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Homotopy Perturbation Method with Reproducing Kernel Method for Third Order Nonlinear Boundary Value Problems

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ABSTRACT

This method is based on a combination of the homotopy perturbation method (HPM) and the reproducing kernel method (RKM). The main advantages of this method is that it can overcome the restriction of the form of nonlinearity term in differential equations and improve the iterative speed of homotopy perturbation method. The solution obtained using the method takes the form of a convergent series with easily computable components. The third order nonlinear boundary value problem is solved using the combination of RKM and HPM. It can avoid additional computation work of HPM in determining the unknown parameters.

KEYWORDS: Nonlinear Boundary Value Problem; Gram-Schmidt Orthogonalization Process; Reproducing Kernel Method (RKM); Homotopy Perturbation Method (HPM).

1 INTRODUCTION

Boundary value problems manifest themselves in many branches of science. For example engineering, technology, control, optimization theory, draining and coating flows and various dynamic systems. Two point boundary-value problems (BVPs) play an important role in mathematical physics and engineering. As the models considered in applied and engineering sciences are non-linear in nature, so the analytical solutions of very few of them are available. Many authors have solved third order boundary value problems with different boundary conditions. Accurate and fast numerical solution of boundary value problems is necessary in many important scientific and engineering applications, e.g. boundary layer theory, the study of stellar interiors, control and optimization theory and flow networks in biology.

Third-order boundary-value problems for differential equation play a very important role in a variety of different areas of applied mathematics and physics. Recently, third-order boundary-value problems have been many scholars' research object. For example, heat conduction, chemical engineering, underground water flow, thermoelasticity, and plasma physics can produce boundary-value.

Recent works on the numerical solution of third order boundary value problems have used spline techniques [4, 5, 14]. Khan and Aziz presented a fourth-order method based on quintic splines for the solution of third order linear and non-linear boundary-value problems [13]. Islam and Tirmizi [18] developed a second-order method based on quartic non polynomial spline to find continuous approximation of a two point boundary-value problem involving a third-order differential equation. Siddiqi and Ghazala [16] presented nonpolynomial spline method for the numerical solution of the fifth-order linear special case boundary value problems. Siddiqi and Ghazala [17] used polynomial sextic spline method for the solution of linear fifth-order boundary value problems and the method is observed to be second-order convergent. J. He developed homotopy perturbation method (HPM) by merging the standard homotopy and perturbation [7, 8, 9, 10]. The homotopy perturbation method (HPM) has been applied to a wide class of diversified physical problems [6, 12, 15]. Reproducing kernel Hilbert space is a useful framework for constructing the approximate solutions of boundary value problems [1, 2, 3].

In this paper, the homotopy perturbation method and reproducing kernel method are combined to solve the third order nonlinear boundary value problem. HPM is used to reduce a nonlinear problem to a sequence of linear problems and RKM is used to solve powerfully linear BVPs. Third order nonlinear BVPs are solved using the combination of these two methods.

Consider the following third order nonlinear two-point boundary value problem (BVP), as

$$\left.\begin{aligned} &u^{(3)}(x)+\sum_{i=0}^{2}a_i(x)u^{(i)}(x)+N(u(x))=f(x),\ \ 0\le x\le 1,\\ &u(0)=A_0, u^{(1)}(0)=A_1, u(1)=A_2, \end{aligned}\right\} \qquad (1.1)$$

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where A_i ,$i = 0, 1, 2$ are finite real constants and the functions $f(x)$, $a_i(x)$; $i = 0, 1, 2$ are continuous on $[0, 1]$ and $N(u(x))$ is nonlinear function of u.

The operator $L : W_2^4[0,1] \to W_2^1[0,1]$ is defined by $L : u^{(3)}(x) + \sum_{i=0}^{2} a_i(x) u^{(i)}(x)$ and it can easily be seen that L is the bounded linear operator.

The system (1.1) can be transformed into the following form after homogenization of the boundary conditions:

$$\left.\begin{array}{l} Lu(x) + N(u(x)) = f(x), \quad 0 \le x \le 1, \\ u(0) = 0, u^{(1)}(0) = 0, u(1) = 0, \end{array}\right\} \qquad (1.2)$$

Thus, the solution of the system (1.2) provides the solution of the system (1.1) and it is noted that $f(x,u(x))=f(x)-N(u(x))$.

The rest of this paper is organized as under:

In Section 2, basic ideas of the homotopy perturbation technique are presented. The reproducing kernel spaces and the reproducing kernel function are given in Section 3. The approximate solution of problems (1.2) with boundary conditions is presented in Section 4. The HPM and RKHSM are applied to (1.2) in Section 5. The numerical examples are presented to demonstrate the accuracy of the method in Section 6.

2 BASIC IDEAS OF THE HOMOTOPY PERTURBATION TECHNIQUE [7]

Following [7], consider the following non-linear differential equation:

$$L(u) + N(u) = f(r), \quad r \in \Omega \qquad (2.1)$$

with the boundary condition

$$B(u, \frac{\partial u}{\partial n}) = 0, \quad r \in \Gamma \qquad (2.2)$$

where L is linear and N is non-linear operators, B is a boundary operator, $f(r)$ is a known analytic function, Γ is the boundary of the domain of Ω.

By He's homotopy perturbation technique, define a homotopy $v(r,p) : \Omega \times [0,1] \to R$ which satisfies

$$H(v, p) = (1-p)[L(v) - L(u_0)] + p[L(v) + N(v) - f(r)] = 0 \qquad (2.3)$$

or

$$H(v, p) = L(v) - L(u_0) + pL(u_0) + p[N(v) - f(r)] = 0 \qquad (2.4)$$

where $r \in \Omega, p \in [0,1]$ is an embedding parameter and u_0 is an initial approximation of Eq. (2.1) which satisfies the boundary conditions. Clearly

$$H(v,0) = L(v) - L(u_0) = 0 \qquad (2.5)$$

$$H(v,1) = L(v) + N(v) - f(r) = 0 \qquad (2.6)$$

As p moves from 0 to 1, $v(r, p)$ moves from $u_0(r)$ to $u(r)$. This is called a deformation and $L(v)$-$L(u_0)$, $L(v) + N(v) - f(r)$ are said to be homotopic in topology. According to the homotopy perturbation method, firstly, the embedding parameter p can be used as a small parameter, and assume that the solution of Eq. (2.3) and Eq. (2.4) can be expressed as a power series in p, that is,

$$v = v_0 + pv_1 + p^2 v_2 + \ldots \qquad (2.7)$$

The approximate solution of Eq. (2.1) therefore, can be expressed as

$$u = \lim_{p \to 1} v = v_0 + v_1 + v_2 + \ldots \tag{2.8}$$

The combination of perturbation method and homotopy method is known as the homotopy perturbation method, which has eliminated the limitations of traditional perturbation methods. Furthermore, this technique has the full advantage of traditional perturbation techniques. The series in Eq. (2.8) is convergent in most cases.

3. REPRODUCING KERNEL SPACES

The reproducing kernel space $W_2^4[0,1]$ is defined by $W_2^4[0,1] = \{u(x) \mid u^{(i)}(x),\ i = 0,1,2,3$ are absolutely continuous real valued functions in [0,1], with $u(0) = u^{(1)} = u(1) = 0,, u^{(4)}(x) \in L^2[0,1]\}$.

The inner product and norm in $W_2^4[0,1]$ are given by

$$\langle u(x), v(x) \rangle = \int_0^1 (u^{(3)}(x) v^{(3)}(x) + u^{(4)}(x) v^{(4)}(x))\, dx, \tag{3.1}$$

$$\|u\| = \sqrt{\langle u(x), u(x) \rangle}, \quad u(x), v(x) \in W_2^4[0,1]. \tag{3.2}$$

Similar to the [2], it can easily be proved that $W_2^4[0,1]$ is reproducing kernel Hilbert space with reproducing kernel *k(x, y).*

Theorem 3.1

The space $W_2^4[0,1]$ is a reproducing kernel Hilbert space. That is $\forall\, u(y) \in W_2^4[0,1]$ and each fixed $x, y \in [0,1]$ there exists $k(x,y) \in W_2^4[0,1]$

s.t. $\langle u(y), k(x,y) \rangle = u(x)$ and $k(x,y)$ is called the reproducing kernel function of space $W_2^4[0,1]$.

The reproducing kernel function *K(x, y)* in [*0, 1*] is given by

$$k(x,y) = R_x(y) = \begin{cases} h(x,y) = \sum_{i=1}^{6} c_i y^{i-1} + c_7 e^y + c_8 e^{-y}, & y \le x \\ h(y,x) = \sum_{i=1}^{6} d_i y^{i-1} + d_7 e^y + d_8 e^{-y}, & y > x \end{cases}$$

Similar to [2], the coefficients c_i *(i=1, 2,…, 8)* and d_i *(i=1, 2,…, 8)* can be calculated.

4 THE EXACT AND APPROXIMATE SOLUTIONS

Using the adjoint operator L^* of L and choose a countable dense subset $T = \{x_1, x_2, x_3, \ldots\} \subset [0,1]$ and let

$$\varphi_i(x) = Q_{x_i}(y), \quad i \in N \tag{4.1}$$

Then $\psi_i(x) = L^* \varphi_i(x)$, where $\psi_i(x) \in W_2^4[0,1]$.

Lemma 4.1 $\psi_i(x) = L_y R_x(y)\,|_{y=x_i}$ and $\{\psi_i(x)\}_{i=1}^{\infty}$ is a complete system of $W_2^4[0,1]$.

Proof: Using the reproducing property, it can be written as

$$\psi_i(x) = \langle \psi_i(y), R_x(y) \rangle = \langle \varphi_i(y), LR_x(y) \rangle = L_y R_x(y)\,|_{y=x_i}\ .$$

For $u(x) \in W_2^4[0,1]$, let $\left\langle u(x), \psi_i(x) \right\rangle = 0, i = 1,2,...$ which implies

$$\left\langle u(x), (L^* \varphi_i)(x) \right\rangle = \left\langle (Lu)(x), Q_{x_i}(x) \right\rangle = (Lu)(x_i) = 0. \tag{4.2}$$

Since $\{x_i\}_{i=1}^{\infty}$ is dense in *[0,1]*, *(Lu)(x) = 0*, which implies $u \equiv 0$ from the existence of L^{-1}.
To orthonormalize the sequence $\{\psi_i(x)\}_{i=1}^{\infty}$ in the reproducing kernel space $W_2^4[0,1]$, Gram Schmidt process can be used, as

$$\overline{\psi}_i(x) = \sum_{k=1}^{i} \beta_{ik} \psi_k(x), \qquad i = 1,2,3,... \tag{4.3}$$

Theorem 4.1 On the other hand, if u(x) is the exact solution of the systems (1.2) then

$$u(x) = \sum_{i=1}^{\infty} \sum_{k=1}^{i} \beta_{ik} f(x_k, u(x_k)) \overline{\psi}_i(x).$$

Proof:

Since $u(x) \in W_2^4[0,1]$ and can be expanded in the form of Fourier series about normal orthogonal system so

$$u(x) = \sum_{i=1}^{\infty} \langle u(x), \overline{\psi}_i(x) \rangle \overline{\psi}_i(x) \tag{4.4}$$

Moreover, the space $W_2^4[0,1]$ is Hilbert space so the series $\sum_{i=1}^{\infty} \langle u(x), \overline{\psi}_i(x) \rangle \overline{\psi}_i(x)$ is convergent in the norm of $\|.\|_{W_2^4}$. From Eqns. (4.3) and (4.4), it can be written as

$$\begin{aligned} u(x) = \sum_{i=1}^{\infty} \langle u(x), \overline{\psi}_i(x) \rangle \overline{\psi}_i(x) &= \sum_{i=1}^{\infty} \left\langle u(x), \sum_{k=1}^{i} \beta_{ik} \psi_k(x) \right\rangle \overline{\psi}_i(x) \\ &= \sum_{i=1}^{\infty} \sum_{k=1}^{i} \beta_{ik} \langle u(x), \psi_k(x) \rangle \overline{\psi}_i(x) \\ &= \sum_{i=1}^{\infty} \sum_{k=1}^{i} \beta_{ik} \langle Lu(x), \varphi_k(x) \rangle \overline{\psi}_i(x). \end{aligned}$$

If $u(x)$ is the exact solution of Eq. (1.2) and $Lu = f(x, u(x))$, then

$$\begin{aligned} u(x) &= \sum_{i=1}^{\infty} \sum_{k=1}^{i} \beta_{ik} \langle f(x, u(x)), \varphi_k(x) \rangle \overline{\psi}_i(x) \\ &= \sum_{i=1}^{\infty} \sum_{k=1}^{i} \beta_{ik} f(x_k, u(x_k)) \overline{\psi}_i(x) \end{aligned}$$

The approximate solution of $u(x)$ is given by

$$u_n(x) = \sum_{i=1}^{n} \sum_{k=1}^{i} \beta_{ik} f(x_k, u(x_k)) \overline{\psi}_i(x) \tag{4.5}$$

5 COMBINATION OF HPM-RKM

A homotopy for Eq. (1.2) can be written as

$$H(u, p) = u^{(3)}(x) + a_2(x) u^{(2)}(x) + a_1(x) u^{(1)}(x) + a_0(x) u(x) - f(x) + pN(u(x)) = 0 \tag{5.1}$$

To expand the solution homotopy parameter p is used as:

$$u = u_0 + pu_1 + p^2 u_2 + p^3 u_3 + ... \tag{5.2}$$

The approximate solution of Equation (1.2) can be obtained by setting $p = 1$

$$u = u_0 + u_1 + u_2 + u_3 + ... \tag{5.3}$$

Substituting Eq. (5.2) into Eq. (5.1), and equating coefficients of like powers of p yields the following equations:

$$p^0 : u_0^{(3)}(x) + a_2(x)u_0^{(2)}(x) + a_1(x)u_0^{(1)}(x) + a_0(x)u_0(x) = f(x),$$

with $u_0(0) = u_0^{(1)}(0) = u_0(1) = 0$

$$p^1 : u_1^{(3)}(x) + a_2(x)u_1^{(2)}(x) + a_1(x)u_1^{(1)}(x) + a_0(x)u_1(x) = -N(u)\big|_{p=0},$$

with $u_1(0) = u_1^{(1)}(0) = u_1(1) = 0$

$$p^2 : u_2^{(3)}(x) + a_2(x)u_2^{(2)}(x) + a_1(x)u_2^{(1)}(x) + a_0(x)u_2(x) = -\frac{d}{dp}N(u)\bigg|_{p=0},$$

with $u_2(0) = u_2^{(1)}(0) = u_2(1) = 0$

$$\vdots$$

$$p^m : u_m^{(3)}(x) + a_2(x)u_m^{(2)}(x) + a_1(x)u_m^{(1)}(x) + a_0(x)u_m(x) = -\frac{d^{m-1}}{(m-1)!dp^{m-1}}N(u)\bigg|_{p=0},$$

with $u_m(0) = u_m^{(1)}(0) = u_m(1) = 0$.

RKM is used to obtain $u_0, u_1, u_2, \ldots$, as

$$u_0(x) = \sum_{i=1}^{n}\sum_{k=1}^{i}\beta_{ik} f_0(x_k)\overline{\psi}_i(x)$$

$$u_1(x) = \sum_{i=1}^{n}\sum_{k=1}^{i}\beta_{ik} f_1(x_k)\overline{\psi}_i(x)$$

$$\vdots$$

$$u_m(x) = \sum_{i=1}^{n}\sum_{k=1}^{i}\beta_{ik} f_m(x_k)\overline{\psi}_i(x)$$

where $f_0(x) = 0$ and for $m \geq 1,\ f_m(x) = -\dfrac{d^{m-1}}{(m-1)!dp^{m-1}}N(u)\bigg|_{p=0}$.

Therefore, the approximate solution of Eq. (1.2) and the m-term approximation to this solution are obtained as

$$U = \sum_{k=1}^{\infty} u_k$$

$$U_m = \sum_{k=1}^{m} u_k.$$

The approximate solution $U_{n,m}(x)$ can be obtained by the n-term intercept of the $u_k(x)$, $k = 0, 1, 2, \ldots$, given by

$$U_{n,m} = \sum_{k=1}^{m}\sum_{i=1}^{n} A_{ik}\overline{\psi}_i(x), \qquad (5.4)$$

where

$$A_{ik} = \sum_{j=1}^{i}\beta_{ij} f_k(t_j).$$

To illustrate the applicability and effectiveness of the method, two numerical examples are constructed. All the numerical computations are performed using Mathematica 5.2.

6 NUMERICAL EXAMPLES

Example 6.1 Consider the following third order nonlinear (BVP) [13, 18]

$$\left.\begin{aligned} &u^{(3)}(x) + 2e^{-3u(x)} = 4(1+x)^{-3}, \quad 0 \leq x \leq 1, \\ &u(0) = 0,\ u^{(1)}(0) = 1,\ u(1) = \ln 2. \end{aligned}\right\} \qquad (6.1)$$

The exact solution of the problem (6.1) is $u(x) = ln(1 + x)$.
The comparison of the errors in absolute values between the method developed in this paper and those of [13, 18] are shown in Tables 1, 2 and Figures 1, 2, which show that the present method is better.
Example 6.2 Consider the following third order nonlinear boundary value problem

$$\left.\begin{aligned} & u^{(3)}(x) + (u(x))^2 = e^x(3 + x(5 + x + e^x(x-1)^2 x)), \quad 0 \le x \le 1, \\ & u(0) = 0, u^{(1)}(0) = -1, u(1) = 0. \end{aligned}\right\} \qquad (6.2)$$

The exact solution of the problem (6.2) is $u(x) = x(x\text{-}1)\, e^x$.
The absolute errors between exact and approximate solution is shown in Table 3 and Figures 3, 4 which show that the present method is very effective tool.
CONCLUSION: The combination of HPM and RKM was employed for solving nonlinear third order boundary value problems and obtain approximate solutions with a high degree of accuracy. With this method, an iterative sequence is obtained which converges to the exact solution uniformly. Numerical results show that the proposed method gives very encouraging results. It is worthy to note that the new method can be used as a very accurate algorithm for solving nonlinear third order boundary value problems. It can avoid additional computation work of HPM in determining the unknown parameters.

RESULTS AND DISCUSSION

The present method is more accurate and reliable than the methods developed in [13, 18] because present method produces good globally smooth approximate solutions as shown by Tables 1-3 and Figs. 1-4. Moreover, the proposed method has an advantage that it is possible to pick any point in the interval of integration and the approximate solutions can be obtained using the same previous partition.
In the present paper, $x_i = \frac{i-1}{n-1}, i = 1,2,...n$ are taken when RKM is used. It is well known that the HPM [7–10] is efficient method to solve linear and nonlinear ODEs. The method provides the solution in a rapid convergent series with computable terms. However, when HPM is used to solve boundary value problems, it is necessary to determine some unknown parameters. The purpose of this paper is to introduce a new reliable method which can avoid, determining some unknown parameters. This method is the combination of the HPM and the RKM. In this paper HPM is used to reduce a nonlinear problem to a sequence of linear problems and RKM is used to solve powerfully linear BVPs.

Table 1: Comparison of the numerical results for problem 6.1

x	Present method $\lvert U_{30,3} - u \rvert$	Present method $\lvert U_{50,3} - u \rvert$	Khan and Aziz [12]
0.	0	0	0
0.1	1.54307 E -6	3.1093 E-7	0.0000056
0.2	1.09794 E -6	1.60517E -7	0.0000095
0.3	4.44358 E -7	1.33811E -7	0.0000032
0.4	2.26811 E -6	3.7426E -7	0.0000175
0.5	3.83418 E -6	4.78398E -7	0.0000292
0.6	4.80906 E -6	4.46364E -7	0.0000288
0.7	5.00156 E -6	3.27257E -7	0.0000132
0.8	4.30592 E -6	1.84741E -7	0.0000051
0.9	2.65649 E -6	6.86286E -8	0
1.0	0	0	0

Table 2: Max. absolute error of our method and others methods for problem 6.1

h	Islam and Tirmizi [17]
1/8	0.534 E -4
1/16	0.971 E -5
1/32	0.199E -5
1/64	0.374 E -6
1/128	0.820 E -7
1/256	0.200 E -7
Our method Max. $\lvert U_{30,3} - u \rvert$	Our method Max. $\lvert U_{56,3} - u \rvert$
1.96274 E-6	2.15351 E-7

Table 3: Absolute error between exact and approximate solution for Example 6.2

x	Present method $\lvert U_{50,3} - u \rvert$	Present method $\lvert U_{70,3} - u \rvert$
0.1	1.80855 E-6	7.85356 E-7
0.2	4.93592 E-6	2.2473 E -6
0.3	8.83914 E-6	4.12348 E -6
0.4	1.29612 E-5	6.13294 E -6
0.5	1.65243E-5	7.95885 E -6
0.6	1.90576 E-5	9.2441 E -6
0.7	1.9655 E-5	9.58481 E -6
0.8	1.73997 E-5	8.52157 E -6
0.9	1.12462 E-5	5.52827 E -6

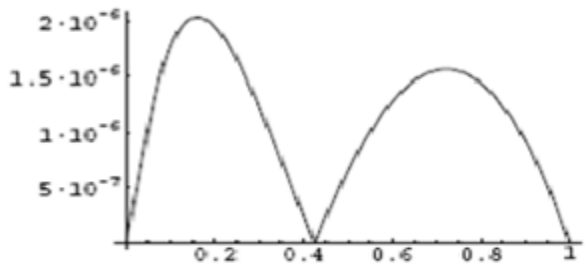

Fig. 1:Absolute Error between Exact Solution u and Approximate Solution $u_{30,3}$ ($|u - u_{30,3}|$)

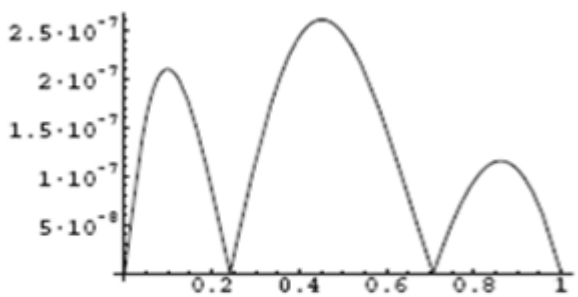

Fig. 2:Absolute Error between Exact Solution u and Approximate Solution $u_{50,3}$ ($|u - u_{50,3}|$)

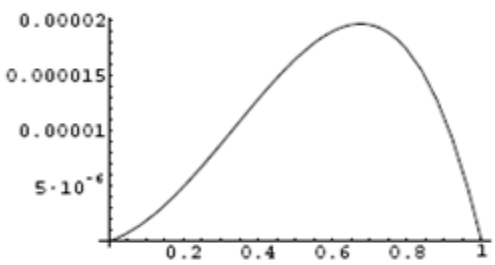

Fig. 3:Absolute Error between Exact Solution u and Approximate Solution $u_{30,3}$ ($|u - u_{50,3}|$)

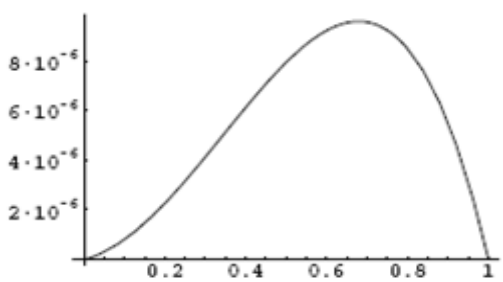

Fig. 4:Absolute Error between Exact Solution u and Approximate Solution $u_{50,3}$ ($|u - u_{70,3}|$)

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