCC_o})$ minimize $(\theta_l - \varepsilon(\sum_{j\in\{DI\}} s_j^- + \sum_{k\in\{DO\}} s_k^+)$

s.t. $\sum_{i=1}^{n} \lambda_i x_{ji} + \lambda_{o'}(x_{jo}+\Delta x_{jo}) + s_j^- = \theta_l x_{jl}\ , \quad j\in\{DI\}$

$\sum_{i=1}^{n} \lambda_i x_{ji} + \lambda_{o'}(x_{jo}) + s_j^- = x_{jl}\ , \quad j\in\{NI\}$

$\sum_{i=1}^{n} \lambda_i y_{ki} + \lambda_{o'}(y_{ko}+\Delta y_{ko}) - s_k^+ = y_{kl}\ , \quad k\in\{Do\}$

$\sum_{i=1}^{n} \lambda_i y_{ki} + \lambda_{o'}(y_{ko}) - s_k^+ = y_{kl}\quad , \quad k\in\{No\}$

$\sum_{i=1}^{n} \lambda_i + \lambda_{o'} = 1$

$\lambda_{o'}, \lambda_i \ge 0, i = 1,\dots,n.$ **(6)**

Note that the $\mathrm{IBCC_o}$ and $\mathrm{DIBCC_o}$ are in the form of multi-objective non-linear programming (MONLP) form.

4. A solution approach to the inverse BCC model in the present of discretionary and non–discretionary data

To solve the inverse BCC model for the resource allocation problem, we need to find the value of $\Delta x_o = (\Delta x_{1o}, \Delta x_{2o}, \dots, \Delta x_{mo})$,which keeps the relative efficiency values of all DMUs unchanged. This can be done by solving the $\mathrm{IBCC_o}$ and $\mathrm{IBCC_l}$ models or by solving $\mathrm{DIBCC_o}$ and $\mathrm{DIBCC_l}$ models. However, these models are in the form of MONLP, which is not easy to solve. In this section, we propose a multi-objective linear programming model ($\mathrm{MLDIBCC}_o$), which gives an optimal solution for the inverse BCC model. Later we propose a linear programming model (LDIBCC_o) in Theorem 2, which gives a Pareto solution to the $\mathrm{MLDIBCC}_o$ model. Therefore,

Theorem 1. Assume that the relative efficiency value of DMU_o with respect to other DMUs in a group of comparable DMUs (i = 1,. . .,n) is θ_o^*. Given the changes in output values of DMU_o, $\Delta y_o \ne 0$, the minimum Δx_o of the perturbed DMU_o ($DMU_{O'}$), which does not make any changes to the relative efficiency values of all DMUs (l = 1,. . .,n, O'), can be obtained by solving the $\mathrm{MLDIBCC}_o$ model. For non–discretionary data let: Δx_{jo}=0 , Δy_{ko}=0.

$(\mathbf{MLDIBCC_o})$ minimize $\Delta x_o = (\Delta x_{1o}, \Delta x_{2o}, \dots, \Delta x_{mo})^T$

s.t. $\sum_{i=1}^{n} \lambda_i x_{ji} + s_j^- = \theta_o^*(x_{jo}+\Delta x_{jo}), \quad j\in\{DI\}$

$\sum_{i=1}^{n} \lambda_i x_{ji} + s_j^- = (x_{jo}), \quad j\in\{NI\}$

$\sum_{i=1}^{n} \lambda_i y_{ki} - s_k^+ = y_{ko} + \Delta y_{ko}, \quad k\in\{Do\}$

$\sum_{i=1}^{n} \lambda_i y_{ki} - s_k^+ = y_{ko}\quad , \quad k\in\{No\}$

$\sum_{i=1}^{n} \lambda_i = 1$

$\Delta x_{jo} = 0 \quad , j\in\{NI\}$

$\lambda_i \ge 0, i = 1,\dots,n.$ **(7)**

Proof. The BCC model for $DMU_{O'}$ relative to other DMUs (l =1,… ,n) is th$\mathrm{DBCC_{O'}}$ model.

$(\mathbf{DBCC_{O'}})$minimize$(\theta_{o'} - \varepsilon(\sum_{j\in\{DI\}} s_j^- + \sum_{k\in\{DO\}} s_k^+))$

s.t. $\sum_{i=1}^{n} \lambda_i x_{ji} + \lambda_{o'}(x_{jo}+\Delta x_{jo}) + s_j^- = \theta_{o'}(x_{jo}+\Delta x_{jo}), \quad j\in\{DI\}$

$\sum_{i=1}^{n} \lambda_i x_{ji} + \lambda_{o'}(x_{jo}+\Delta x_{jo}) + s_j^- = (x_{jo}), \quad j\in\{NI\}$

$\sum_{i=1}^{n} \lambda_i y_{ki} + \lambda_{o'}(y_{ko}+\Delta y_{ko}) - s_k^+ = y_{ko} + \Delta y_{ko}, \quad k\in\{Do\}$

$\sum_{i=1}^{n} \lambda_i y_{ki} + \lambda_{o'}(y_{ko}+\Delta y_{ko}) - s_k^+ = y_{ko}\quad , \quad k\in\{No\}$

$$\sum_{i=1}^{n} \lambda_i + \lambda_{o'} = 1$$
$$\lambda_{o'}, \lambda_i \geq 0, i = 1, \dots, n. \quad (8)$$

The set of constraints in the $DBCC_{O'}$ model can be rearranged in the following form:

$$\sum_{i=1}^{n} \lambda_i x_{ji} + s_j^- = (\theta_{o'} - \lambda_{o'})(x_{jo} + \Delta x_{jo}), \quad j \in \{DI\}$$
$$\sum_{i=1}^{n} \lambda_i x_{ji} + s_j^- = (1 - \lambda_{o'})(x_{jo}), \quad j \in \{NI\}$$
$$\sum_{i=1}^{n} \lambda_i y_{ki} - s_k^+ = (1 - \lambda_{o'})(y_{ko} + \Delta y_{ko}), \quad k \in \{Do\}$$
$$\sum_{i=1}^{n} \lambda_i y_{ki} - s_k^+ = (1 - \lambda_{o'})(y_{ko}), \quad k \in \{No\}$$
$$\sum_{i=1}^{n} \lambda_i + \lambda_{o'} = 1$$
$$\lambda_{o'}, \lambda_i \geq 0, i = 1, \dots, n. \quad (9)$$

Case 1: $\theta_o^* < 1$

From the set of constraints in (9), if $\lambda_{o'} = 1$, then $\lambda_i = 0$,for i = 1,2,. . . ,n and $\theta_{o'} = 1$. The solution ($\lambda_i = 0$ for i =1, 2, . . ., n; $\lambda_{o'} = 1, \theta_{o'} = 1$)is not the optimal solution to the $DBCC_{O'}$ model When $\theta_o^* < 1$,because we can find a better solution at $\lambda_{o'} = 0$ .When $\lambda_{o'} = 0$, the constraints of the $DBCC_{O'}$ model are in the same form as the constraints in the MLDIBCC_o model. Thus, the objective value is θ_o^*, which is less than 1.Since $(1 - \lambda_{o'}) \neq 0$, we divide all constraints in the $DBCC_{O'}$ model by $(1 - \lambda_{o'})$ and set $\theta_{o'}^- = \frac{(\theta_{o'} - \lambda_{o'})}{(1-\lambda_{o'})}$and $\lambda_i^- = \frac{\lambda_i}{(1-\lambda_{o'})}$ for i=1, 2, . . .,n. Then the $DBCC_{O'}$ model becomes:

$$\text{minimize}(\theta_{o'} - \varepsilon(\textstyle\sum_{j \in \{DI\}} s_j^- + \sum_{k \in \{DO\}} s_k^+))$$
s.t. $$\sum_{i=1}^{n} \lambda_i^- {}_i x_{ji} + \frac{s_j^-}{(1-\lambda_{o'})} = (\theta_{o'}^-)(x_{jo} + \Delta x_{jo}), \quad j \in \{DI\}$$
$$\sum_{i=1}^{n} \lambda_i^- x_{ji} + \frac{s_j^-}{(1-\lambda_{o'})} = (x_{jo}), \quad j \in \{NI\}$$
$$\sum_{i=1}^{n} \lambda_i^- y_{ki} - \frac{s_k^+}{(1-\lambda_{o'})} = (y_{ko} + \Delta y_{ko}), \quad k \in \{Do\}$$
$$\sum_{i=1}^{n} \lambda_i^- y_{ki} - \frac{s_k^+}{(1-\lambda_{o'})} = (y_{ko}), \quad k \in \{No\}$$
$$\sum_{i=1}^{n} \lambda_i^- + \frac{\lambda_{o'}}{(1-\lambda_{o'})} = 1$$
$$\lambda_{o'}, \lambda_i \geq 0, i = 1, \dots, n.$$

From $\sum_{i=1}^{n} \lambda_i + \frac{\lambda_{o'}}{(1-\lambda_{o'})} = 1$, we find that $\lambda_{o'} = \frac{(1-\sum_{i=1}^{n} \lambda_i^-)}{(2-\sum_{i=1}^{n} \lambda_i^-)}$.Let:

$\forall j = 1, \dots, m \quad s_j^- = \frac{s_j^-}{(1-\lambda_{o'})}$, $\forall k = 1, \dots . r \quad s_k^+ = \frac{s_k^+}{(1-\lambda_{o'})}$

By substituting that $\lambda_{o'} = \frac{(1-\sum_{i=1}^{n} \lambda_i^-)}{(2-\sum_{i=1}^{n} \lambda_i^-)}$ into $\theta_{o'}^- = \frac{(\theta_{o'} - \lambda_{o'})}{(1-\lambda_{o'})}$,we can find that the objective function of the $DBCC_{O'}$ model is to minimize $\theta_{o'} = \frac{(\theta_{o'}^- + 1 - \sum_{i=1}^{n} \lambda_i^-)}{(2-\sum_{i=1}^{n} \lambda_i^-)}$. thus, the $DBCC_{O'}$ model becomes:

$$\text{minimize } \left(\frac{(\theta_{o'}^- + 1 - \sum_{i=1}^{n} \lambda_i^-)}{(2-\sum_{i=1}^{n} \lambda_i^-)} - \varepsilon(\textstyle\sum_{j \in \{DI\}} s_j^- + \sum_{k \in \{DO\}} s_k^+)\right)$$
s.t. $$\sum_{i=1}^{n} \lambda_i^- x_{ji} + s_j^- = (\theta_{o'}^-)(x_{jo} + \Delta x_{jo}), \quad j \in \{DI\}$$
$$\sum_{i=1}^{n} \lambda_i^- x_{ji} + s_j^- = (x_{jo}), \quad j \in \{NI\}$$
$$\sum_{i=1}^{n} \lambda_i^- y_{ki} - s_k^+ = (y_{ko} + \Delta y_{ko}), \quad k \in \{Do\}$$
$$\sum_{i=1}^{n} \lambda_i^- y_{ki} - s_k^+ = (y_{ko}), \quad k \in \{No\}$$
$$\lambda_i^- \geq 0, i = 1, \dots, n. \quad (10)$$

Note that a fractional number is invariant under multiplication of both numerator and denominator by the same nonzero number. We set the denominator of the model (10) equal to 1, move it to a constraint, and minimize the numerator. This results in the following model.

$$\text{minimize } ((\theta_{o'}^- + 1 - \textstyle\sum_{i=1}^{n} \lambda_i^-) - \varepsilon(\sum_{j \in \{DI\}} s_j^- + \sum_{k \in \{DO\}} s_k^+))$$
s.t. $$\sum_{i=1}^{n} \lambda_i^- x_{ji} + s_j^- = (\theta_{o'}^-)(x_{jo} + \Delta x_{jo}), \quad j \in \{DI\}$$
$$\sum_{i=1}^{n} \lambda_i^- x_{ji} + s_j^- = (x_{jo}), \quad j \in \{NI\}$$
$$\sum_{i=1}^{n} \lambda_i^- y_{ki} - s_k^+ = (y_{ko} + \Delta y_{ko}), \quad k \in \{Do\}$$
$$\sum_{i=1}^{n} \lambda_i^- y_{ki} - s_k^+ = (y_{ko}), \quad k \in \{No\}$$

$$(2 - \sum_{i=1}^{n} \lambda_i^-) = 1$$
$$\lambda_i^- \geq 0, i = 1, \dots, n.$$

Since $\sum_{i=1}^{n} \lambda_i^-) = 1$, the above model becomes the model (11).

$$\begin{aligned} &\text{minimize } (\theta_{o'}^- - \varepsilon(\sum_{j\in\{DI\}} s_j^- + \sum_{k\in\{DO\}} s_k^+)) \\ &\text{s.t. } \sum_{i=1}^{n} \lambda_i^- x_{ji} + s_j^- = (\theta_{o'}^-)(x_{jo} + \Delta x_{jo}), \ j \in \{DI\} \\ &\sum_{i=1}^{n} \lambda_i^- x_{ji} + s_j^- = (x_{jo}), \qquad j \in \{NI\} \\ &\sum_{i=1}^{n} \lambda_i^- y_{ki} - s_k^+ = (y_{ko} + \Delta y_{ko}), \qquad k \in \{Do\} \\ &\sum_{i=1}^{n} \lambda_i^- y_{ki} - s_k^+ = (y_{ko}) \qquad , \qquad k \in \{No\} \\ &\sum_{i=1}^{n} \lambda_i^- = 1 \\ &\lambda_i^- \geq 0, i = 1, \dots, n. \end{aligned} \tag{11}$$

The optimal solution of the model (11) is also optimal for the model (10) since the above transformation is reversible. The models (10) and (11) therefore have the same optimal objective value. Note that the constraints in the model (11) are in the same form of the constraints in the MLDIBCC_o model. Using the minimum Δx_o obtained from solving the MLDIBCC_o model, all constraints in the model (11) are satisfied and the objective function is minimized at $\theta_{o'}^- = \theta_o^*$, otherwise Δx_o is not optimal for the MLDIBCC_o model.Therefore, the relative efficiency value of $DMU_{O'}$ with respect to the set of DMU_l(l = 1, . . . ,n,o') will remain equal to θ_o^* .

Case 2: θ_o^* =1

If $\lambda_{o'} \neq 1$in the $DBCC_{O'}$ model, we can prove that the optimal objective value of the $DBCC_{O'}$ model ($\theta_{o'}$) is equal to θ_o^* =1 by using the same way for the proof of case 1. And if $\lambda_{o'} = 1$, then $\lambda_i = 0$ for i = 1,2,. . .,n and $\theta_{o'} = \theta_o^* = 1$ .Now let us consider the inverse BCC ($DIBCC_o$) model. If $\lambda_{o'} = 1$, then, $\lambda_i = 0$ for i = 1,2,. . .,n , $\theta_{o'} = 1$ and $x_o + \Delta x_o = 0$. However, we assume at the beginning that $x_o + \Delta x_o$ must be a semi-positive vector. Therefore, the solution $\lambda_{o'} = 1$, $\lambda_i = 0$ for i = 1,2,. . .,n, $\theta_{o'} = 1$ and $x_o + \Delta x_o = 0$ is not an optimal solution for the inverse BCC model. Consequently, given the changes in output values of $DMU_o, \Delta y_o \neq 0$, the minimum Δx_o of $DMU_{O'}$, which does not make any change to the relative efficiency value of $DMU_{O'}$, can be obtained by solving the MLDIBCC_o model. For other DMUs, the $IBCC_l$model in (4) can be written in the vector-metric form as follows:

$$\begin{aligned} &\max u^T y_l - \sum_{j\in\{NI\}} x_{jo} v_j - u_o \\ &s.t. \ \sum_{j\in\{DI\}} v_j x_{jl} = 1 \\ &-v^T x_i + u^T y_i - u_o \leq 0 \qquad i = 1, \dots, n \\ &-v^T(x_{jo} + \Delta x_{jo}) - v^T(x_{jo}) + u^T(y_{ko} + \Delta y_{ko}) + u^T(y_{ko}) - u_o \leq 0 \\ &\quad \{j \in DI\} \qquad \{j \in NI\} \qquad \{k \in DO\} \qquad \{k \in NO\} \\ &u_o \ is\ free, U, V \geq 0 \end{aligned}$$

where $U^T = [u_1, u_2, \dots, u_r]$, $V^T = [v_1, v_2, \dots, v_m]$

$$x_i = \begin{bmatrix} x_{1i} \\ x_{2i} \\ \vdots \\ x_{mi} \end{bmatrix}, \ y_i = \begin{bmatrix} y_{1i} \\ y_{2i} \\ \vdots \\ y_{ri} \end{bmatrix}, x_o = \begin{bmatrix} x_{1o} \\ x_{2o} \\ \vdots \\ x_{mo} \end{bmatrix}, \Delta x_o = \begin{bmatrix} \Delta x_{1o} \\ \Delta x_{2o} \\ \vdots \\ \Delta x_{mo} \end{bmatrix}, \Delta x_i = \begin{cases} \Delta x_{ji} & j \in DI \\ 0 & j \in NI \end{cases}$$

From the constraints in the MLDIBCC_o model , $(x_o + \Delta x_o, y_o + \Delta y_o) \in p$ where p is a production possibility set of all DMU_i , $i = 1, \dots, n$ and $p = \{(x, y) | \ x \geq X\lambda, y \leq Y\lambda, e^T\lambda = 1, \lambda \geq 0\}$, $X = [x_{ji}]_{m*n}$, $Y = [y_{ki}]_{r*n}$, $\lambda = (\lambda_i)_{n*1}, \lambda \in R^n$.if ($x_o + \Delta x_o, y_o + \Delta y_o) \in p$, then we have:

$$\begin{aligned} &-v^T(x_{jo} + \Delta x_{jo}) - v^T(x_{jo}) + u^T(y_{ko} + \Delta y_{ko}) + u^T(y_{ko}) - u_o \leq \\ &\quad \{j \in DI\} \qquad \{j \in NI\} \qquad \{k \in DO\} \qquad \{k \in NO\} \\ &\leq -V^T(X\lambda) + U^T(Y\lambda) - u_o \leq -\sum_{i=1}^{n} V^T x_i \lambda_i + \sum_{i=1}^{n} U^T y_i \lambda_i - u_o \\ &\leq \sum_{i=1}^{n}(-V^T x_i + U^T y_i)\lambda_i - u_o \end{aligned}$$

From the IBCC_l model $-V^T x_i + U^T y_i \leq u_o$ for i = 1, . . . ,n. Therefore $-V^T(x_o + \Delta x_o) + U^T(y_o + \Delta y_o) - u_o \leq \sum_{i=1}^{n} u_o\lambda_i - u_o \leq 0$ .This shows that $-V^T(x_o + \Delta x_o) + U^T(y_o + \Delta y_o) - u_o \leq 0$ in the IBCC_l model is redundant and can be dropped out from the model without changing the solution set and the optimal objective value. In other words, the IBCC_lmodel is equivalent to the BCC model for DMU_l before DMU_o changes its output values. This implies that the relative efficiency values of allDMU_l, (l = 1,. . . ,n) remains unchanged.

Lemma 1. Assume that the relative efficiency value of DMUo with respect to other DMUs in a group of comparable DMUs (i = 1, . . .,n) is θ_o^* . Also, assume the output values of DMU_o are changed from y_o to $y_o + \Delta y_o \geq 0$. $(\Delta y_o \neq 0)$ .There exists at least an optimal solution to the $MLDIBCC_o$ model, if and only if $y_o + \Delta y_o \in P_{out}$, where $P_{out} = \{y | y \leq Y\lambda, e^T\lambda = 1, \lambda \geq 0\}$, $Y = [y_{ki}]_{r\times n}, \lambda = (\lambda_i)_{n\times 1}, \lambda \in R^n$.

Proof. If $y_o + \Delta y_o \in P_{out}$, then the constraints $\sum_{i=1}^n \lambda_i y_{ki} \geq y_{ko} + \Delta y_{ko}$, $k \in \{DO\}$, $\sum_{i=1}^n \lambda_i y_{ki} \geq y_{ko}$, $k \in \{NO\}$, $\sum_{i=1}^n \lambda_i = 1$, $\lambda_i \geq 0$ $(i = 1, \dots, n)$ in the $MLDIBCC_o$ model are satisfied. The constraints $\sum_{i=1}^n \lambda_i x_{ji} \leq \theta_o^*(x_{jo})$, $k \in \{NI\}$, $\sum_{i=1}^n \lambda_i x_{ji} \leq \theta_o^*(x_{jo} + \Delta x_{jo})$, $k \in \{DI\}$ can be satisfied by finding the appropriate value of Δx_{jo}. Also, from the constraints $\sum_{i=1}^n \lambda_i x_{ji} \leq \theta_o^*(x_{jo})$, $j \in \{NI\}$, $\sum_{i=1}^n \lambda_i x_{ji} \leq \theta_o^*(x_{jo} + \Delta x_{jo})$, $j \in \{DI\}$, we know that $x_{jo} + \Delta x_{jo} \geq 0$, $j = 1, \dots, m$ .The objective of the $MLDIBCC_o$ model is to minimize Δx_o, therefore,there exists at least an optimal solution to the $MLDIBCC_o$ model. This proves that there exits at least an optimal solution to the $MLDIBCC_o$ model if $y_o + \Delta y_o \in P_{out}$.

If there exists at least an optimal solution to the $MLDIBCC_o$ model, from the set of constraints $\sum_{i=1}^n \lambda_i y_{ki} - s_k^+ = y_{ko}$, $k \in \{NO\}$, $\sum_{i=1}^n \lambda_i y_{ki} - s_k^+ = y_{ko} + \Delta y_{ko}$, $k \in \{DO\}$ in the $MLDIBCC_o$ model, then $y_o + \Delta y_o \in P_{out}$.

From Lemma 1, we can check whether $y_o + \Delta y_o$ is in P_{out} or not by determining a set of non-dominated DMUs based on the output comparison. Then if all elements of $y_o + \Delta y_o$ is less than or equal to all elements of the outputs of at least one DMU in the non-dominated set, then $y_o + \Delta y_o$ is in P_{out}.

Theorem 2. $\Delta x_o = (\Delta x_1, \dots, \Delta x_{mo})$ obtained by solving the $LDIBCC_o$ model is a Pareto solution for the $MLDIBCC_o$ model.

$$
\begin{aligned}
(\mathbf{LDIBCC_o})\quad & \min\ w^T \Delta x_o \\
\text{s.t.}\quad & \textstyle\sum_{i=1}^n \lambda_i x_{ji} \leq \theta_o^*(x_{jo} + \Delta x_{jo}), && j \in \{DI\} \\
& \textstyle\sum_{i=1}^n \lambda_i x_{ji} \leq \theta_o^*(x_{jo}), && j \in \{NI\} \\
& \textstyle\sum_{i=1}^n \lambda_i y_{ki} \geq y_{ko} + \Delta y_{ko}, && k \in \{DO\} \\
& \textstyle\sum_{i=1}^n \lambda_i y_{ki} \geq y_{ko}, && k \in \{NO\} \\
& \textstyle\sum_{i=1}^n \lambda_i = 1 \\
& \lambda_i \geq 0, && i = 1, \dots, n \\
& (\Delta x_{jo}) = 0, && j \in \{NI\}\}
\end{aligned}
$$

Where $w^T \in R^m$.

Proof. Assume that $\lambda^* = (\lambda_1^*, \dots, \lambda_n^*)$, $\Delta x_o^* \in R^m$ are the optimal solution from solving the $LDIBCC_o$ model but they were not Pareto solution to the $MLDIBCC_o$ model. There should be a possible $\overline{\Delta x}_o \in R^m$, $\bar{\lambda} = (\bar{\lambda}_1, \dots, \bar{\lambda}_n)$ from the $MLDIBCC_o$ model where $\overline{\Delta x}_o \leq_p \Delta x_o^*$ and thus $w^T\overline{\Delta x}_o < w^T \Delta x_o^*, w^T > 0$ .Note that $\Delta x_o \leq_p \Delta x_o^*$ represents a set of inequalities $\overline{\Delta x}_{jo} \leq \Delta x_{jo}^*$, $j = 1, \dots, m$ with at least one strict inequality, $\overline{\Delta x}_{jo} \leq \Delta x_{jo}^*$ Since the $MLDIBCC_o$ model and the $LDIBCC_o$ model have the same constraint sets, $\overline{\Delta x}_o \in R^m$ and $\bar{\lambda}$ are also the solution to the $LDIBCC_o$ model. This leads to a contradiction; therefore, $\Delta x_o^* \in R^m$ and λ^* from the $LDIBCC_o$ model would also be a Pareto solution to the $MLDIBCC_o$ model. ❑

From Theorem 2, if we find any positive vector, $w^T \epsilon R$, we would be able to find a Pareto solution for the $MLDIBCC_o$ model from solving the $LDIBCC_o$ model, which is a linear programming model. Consequently, the input and output vector of $(x_o + \Delta x_o, y_o + \Delta y_o)$ ، $DMU_ó$ obtained from the $LDIBCC_o$ model will be a Paretoefficient solution to the inverse BCC model.

5. Numerical example

There are 15 decision making units in this study that use 2 inputs to produce 2 outputs. The second input and output are non-discretionary .Input and output values of DMUs are given in Table 1 for the efficiency analysis. After solving the BCC models ($PBCC_o$ or $DBCC_o$) for all DMUs with the data from Table 1, the relative efficiency values θ_i^* of all DMUs are given in Table 2.From Table 2, there are only 5 technically efficient DMus, which are DMU_4 , DMU_7 , DMU_9, DMU_{10}, DMU_{15}. All other DMUs are technically inefficient. If we compare the performance of all DMUs based on outputs only, the set of non-dominated DMUs includes DMU_{15} .Let us consider an inefficient DMU_1, the optimal objective value is $\theta_1^* = 0/42$ which is less than 1. Suppose that the output vector of DMU_1 is changed from $(10,1)^T$ to $(11/7,1)^T$ and let $w^T = (1, 1)$ for input weights. Solving the $LDIBCC_o$ model we can observe that the new output values are in P_{out} the first output of DMU_1 is less than or equal to the first output of

DMU_{15} .By solving the $LDIBCC_o$ model forDMU_1, we find that the optimal solution is Δx^*_{11}=13/37 , Δx^*_{21}=0 and the optimal objective value is equal to 13/37 . Therefore, the new input vector is $(18/37,100)^T$. Using the new input and output vectors for DMU_1, the relative efficiency values of all DMUs still remain the same .

Table 1:Input and output values of 15 DMUs.

DMU_i	Input1	Input2	Output1	Output2
DMU_1	5	100	10	1
DMU_2	7	95	12	10
DMU_3	3	120	11	20
DMU_4	2	70	9	3
DMU_5	8	30	11	17
DMU_6	2/2	100	10	2
DMU_7	3	20	12	15
DMU_8	3/5	55	12/3	5
DMU_9	2/5	67	13/7	67.1
DMU_{10}	6/6	15	14	67
DMU_{11}	7/5	66	9/8	43
DMU_{12}	9	33	9/96	6
DMU_{13}	5/5	76	14/4	53
DMU_{14}	5/8	90	7/8	40
DMU_{15}	4	100	15	50

Table 2.The relative efficient values θ^*_i of DMU_i , i = 1, . . ., 15.

DMU_i	θ^*_i	DMU_i	θ^*_i	DMU_i	θ^*_i
DMU_1	.42	DMU_6	.96	DMU_{11}	.30
DMU_2	.33	DMU_7	1	DMU_{12}	.30
DMU_3	.74	DMU_8	.72	DMU_{13}	.70
DMU_4	1	DMU_9	1	DMU_{14}	.38
DMU_5	.35	DMU_{10}	1	DMU_{15}	1

6.Conclusions

In this paper, we extended the proposed models by S. Lertworasirikul et al.(2011) in the present of non-discretionary and discretionary data. The traditional inverse DEA model is used to determine the best possible values of inputs (outputs) for given values of outputs (inputs) of a considered DMU such that relative efficiency value of a considered DMU with respect to other DMUs remain unchanged. We study the inverse BCC model for the resource allocation problem. We propose a linear programming model, which gives a Pareto-efficient solution to the inverse BCC problem.

REFERENCES

[1]Alinezhad, A., Makui, A., & Kiani Mavi, R. (2007). An inverse DEA model for inputs/ outputs estimation with respect to decision maker ′ s preferences: The case of Refah bank of IRAN. *Journal of Mathematical Science,* 1(1–2), 61–70.

[2]Arnade, C.A. (1994). Using data envelopment analysis to measure international agricultural efficiency and productivity. *United Stated Department of Agriculture,Economic Research Service. Technical Bulletin*, 1831, 1–30.

[3]Banker, R. D., Charnes, A., & Cooper, W. W. (1984). Some models for estimating technical and scale efficiencies in data envelopment analysis. *Management Science*, 30(9), 1078–1092.

[4]Barros, C. P., & Peypoch, N. (2009). An evaluation of European airlines ′ operational performance. *International Journal of Production Economics*, 122(2), 525–533.

[5]Chandra, P., Cooper, W. W., Li, S., & Rahman, A. (1998). Using DEA to evaluate 29 Canadian textile companies – Considering returns to scale. *International Journal of Production Economics,* 54(2), 129–141.

[6]Charnes, A., Cooper, W. W., & Rhodes, E. (1978). Measuring the efficiency of decision making units. *European Journal of Operational Research*, 2(6), 429–444.

[7]Charnes, A., Cooper, W. W., Lewin, A. Y., & Seiford, L. M. (1994). Data envelopment analysis: Theory, methodology, and application. *London: Kluwer Academic Publishers.*

[8]Chen, T-Y. (1997). A measurement of the resource utilization efficiency of university libraries. *International Journal of Production Economics,* 53(1), 71–80.

[9]Cooper, W. W., Seiford, L. M., & Tone, K. (2000). Data envelopment analysis: A comprehensive text with models, applications, reference and DEA-solver software.*London: Kluwer Academic Publishers*.

[10]Hadi-Vencheh, A., & Foroughi, A. A. (2006). A generalized DEA model for inputs/outputs estimation. *Mathematical and Computer Modelling,* 43(5–6), 447–457.

[11]Jahanshahloo, G. R., Lotfi, F. H., Shoja, N., Tohidi, G., & Razavyan, S. (2004a). The outputs estimation of a DMU according to improvement of its efficiency. *Applied Mathematics and Computation,* 147(2), 409–413.

[12] Jahanshahloo, G.R.,Lotfi, F.H., Shoja,N.,Tohidi, G., & Razavyan, S. (2004b). Input estimation and identification of extra inputs in inverse DEA models. *Applied Mathematics and Computation,* 156(2), 427–437.

[13] Jahanshahloo, G. R., Lotfi, F. H., Shoja, N., Tohidi, G., & Razavyan, S. (2005). Sensitivity of efficiency classifications in the inverse DEA models. *Applied Mathematics and Computation,* 169(2), 905–916.

[14] Lertworasirikul, S., Fang, S.-C., Nuttle, H. L. W., & Joines, J. A. (2003). Fuzzy BCC model for data envelopment analysis. *Fuzzy Optimization and Decision Making*,2(4), 337–358.

[15]Lertworasirikul,S., Charnsethikul,P., Fang,S.C. (2011). Inverse data envelopment analysis model to preserve relative efficiency values:The case of variable returns to scale .*Computera & Industrial Engineering* ,61,1017- 1023-

[16]Thanassoulis, E. (2001). Introduction to the theory and application of data envelopment analysis: A foundation text with integrated software. *Massachusetts: Kluwer Academic Publishers.*

[17]Wei, Q., Zhang, J., & Zhang, X. (2000). An inverse DEA model for inputs/outputs estimate. *European Journal of Operational Research*,121(1),151–163.

[18]Yan, H., Wei, Q., & Hao, G. (2002). DEA models for resource reallocation and production input/output estimation. *European Journal of Operational Research*,136(1), 19–31.