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Exact Solutions of Steady Thin Film Flows of Modified Second Grade Fluid on a Vertical Belt

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ABSTRACT

This paper explores the problem of thin film for two types of flows i.e., (i) lifting and (ii) drainage of a steady, incompressible, non-isothermal modified second grade fluid. Exact solutions are obtained for the modeled highly non-linear ordinary differential equations subject to suitable boundary conditions. The Newtonian solutions are retrieved for the flow behavior index m=0 [1]. Explicit expressions for velocity profile, temperature distribution, average velocity, volume flow rate and shear stress on the moving and stationary belt are obtained in both cases. It is observed from this study that the flow of thin films is strongly depending on m (flow behavior index of modified second grade fluid), S_t (Stokes number) and B_r (Brinkman number).Also the effects of various emerging parameters on the velocity and temperature of the fluid are presented graphically.
KEYWORDS: Lifting, Drainage, Modified second grade fluid, Exact solution.

1. INTRODUCTION

The fluids of non-Newtonian behavior are of great interest to researchers [1, 2, 17, 21-22] for their significance in industry and engineering sector. The Navier-Stokes equations are proved to be insufficient to depict and obtain the distinctiveness of complex rheological fluids as well as polymers [3]. A large amount of the industrial and biological fluids are of non-Newtonian nature. Honey, ketchup of tomato, blood, plastic, shampoo, certain oils and greases, paint, mud and polymer solutions are common examples of such fluids. In general non-Newtonian fluids show shear-thinning/shear-thickening behavior [4]. The meagerness of classical theories to explain such liquids has led to the growth of new theory to study fluids of non-Newtonian nature. It is a known fact that because of complexity of liquids, a single model cannot be used to describe the effects of all non-Newtonian fluids. For this reason several constitutive models have been proposed in fluid mechanics. Among these the differential type fluids have attained special status. A subclass of the fluids of the differential type is the modified second grade fluid, which have been deliberated successfully in a variety of flow situations. The non-linear response of these liquids constitutes a significant field of mathematical modeling. The governing equations of motion are highly non-linear and finding exact solutions is a difficult task.

Thin film flows are very important due to their applications in engineering, biomedical, material processing and industries. Such applications include microchip production, lining of mammalian lungs. Many researchers have studied thin film flow to understand different types of its configuration. The flow of a thin film can be produced by various means; for instance gravity, thermal effects and configuration forces. The literature on thin film flows is extensive for viscous fluids [5], but no proper attention has been given to such problems involving non-Newtonian fluids. Few attempts [6 – 9, 14, 15] have been made which deal with the non-Newtonian thin film flows. In all these studies,the authors have either used the homotopy perturbation technique or the homotopy analysis method. There are very few cases, especially for the non-Newtonian fluids, in which the exact solutions of momentum equations were found.

The study of heat transfer to a falling liquid layer has been a matter of broad research in the last few decades [10 – 13, 16]. In most of these studies, the power-law fluid model is taken as the non-Newtonian fluid. Only modest attention has been devoted to the studies where the effects of viscousdissipation areincorporated, althoughthis has been shown to be very significant in various cases such as polymer processing.

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Because of the nonlinearity of the Navier-Stokes equations, a small number of exact solutions are found in literature. This is because of the nonlinearity in the inertial part and higher order in viscosity part. Numerical solutions of differential equations, no matter how accurate, are still not exact solutions because the parameters involved in equation have to be assigned values for each solution. Exact solutions are important for the following reasons [18-20].

(i). These solutions represent fundamental fluid-dynamic flows. Also the basic phenomena described by the Navier-Stokes equations can be studied more closely because of the uniform validity of exact solutions.

(ii). Exact solutions serve as accuracy checks for experimental, asymptotic and numerical or empirical methods. Although computer techniques make the complete integration of the equation of motion feasible, the accuracy of the results can be established by comparison with an exact solution.An excellent review of exact solutions of the Navier-Stokes equation has been given by Wang [18].

In the present paper, we considered modified second grade fluid, and discussed the flow problems on a vertically upward moving belt and down a stationary vertical belt. Exact solutions of the modeled nonlinear ordinary differential equations subject to suitable boundary conditions, are achieved in both cases. For$m=0$, we recover velocity of Newtonian case [5]. Expressions for velocity, temperature, volume flow rate, average velocity and shear stress on the moving and stationary belt are also calculated. It is observed that for steady case, generalized (modified) second grade fluid shows the results of power law model.

This paper is organized as follows: Section 2 presents basic equations for modified second grade fluid, Section 3 transforms equations in section 2 for lifting problem and obtains exact solutions for velocity profile and temperature distribution. Section 4 calculates volume flux, average velocity and belt shear stress for lifting problem. Section 5 used equations from section 2 for drainage problem and found exact solutions for velocity profile and temperature distribution. Section 6 calculates volume flow rate, average velocity and shear stress on the belt for drainage problem. Results and discussion are given in section 7 and conclusion is provided in section 8.

2. BASIC FLOW EQUATIONS

The basic equations governing the flow of an incompressible fluid including thermal effects are:

$$\nabla \cdot \boldsymbol{V} = 0, \tag{1}$$

$$\rho \frac{d\boldsymbol{V}}{dt} = \rho \boldsymbol{f} - \nabla p + \nabla \cdot \boldsymbol{T}, \tag{2}$$

$$\rho C_p \frac{d\Theta}{dt} = \kappa \nabla^2 \Theta + \boldsymbol{T} \cdot \boldsymbol{L}, \tag{3}$$

where**V** is the velocity vector, $\frac{d}{dt}$ is the material time derivative defined as

$$\frac{d}{dt}(*) = \left(\frac{\partial}{\partial t} + \boldsymbol{V} \cdot \nabla \right)(*),$$

ρ is the constant density, $\boldsymbol{f}$ is the body force, p is the dynamic pressure, Θ is the temperature, C_p is the specific heat constant, κ is the thermal conductivity and **T** is the extra stress tensor which is defined differently for different fluids. We choose the modified second grade fluid for our case. The extra stress tensor for such a fluid is given by

$$\boldsymbol{T} = \mu_{eff} \boldsymbol{A_1} + \alpha_1 \boldsymbol{A_2} + \alpha_2 \boldsymbol{A_1}^2, \tag{4}$$

where α_1 and α_2 are the normal stress coefficients, μ_{eff} is an effective viscosity for generalized second grade fluid, and a function of the shear rate, defined as

$$\mu_{eff} = \eta \left(\frac{1}{2} tr \boldsymbol{A}_1^2 \right)^{m/2},$$

η is the flow consistency index and m is the flow behavior index. The RivlinEricksen tensors, $\boldsymbol{A}_1$ and $\boldsymbol{A_2}$ are defined as

$$\boldsymbol{A}_1 = \boldsymbol{L} + \boldsymbol{L}^T, \qquad \boldsymbol{L} = \nabla \boldsymbol{V},$$

$$\boldsymbol{A}_2 = \frac{d\boldsymbol{A}_1}{dt} + \boldsymbol{A}_1\boldsymbol{L} + \boldsymbol{L}^T\boldsymbol{A}_1 .$$

For $m < 0$, the fluid is shear thinning, for $m > 0$ the fluid is shear thickening and for $m = 0$ we obtain second grade fluid model. On the other hand, if $\alpha_1 = \alpha_2 = 0$, equation (4) reduces to the power-law model and for $m = \alpha_1 = \alpha_2 = 0$, we obtain the classical Newtonian model. It is worth mentioning that the flow behavior index *m* must have the limits $-1 < m < 1$ [13].

3. LIFTING PROBLEM

Consider a pot filled with an incompressible non-isothermal, modified second grade fluid andabelt moves vertically upward with a constant speed V_0 all the way through this pot as shown in Figure 1(a). As the belt travels it picks up a thin layer of liquid of the samethickness δ. Gravity tries to make the liquidsap down the belt. Select*x*-axis normal to the belt, *y*-axis along the belt with origin at a point *O,* as exposed in Fig.ure1(a). Suppose that the flow is uniform, laminar and steady.Velocity and temperature profilesare of the form

$$\boldsymbol{V} = [0, v(x), 0], \quad \Theta = \Theta(x). \tag{5}$$

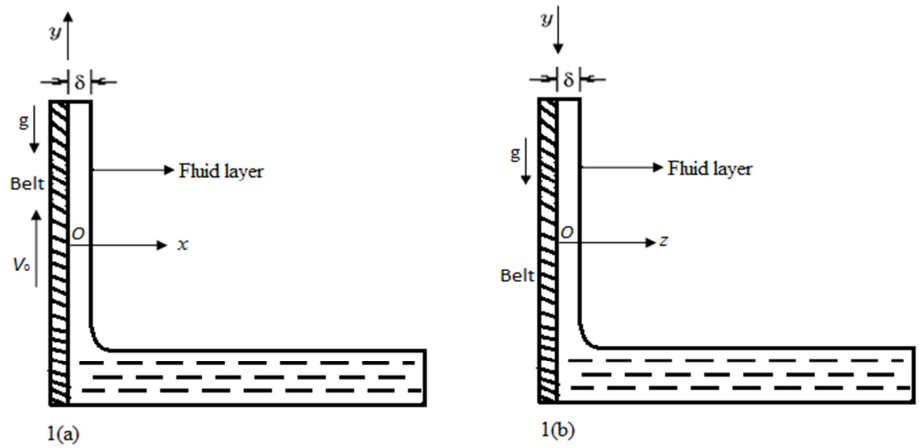

Figure 1: Geometry of the lifting, 1(a), and drainage, 1(b), problems.

Equation (1) is satisfied identically and equation (2) becomes

$$\frac{\partial p}{\partial x} = (2\alpha_1 + \alpha_2)\frac{d}{dx}\left[\left(\frac{dv}{dx}\right)^2\right], \tag{6a}$$

$$\frac{\partial p}{\partial y} = -\rho g + \eta\frac{d}{dx}\left[\left(\frac{dv}{dx}\right)^{m+1}\right], \tag{6b}$$

$$\frac{\partial p}{\partial z} = 0. \tag{6c}$$

For the problem under discussion the pressure *p*is assumed to be atmospheric pressure which is constant, so we can take $\frac{\partial p}{\partial y} = 0$, and equation (6b) reduces to

$$\frac{d}{dx}\left[\left(\frac{dv}{dx}\right)^{m+1}\right] = \frac{\rho g}{\eta}. \tag{7}$$

By means of(5) the energy equation (3) we get

$$\frac{d^2\Theta}{dx^2} = -\frac{\eta}{\kappa}\left(\frac{dv}{dx}\right)^{m+2}. \tag{8}$$

The associated boundary conditions for equations (7) and (8) are

$$\begin{aligned} &v = V_0, \qquad \Theta = \Theta_0 \qquad at \qquad x = 0, \\ &\frac{dv}{dx} = 0, \qquad \frac{d\Theta}{dx} = 0 \qquad at \qquad x = \delta. \end{aligned} \tag{9}$$

Integrating equation (7) with respect to x and usingthe appropriate boundary condition (9) we find that

$$\frac{dv}{dx} = -\left(\frac{\rho g}{\eta}\right)^{\frac{1}{m+1}}(\delta - x)^{\frac{1}{m+1}}. \tag{10}$$

Since velocity decreases with the increase of x so the sign of $\frac{dv}{dx}$ is always opposite to that of x as can be seen in equation (10). Now solving equations (10) together with (8) andapplingthe boundary conditions (9) we obtain

$$v(x) = V_0 - \left(\frac{\rho g}{\eta}\right)^{\frac{1}{m+1}}\left(\frac{m+1}{m+2}\right)\left[\delta^{\frac{m+2}{m+1}} - (\delta - x)^{\frac{m+2}{m+1}}\right], \tag{11}$$

$$\Theta(x) = \Theta_0 + \frac{\eta(m+1)^2}{\kappa(2m+3)(3m+4)}\left(\frac{\rho g}{\eta}\right)^{\frac{m+2}{m+1}}\left[\delta^{\frac{3m+4}{m+1}} - (\delta - x)^{\frac{3m+4}{m+1}}\right]. \tag{12}$$

In solution (12), the second term inside the square brackets confirmdecline in temperature during the motion for any x by keeping η, κ etc. fixed. This is true for the velocity profile (11) also.Velocity profile for Newtonian fluids(Munson[5]) is achieved by taking $m=0$, that is

$$v(x) = V_0 - \frac{\rho g}{\eta}\left(\delta x - \frac{1}{2}x^2\right), \tag{11a}$$

where $\eta = \mu_{eff}$ = constant, kinematic viscosity for Newtonian fluid, and temperature for Newtonian fluid is

$$\Theta(x) = \Theta_0 + \frac{\eta}{12\kappa}\left(\frac{\rho g}{\eta}\right)^2\left[\delta^4 - (\delta - x)^4\right]. \tag{12a}$$

4. VOLUME FLOW RATE, AVERAGE VELOCITY AND SHEAR STRESS FOR LIFTING PROBLEM

Volume flux is defined as

$$Q = \int_0^{\delta} v(x)\, dx, \tag{13}$$

using velocity profile (11) becomes

$$Q = V_0\delta - \left(\frac{m+1}{2m+3}\right)\left(\frac{\rho g}{\eta}\right)^{\frac{1}{m+1}}\delta^{\frac{2m+3}{m+1}}. \tag{14}$$

Average velocity, $\bar{v}$, is given by

$$\bar{v} = \frac{Q}{\delta}. \tag{15}$$

Thus, average film velocity for the vertically upward moving belt is

$$\bar{v} = V_0 - \left(\frac{m+1}{2m+3}\right)\left(\frac{\rho g}{\eta}\right)^{\frac{1}{m+1}} \delta^{\frac{m+2}{m+1}}. \tag{16}$$

If we take $\bar{v} > 0$ in equation (16) the net upward flow of fluid can be obtained, i.e.,

$$V_0 > \left(\frac{m+1}{2m+3}\right)\left(\frac{\rho g}{\eta}\right)^{\frac{1}{m+1}} \delta^{\frac{m+2}{m+1}},$$

whichgives a sensible estimate for the belt velocity to withdraw a modified second grade fluid.
Shear stress on the surface of the belt is

$$T_{xy}\Big|_{x=0} = \eta\left(\frac{dv}{dx}\right)^{m+1}\Bigg|_{x=0} = -\rho g \delta. \tag{17}$$

Introducing the following non-dimensional values

$$v^* = \frac{v}{U},\ x^* = \frac{x}{\delta},\ V_0^* = \frac{V_0}{U}, \Theta^* = \frac{\Theta - \Theta_0}{\Theta_1 - \Theta_0}, S_t = \frac{\rho\, g\, \delta^2}{U\, \mu_{eff}},\ B_r = \frac{\mu_{eff}\, U^2}{\kappa\left(\Theta_1 - \Theta_0\right)}, \tag{18}$$

where S_t is Stokes number, B_r isBrinkman number,$U$is characteristic velocity and Θ_1 is characteristic temperature.
Using these dimensionless parameters, equations (11) and (12), after dropping the asterisk, become

$$v(x) = V_0 - S_t^{\frac{1}{m+1}}\left(\frac{m+1}{m+2}\right)\left[1 - (1-x)^{\frac{m+2}{m+1}}\right], \tag{19}$$

$$\Theta(x) = B_r\, S_t^{\frac{m+2}{m+1}} \frac{(m+1)^2}{(2m+3)(3m+4)}\left[1 - (1-x)^{\frac{3m+4}{m+1}}\right], \tag{20}$$

These are the non-dimensional velocity profile (equation (19)) and temperature distribution (equation (20))for modified second grade fluid, respectively.

5. DRAINAGE PROBLEM

Now consider modified second grade fluid, falling on infinite vertical belt which is at rest as shown Figure 1(b). Here we choose *y*-axis along the belt in downward direction, *z*-axis perpendicular to the belt with origin at a point *O,* as shown in Figure 1(b). The flow is in the downward direction due to gravity. Governing equations (2) and (3) become

$$\frac{d}{dz}\left[\left(\frac{dv}{dz}\right)^{m+1}\right] = -\frac{\rho g}{\eta}, \tag{21}$$

$$\frac{d^2\Theta}{dz^2} = -\frac{\eta}{\kappa}\left(\frac{dv}{dz}\right)^{m+2}, \tag{22}$$

along with the boundary conditions

$$\begin{aligned} &v = 0, \quad \Theta = \Theta_0 \quad at \quad z = 0, \\ &\frac{dv}{dz} = 0, \quad \frac{d\Theta}{dz} = 0 \quad at \quad z = \delta. \end{aligned} \tag{23}$$

Solving equations (21) and (22) with boundary conditions (23) we obtain

$$v(z) = \left(\frac{\rho g}{\eta}\right)^{\frac{1}{m+1}}\left(\frac{m+1}{m+2}\right)\left[\delta^{\frac{m+2}{m+1}} - (\delta - z)^{\frac{m+2}{m+1}}\right], \tag{24}$$

$$\Theta(z)=\Theta_0+\frac{\eta(m+1)^2}{\kappa(2m+3)(3m+4)}\left(\frac{\rho g}{\eta}\right)^{\frac{m+2}{m+1}}\left[\delta^{\frac{3m+4}{m+1}}-(\delta-z)^{\frac{3m+4}{m+1}}\right]. \tag{25}$$

For *m=0*, we obtain velocity profile for fluid as

$$v(z)=\frac{\rho g}{\eta}\left(\delta z-\frac{1}{2}z^2\right), \tag{24a}$$

and temperature profile becomes

$$\Theta(z)=\Theta_0+\frac{\eta}{12\kappa}\left(\frac{\rho g}{\eta}\right)^2\left[\delta^4-(\delta-z)^4\right]. \tag{25a}$$

for Newtonian case.

6. VOLUME FLUX, AVERAGE FILM VELOCITY AND BELT SHEAR STRESS FOR DRAINAGE PROBLEM

To calculate volume flux, Q, we use equation (24) in equation (13) to get

$$Q=\left(\frac{m+1}{2m+3}\right)\left(\frac{\rho g}{\eta}\right)^{\frac{1}{m+1}}\delta^{\frac{2m+3}{m+1}}. \tag{26}$$

The average film velocity, $\bar{v}$, is calculated as

$$\bar{v}=\left(\frac{m+1}{2m+3}\right)\left(\frac{\rho g}{\eta}\right)^{\frac{1}{m+1}}\delta^{\frac{m+2}{m+1}}. \tag{27}$$

Shear stress on the belt becomes

$$T_{yz}\Big|_{z=0}=\eta\left(\frac{dv}{dz}\right)^{m+1}\Bigg|_{z=0}=\rho g\delta. \tag{28}$$

With the help of dimensionless parameters (18) equations (24) and (25) in the non-dimensional forms, after dropping the '*', become

$$v(z)=\left(\frac{m+1}{m+2}\right)S_t^{\frac{1}{m+1}}\left[1-(1-z)^{\frac{m+2}{m+1}}\right], \tag{29}$$

$$\Theta(z)=B_r\,S_t^{\frac{m+2}{m+1}}\frac{(m+1)^2}{(2m+3)(3m+4)}\left[1-(1-z)^{\frac{3m+4}{m+1}}\right], \tag{30}$$

which are the velocity profile and temperature distribution for drainage problem, in the non-dimensional forms, respectively.

7. RESULTS AND DISCUSSION

In the above sections we studied the thin film flows for lifting and drainage problems using a non-isothermal, incompressible modified second grade fluid. We developed differential equations. The exact solutions for both cases are obtained. We found that these solutions are dependent on the flow behavior index *m*, Stokes number S_t and Brinkman number B_r .The effect of flow behavior index *m*on the velocity field and temperature distribution for shear thickening fluids is illustrated graphically through figures 2-3 and 6-7. For shear thinning fluids figures 5 and 9 are presented. Graphs for the Newtonian fluids are shown in figures 4 and 8. If the belt is moving vertically upward, the magnitude of both velocity and temperature decrease as the fluid is becoming thicker and vice versa, this can be observed in figures 2(a) and 2(b), while figures 6(a) and 6(b) are showing results for the drainage case. Figure 3(a) is showing the effect of Stokes number, S_t, on velocity profile for lifting problem. We see that as S_t is

increasing, fluid is becoming thicker and gradient of velocity decreases. The effect of Stokes number, S_t, and Brinkman number, B_r, on heat transfer for lifting problem can be observed in figures 3(b) and 3(c) respectively. For $m = 0$ the effect of S_t on velocity and S_t along with B_r on heat transfer are shown in figures 4(a-c) respectively, while for drainage problem we obtained figures 8(a-c). Respective graphs for shear thinning fluid are shown in figures 5 and 9. It is observed from graphs that velocity of the fluid is higher for lifting problem than that of the velocity for drainage problem. Similarly temperature is low for the lifting problem as compared to drainage problem.

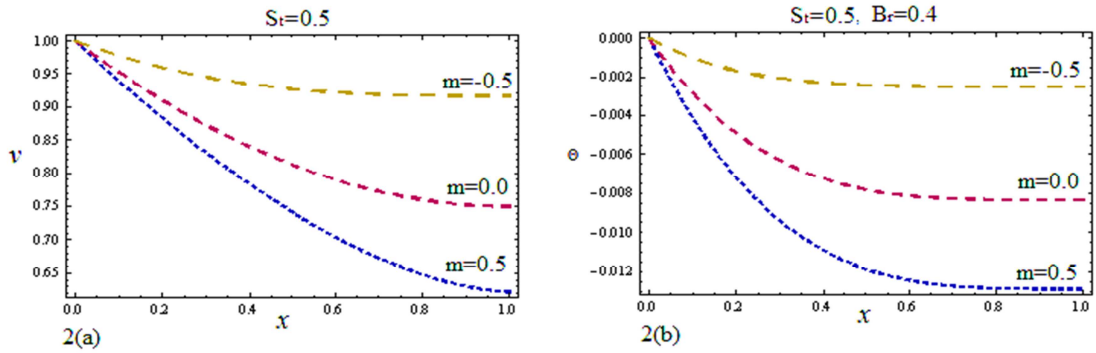

Figure 2: Effect of *m* on velocity 2(a) and temperature 2(b) for lifting problem.

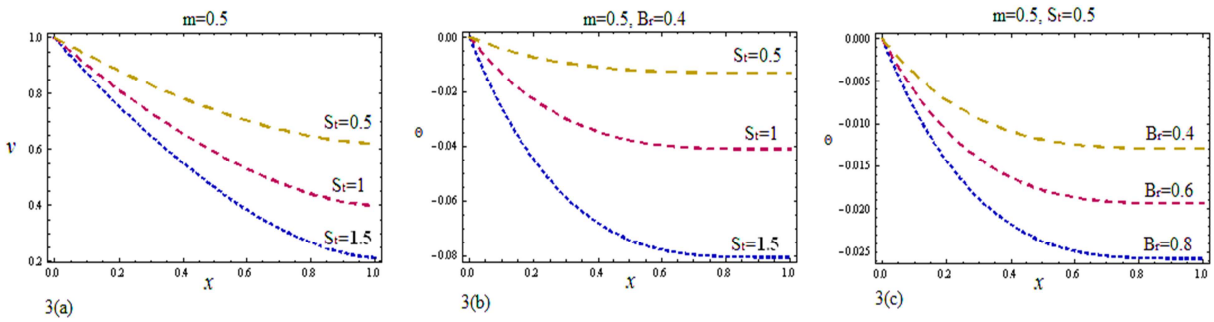

Figure 3: Effect, on shear thickening fluid, of S_t and B_r numbers for lifting problem.

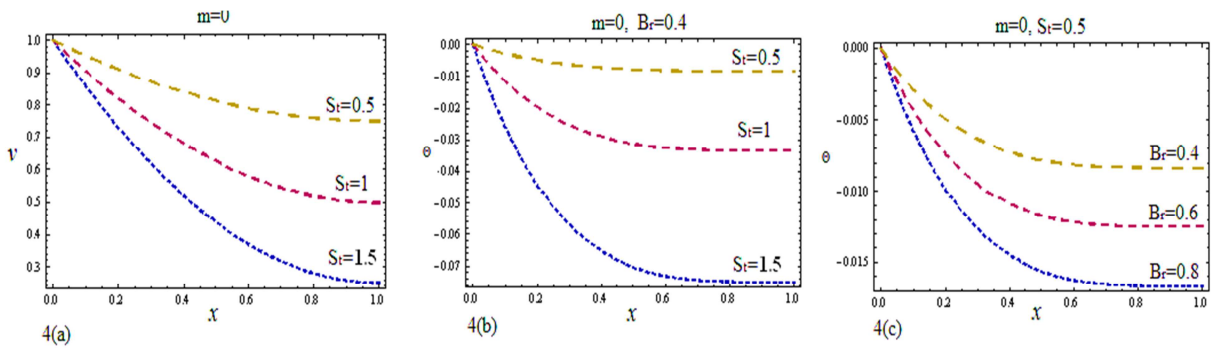

Figure 4: Effect, on Newtonian fluid, of S_t and B_r numbers for lifting problem.

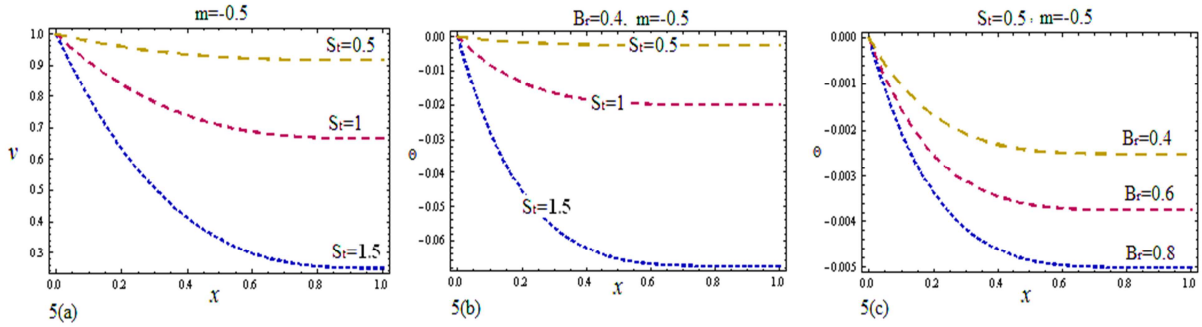

Figure 5: Effect, on shear thinning fluid, of S_t and B_r numbers for lifting problem.

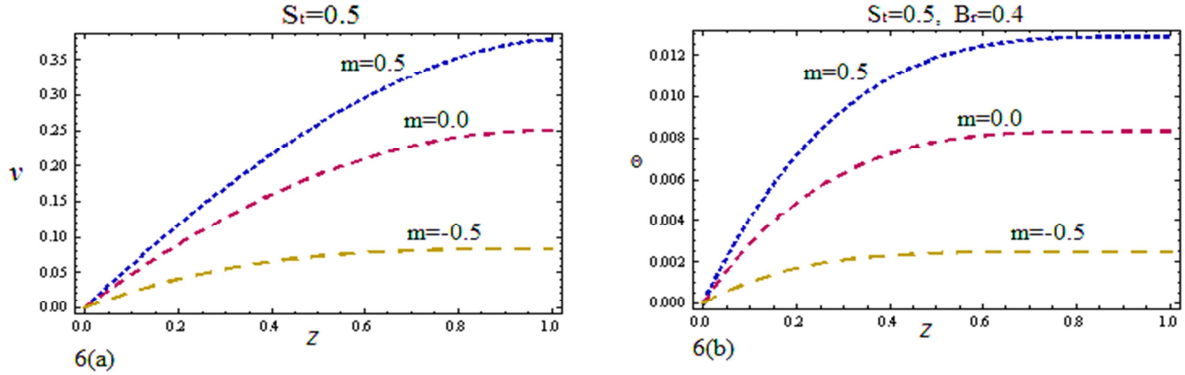

Figure 6: Effect of *m* on velocity 6(a) and temperature 6(b) for drainage problem.

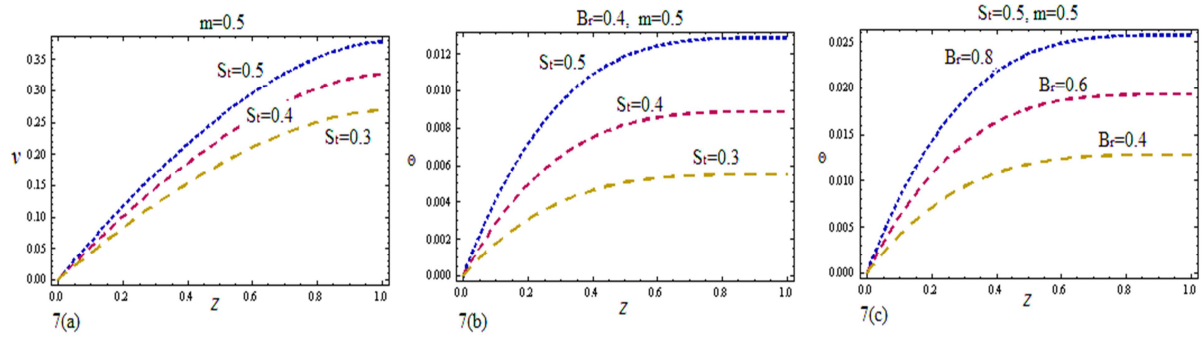

Figure 7: Effect, on shear thickening fluid, of S_t and B_r numbers for drainage problem.

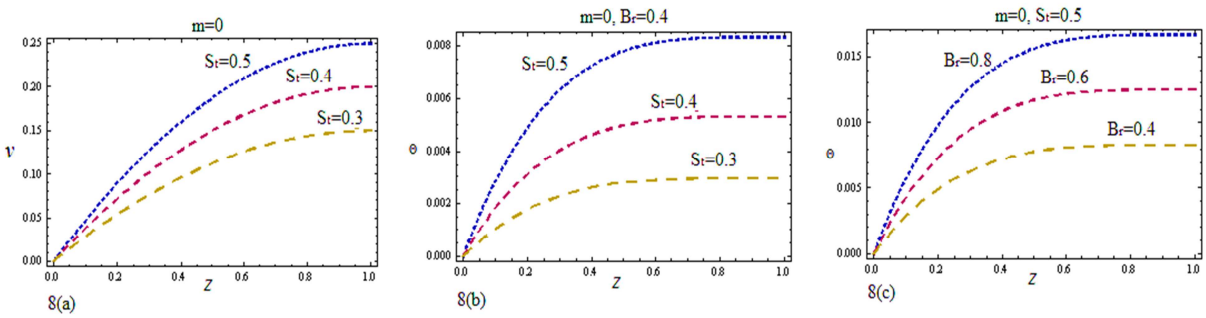

Figure 8: Effect, on Newtonian fluid, of S_t and B_r numbers for drainage problem.

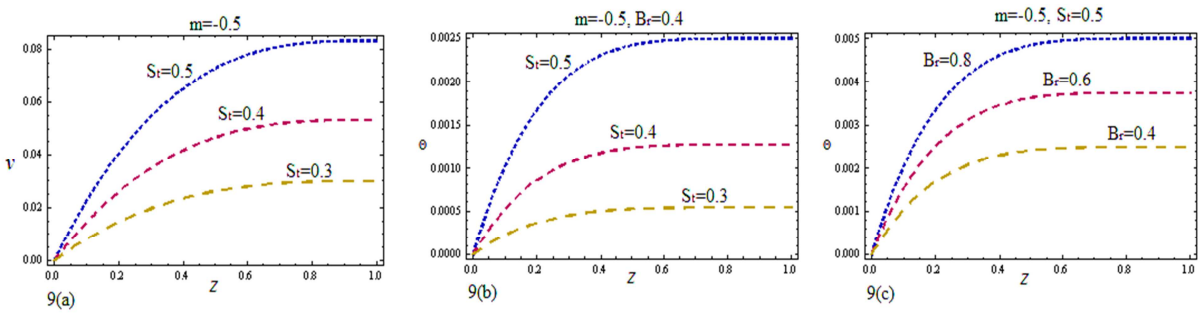

Figure 9: Effect, on shear thinning fluid, of S_t and B_r numbers for drainage problem.

8. CONCLUSIONS

We have considered equations for steady, non-isothermal thin film flow for lifting and drainage problems of modified second grade fluid and obtained exact solutions for both cases. We note that for *m=0*, solutions (11), (12) for lifting problem and (24), (25) for drainage problem of the modified second grade fluid reduce to Newtonian solutions (11a), (12a) and (24a), (25a) of respective problems. It is worthwhile to mention here that normal stresses do not contribute for steady modified second grade fluid flow. Solutions (11) and (24) for velocity profiles and also solutions (12) and (25) for temperature profiles are same as that of power law fluid, we do not see any contribution of the modified second grade fluid model verses power law model. There will be a net upward flow if $\bar{v} > 0$, that is, non-dimensional V_0 must be greater than $\left(\frac{m+1}{2m+3}\right) S_t^{\frac{1}{m+1}}$. For Newtonian fluid V_0 must be greater than $\frac{1}{3} S_t$. For fixed value of the Stokes number, S_t, the inverse relation has been observed between the flow behavior index *m* and the nature of the velocity, as illustrated in figures 2(a) and 6(b) respectively.

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