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AN INEQUALITY OF OSTROWSKI TYPE VIA VARIANT OF POMPEIU'S MEAN VALUE THEOREM

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ABSTRACT

An inequality providing some better bounds for integral mean via variant of Pompeiu's mean value theorem and applications for quadrature rules and special means are given. Our results are of independent interest.

KEY WORDS: Ostrowski Type inequality, Pompeiu's means value theorem, Better bounds, Quadrature rules

I INTRODUCTION

In 1938, the classical integral inequality established by Ostrowski [7] as follows:

Theorem 1. Let $f:[a,b]\to R$ be a differentiable mapping on (a,b) with the property that $\left|f'(t)\right|\le M$ for all $t\in(a,b)$. Then

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(t)dt\right|\le\left[\frac{1}{4}+\left(\frac{x-\frac{a+b}{2}}{b-a}\right)^2\right](b-a)M, \qquad (1.1)$$

for all $x\in(a,b)$.

The constant $\frac{1}{4}$ is best possible in the sense that it cannot be replaced by a smaller constant.

In [3], the author has proved the following Ostrowski type inequality.

Theorem 2. Let $f:[a,b]\to\mathrm{R}$ be a continuous mapping on (a,b) with $a>0$ and differentiable on (a,b). Let $p\in\mathrm{R}\setminus\{0\}$ and assume that

$$K_p(f'):=\sup_{u\in(a,b)}\left(u^{1-p}\left|f'(u)\right|\right)<\infty.$$

Then, we have the inequality

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(t)dt\right|$$
$$\le\frac{K_p(f')}{|p|(b-a)}$$
$$\times\begin{cases}2x^p(x-A)+(b-x)L_p^p(b,x)-(x-a)L_p^p(x,a), \text{ if } p\in(0,\infty)\\ (x-a)L_p^p(x,a)-(b-x)L_p^p(b,x)-2x^p(x-A),\\ \qquad\text{if } p\in(-\infty,-1)\cup(-1,0)\\ (x-a)L^{-1}(x,a)-(b-x)L^{-1}(b,x)-\frac{2}{x}(x-A), \text{ if } p=-1,\end{cases} \qquad (1.2)$$

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for any $x \in (a,b)$, where for $a \neq b, A$ is the arithmetic mean, L_p is the p-logarithmic mean , $p \in R \setminus \{-1,0\}$ and L is the logarithmic mean.

Another result of this type obtained in the same paper is as follows:

Theorem 3. Let $f : [a,b] \to \mathrm{R}$ be a continuous mapping on (a,b) with $a > 0$ and differentiable on (a,b). If

$$P(f') := \sup_{u \in (a,b)} \left| u f'(x) \right| < \infty,$$

then we have the inequality

$$\left| f(x) - \frac{1}{b-a} \int_a^b f(t)dt \right| \leq \frac{P(f')}{(b-a)} \left[\ln \left[\frac{I(x,b)^{b-x}}{I(a,x)^{x-a}} \right] + 2(x-A) \ln x \right], \tag{1.3}$$

for any $x \in (a,b)$, where for $a \neq b$, I is the identric mean.

If some local information around the point $x \in (a,b)$ is available, then we may state the following result as well [3].

Theorem 4. Let $f : [a,b] \to R$ be continuous on (a,b) with $a > 0$ and differentiable on (a,b). Let $p \in (0,\infty)$ and assume, for a given $x \in (a,b)$, we have that

$$M_p(x) := \sup_{u \in (a,b)} \left(\left| x-u \right|^{1-p} \left| f'(x) \right| \right) < \infty,$$

then we have the inequality

$$\left| f(x) - \frac{1}{b-a} \int_a^b f(t)dt \right| \leq \frac{M_p(x)}{p(p+1)(b-a)} \left[(x-a)^{p+1} + (b-x)^{p+1} \right]. \tag{1.4}$$

For some recent results in connection with Ostrowski's inequality, see the papers [1-2] and monograph [4]. Instead of using Cauchy mean value theorem, S. S. Dragomir [5] and Anna Maria [6] used Pompeiu mean value theorem to evaluate the integral mean of an absolutely continuous function. He further gave applications and particular instances of functions. The result is provided in the following theorem:

Theorem 5. Let $f : [a,b] \to R$ be continuous on (a,b) with $a > 0$ and differentiable on (a,b) with $[a,b]$ not containing 0. Then for any $x \in [a,b]$, we have the inequality

$$\left| \frac{a+b}{2} . \frac{f(x)}{x} - \frac{1}{b-a} \int_a^b f(t)dt \right| \leq \frac{b-a}{|x|} \left[\frac{1}{4} + \left(\frac{x - \frac{a+b}{2}}{b-a} \right)^2 \right] \left\| f - l f' \right\|_\infty, \tag{1.5}$$

where $l(t) = t, \quad t \in [a,b]$. The constant $\frac{1}{4}$ is sharp in the sense that it cannot be replaced by a smaller constant.

In 1946 , Pompeiu [8] derived a variant of Lagrange's mean value theorem, now known as Pompeiu's mean value theorem (see also [8, p.83]).

The main aim of this paper is to provide better bounds for integral mean using variant of Pompeiu's mean value theorem. Applications to quadrature rules and some special means are also given.

II MAIN RESULTS

Theorem 6. Let $f : [a,b] \to R$ be continuous on (a,b) with $a > 0$ and twice differentiable on (a,b) with $[a,b]$ not containing 0. Then for any $x \in [a,b]$, we have the inequality

$$\begin{aligned} &\left| \frac{a+b}{2} \left(\frac{2f(x)}{3x} - \frac{f'(x)}{3} \right) + \frac{1}{3} \left(\frac{bf(b) - af(a)}{b-a} \right) - \frac{1}{b-a} \int_a^b f(t)dt \right| \\ &\leq \frac{b-a}{3|x|} \left[\frac{1}{4} + \left(\frac{x - \frac{a+b}{2}}{b-a} \right)^2 \right] \left\| l^2 f'' - 2lf' + 2f \right\|_\infty, \end{aligned} \tag{2.1}$$

where $l(t)=t, \quad t\in[a,b].$

Proof. Define a real valued function F on the interval $[\frac{1}{b},\frac{1}{a}]$, then applying mean value theorem to F on the interval $[x,y]\subset[\frac{1}{b},\frac{1}{a}]$, we get

$$\frac{F(x)-F(y)}{x-y}=F^{/}(\eta^{/}),\ \text{for some}\ \ \eta'\in(x,y).$$

Let $F^{/}(t)=\phi(t)$, for all $t\in[\frac{1}{b},\frac{1}{a}]$, then applying mean value theorem to ϕ on the interval $[x,y]\subset[\frac{1}{b},\frac{1}{a}]$, we get

$$\frac{\phi(x)-\phi(y)}{x-y}=\phi'(\eta),\ \text{for some}\ \eta\in(x,y).$$

This implies

$$\frac{F'(x)-F'(y)}{x-y}=F''(\eta),\ \text{for some}\ \eta\in(x,y). \tag{2.2}$$

Accordingly, the theorem can be extended for higher derivatives. Let $F(t)=t^2 f(\frac{1}{t})$ be a real valued function defined on the interval $[\frac{1}{b},\frac{1}{a}]$.

Now

$$F'(t)=-f'(\frac{1}{t})+2tf(\frac{1}{t}),\ \ F''(t)=\frac{1}{t^2}f''(\frac{1}{t})-\frac{2}{t}f'(\frac{1}{t})+2f(\frac{1}{t}).$$

(2.2) gives

$$\frac{-f'(\frac{1}{x})+2xf(\frac{1}{x})+f'(\frac{1}{y})-2yf(\frac{1}{y})}{x-y}=\frac{1}{\eta^2}f''(\frac{1}{\eta})-\frac{2}{\eta}f'(\frac{1}{\eta})+2f(\frac{1}{\eta}).$$

Let $x_2=\frac{1}{x}$, $x_1=\frac{1}{y}$ and $\xi=\frac{1}{\eta}$. Then $\eta\in(x,y), x_1<\xi<x_2$, implies

$$\frac{-f'(x_2)+\frac{2}{x_2}f(x_2)+f'(x_1)-\frac{2}{x1}f(x_1)}{\frac{1}{x_2}-\frac{1}{x_1}}=\xi^2 f''(\xi)-2\xi f'(\xi)+2f(\xi),$$

or

$$\begin{aligned}&-x_1x_2 f^{/}(x_2)+2x_1 f(x_2)+x_1x_2 f^{/}(x_1)-2x_2 f(x_1)\\&=(x_1-x_2)\left[\xi^2 f^{//}(\xi)-2\xi f^{/}(\xi)+2f(\xi)\right].\end{aligned}$$

Let $l(t)=t,\ t\in[a,b]$. Putting $x_1=t,\ x_2=x$ in the above result, we have:

$$\begin{aligned}&-txf^{/}(x)+2tf(x)+txf^{/}(t)-2xf(t)\\&=(t-x)\left[\xi^2 f^{//}(\xi)-2\xi f^{/}(\xi)+2f(\xi)\right].\end{aligned} \tag{2.3}$$

Integrating (2.3) with respect to t over $[a,b]$, we obtain

$$\begin{aligned}&-xf^{/}(x)\int_a^b tdt+2f(x)\int_a^b tdt+x\int_a^b tf^{'}(t)dt-2x\int_a^b f(t)\\&=\int_a^b (t-x)\left[\xi^2 f^{//}(\xi)-2\xi f^{/}(\xi)+2f(\xi)\right]dt.\end{aligned}$$

This gives

$$\frac{b^2-a^2}{2}\left[2f(x)-xf'(x)\right]+x\left[bf(b)-af(a)\right]-3x\int_a^b f(t)$$
$$=\int_a^b (t-x)\left[\xi^2 f''(\xi)-2\xi f'(\xi)+2f(\xi)\right]dt.$$

This implies

$$\left|\frac{b+a}{2}\left(\frac{2f(x)}{3x}-\frac{f'(x)}{3}\right)+\frac{1}{3}\left(\frac{bf(b)-af(a)}{b-a}\right)-\frac{1}{b-a}\int_a^b f(t)dt\right|$$
$$\le \frac{1}{3|x|(b-a)}\left\|l^2 f''-2lf'+2f\right\|_\infty \int_a^b |t-x|\,dt,$$

implies

$$\left|\frac{b+a}{2}\left(\frac{2f(x)}{3x}-\frac{f'(x)}{3}\right)+\frac{1}{3}\left(\frac{bf(b)-af(a)}{b-a}\right)-\frac{1}{b-a}\int_a^b f(t)dt\right|$$
$$\le \frac{(b-a)}{3|x|}\left\|l^2 f''-2lf'+2f\right\|_\infty \left[\frac{1}{4}+\left(\frac{x-\frac{a+b}{2}}{b-a}\right)^2\right].$$

Hence the theorem,

Corollary 1. With the assumptions of Theorem 6, we have the following inequality:

$$\left|\frac{b+a}{2}\left(\frac{f(\frac{a+b}{2})}{3(b+a)}-\frac{f'(\frac{a+b}{2})}{3}\right)+\frac{1}{3}\left(\frac{bf(b)-af(a)}{b-a}\right)-\frac{1}{b-a}\int_a^b f(t)dt\right|$$
$$=\frac{(b-a)}{6(a+b)}\left\|l^2 f''-2lf'+2f\right\|_\infty. \tag{2.4}$$

III THE CASE OF WEIGHTED INTEGRALS

We consider here the following weighted integral case:

Theorem 7. Let $f:[a,b]\to R$ be continuous on (a,b) with $a>0$ and twice differentiable on (a,b) with $[a,b]$ not containing 0. If $w:[a,b]\to R$ is non-negative and integrable on $[a,b]$, then for each $x\in[a,b]$, we have the inequality:

$$\left|\int_a^b w(t)f(t)dt-\left(\frac{2f(x)-xf'(x)}{2x}\right)\int_a^b tw(t)dt-\frac{1}{2}\int_a^b tw(t)f'(t)dt\right|$$
$$\le \frac{1}{2}\left\|l^2 f''-2lf'+2f\right\|_\infty$$
$$\times\left[\operatorname{sgn}(x)\left(\int_a^x w(t)dt-\int_x^b w(t)dt\right)+\frac{1}{|x|}\left(\int_x^b tw(t)dt-\int_a^x tw(t)dt\right)\right]. \tag{3.1}$$

Proof. Using the inequality (2.3), we have:

$$\left|\left(2f(x)-xf'(x)\right)\int_a^b tw(t)dt+x\int_a^b tw(t)f'(t)dt-2x\int_a^b w(t)f(t)dt\right|$$

$$\leq \left\|l^2 f''-2lf'+2f\right\|_\infty \int_a^b w(t)|x-t|\,dt$$

$$=\left\|l^2 f''-2lf'+2f\right\|_\infty \left(\int_a^x w(t)(x-t)dt+\int_x^b w(t)(t-x)dt\right)$$

$$=\left\|l^2 f''-2lf'+2f\right\|_\infty \left[x\left(\int_a^x w(t)dt-\int_x^b w(t)dt\right)+\int_x^b tw(t)dt-\int_a^x tw(t)dt\right],$$

implies

$$\left|\int_a^b w(t)f(t)dt-\left(\frac{2f(x)-xf'(x)}{2x}\right)\int_a^b tw(t)dt-\frac{1}{2}\int_a^b tw(t)f'(t)dt\right|$$

$$\leq \frac{1}{2}\left\|l^2 f''-2lf'+2f\right\|_\infty$$

$$\times\left[\operatorname{sgn}(x)\left(\int_a^x w(t)dt-\int_x^b w(t)dt\right)+\frac{1}{|x|}\left(\int_x^b tw(t)dt-\int_a^x tw(t)dt\right)\right].$$

Assume that $0<a<b,$ then

$$a\leq \frac{\int_a^b tw(t)dt}{\int_a^b w(t)dt}\leq b,\ \text{provided}\ \int_a^b w(t)dt>0. \tag{3.2}$$

Corollary 2. With the assumptions of Theorem 7, we have the following corollary:

$$\left|\frac{1}{\int_a^b w(t)dt}\int_a^b w(t)f(t)dt\right.$$

$$\left.-\frac{1}{2}\left[2f\left(\frac{\int_a^b tw(t)dt}{\int_a^b w(t)dt}\right)-\frac{\int_a^b tw(t)dt}{\int_a^b w(t)dt}f'\left(\frac{\int_a^b tw(t)dt}{\int_a^b w(t)dt}\right)\right]-\frac{1}{2}\int_a^b tw(t)f'(t)dt\right|$$

$$\leq \frac{1}{2}\left\|l^2 f''-2lf'+2f\right\|_\infty$$

$$\times\left[\operatorname{sgn}(x)\left(\frac{\int_a^x w(t)dt-\int_x^b w(t)dt}{\int_a^b w(t)dt}\right)+\frac{\left(\int_x^b tw(t)dt-\int_a^x tw(t)dt\right)}{\left|\int_a^b tw(t)dt\right|}\right]. \tag{3.3}$$

IV A Quadrature Formula

We may utilize the previous inequality to give us estimates of composite quadrature rules which, it turns out to have remarkably smaller error than that which may be obtained by the classical results and by S. S. Dragomir [1].

Theorem 8. Let $I_n : a = x_0 < x_1 < x_2 < ... < x_n = b$ be a partition of the interval $[a,b]$, $h_i = x_{i+1} - x_i$, $\xi_i \in [x_i, x_{i+1}]$, $i = 0,1,..., n-1$, a sequence of intermediate points. Define the quadrature

$$S(f, I_n, \xi) = \sum_{i-0}^{n-1} h_i \left[\frac{x_i + x_{i+1}}{2} \left(\frac{2f(\xi_i)}{3\xi_i} - \frac{f'(\xi_i)}{3} \right) + \frac{1}{3} \left(\frac{x_{i+1} f(x_{i+1}) - x_i f(x_i)}{h_i} \right) \right],$$

then

$$\int_a^b f(x)dx = S(f, I_n, \xi) + R(f, I_n, \xi),$$

where

$$\begin{aligned} &|R(f, I_n, \xi)| \\ &\le \left\| l^2 f'' - 2lf' + 2f \right\|_\infty \left[\frac{1}{3} \sum_{i=0}^{n-1} \frac{h_i^2}{|\xi_i|} \left\{ \frac{1}{4} + \left(\xi_i - \frac{\frac{x_i + x_{i+1}}{2}}{h_i} \right)^2 \right\} \right] \qquad (4.1) \\ &\le \frac{\left\| l^2 f'' - 2lf' + 2f \right\|_\infty}{12} \sum_{i=0}^{n-1} \frac{h_i^2}{\xi_i} \le \frac{\left\| l^2 f'' - 2lf' + 2f \right\|_\infty}{6a} \sum_{i=0}^{n-1} h_i^2. \end{aligned}$$

Proof. Applying the inequality (2.1) on $\xi_i \in [x_i, x_{i+1}]$, summing over $i = 0,1,..., n-1$ and using the generalized triangular inequality, we deduce the desired estimate.

Choosing $\xi_i = \frac{x_i + x_{i+1}}{2}$, $i = 0,1, ..., n-1$, for mid-point rule and with assumptions of Theorem 8, we have:

$$\int_a^b f(x)dx = M_n(f, I_n) + R(f, I_n),$$

and

$$\begin{aligned} &M_n(f, I_n) \\ &= \sum_{i-0}^{n-1} h_i \\ &\times \left[\frac{x_i + x_{i+1}}{2} \left(\frac{4f(\frac{x_i + x_{i+1}}{2})}{3(x_i + x_{i+1})} - \frac{f'(\frac{x_i + x_{i+1}}{2})}{3} \right) + \frac{1}{3} \left(\frac{x_{i+1} f(x_{i+1}) - x_i f(x_i)}{h_i} \right) \right], \end{aligned}$$

where the remainder satisfies the estimate

$$|R(f, I_n)| \le \frac{\left\| l^2 f'' - 2lf' + 2f \right\|_\infty}{12} \sum_{i=0}^{n-1} \frac{h_i^2}{x_i + x_{i+1}} \le \frac{\left\| l^2 f'' - 2lf' + 2f \right\|_\infty}{6a} \sum_{i=0}^{n-1} h_i^2.$$

V APPLICATIONS FOR SPECIAL MEANS

Consider Corollary 1 for applications.

$$\begin{aligned} &\left| \frac{b+a}{2} \left(\frac{f(\frac{a+b}{2})}{3(b+a)} - \frac{f'(\frac{a+b}{2})}{3} \right) + \frac{1}{3} \left(\frac{bf(b) - af(a)}{b-a} \right) - \frac{1}{b-a} \int_a^b f(t)dt \right| \qquad (5.1) \\ &= \frac{(b-a)}{6|b+a|} \left\| l^2 f'' - 2lf' + 2f \right\|_\infty, \end{aligned}$$

provide $0 < a < b$.

1. Consider the function $f\ :[a,b] \subset (0,\infty) \to R, \quad f(t) = t^p, \quad p \in R \setminus \{-1,0\}.$ Then

$$f\left(\frac{a+b}{2}\right) = A^p, \qquad \frac{1}{b-a}\int_a^b f(t)dt = L_p^p,$$

$$\left\| l^2 f^{//} - 2lf^{/} + 2f \right\|_\infty = \begin{cases} (p^2 - 3p + 2)a^p, \text{ if } p \in (-\infty, 0) \setminus \{-1\} \\ (p^2 - 3p + 2)b^p, \text{ if } p \in (0,1) \cup (1,\infty). \end{cases}$$

Consequently by (5.1), we have:

$$\left| \frac{1}{6} A^p (1-2p) + \frac{bf(b) - af(a)}{3(b-a)} - L_p^p \right|$$

$$\leq \frac{1}{6} \begin{cases} (p^2 - 3p + 2)a^p, \text{if } p \in (-\infty, 0) \setminus \{-1\} \\ (p^2 - 3p + 2)b^p, \text{if } p \in (0,1) \cup (1,\infty) \end{cases}$$

2. Consider the function $f\ :[a,b] \subset (0,\infty) \to R, \quad f(t) = \frac{1}{t}.$ Then

$$f\left(\frac{a+b}{2}\right) = A^{-1}, \ \frac{1}{b-a}\int_a^b f(t)dt = L^{-1},$$

$$f^{/}\left(\frac{a+b}{2}\right) = -A^{-2}, \ \left\| l^2 f^{//} - 2lf^{/} + 2f \right\|_\infty = \frac{6}{a}.$$

Consequently by (5.1), we have:

$$\left| A\left(\frac{A^{-1}}{6A} - \frac{-A^2}{3}\right) + \frac{bf(b) - af(a)}{3(b-a)} - L^{-1} \right| \leq \frac{6}{a}\left(\frac{b-a}{12a}\right).$$

This gives

$$\left| \frac{L}{2} + \frac{bf(b) - af(a)}{3(b-a)} AL - A \right| \leq \frac{b-a}{2a^2} AL.$$

Similar applications for other functions can also be quoted.

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