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Global Dynamics and Stability of SIR Epidemic Model

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ABSTRACT

To understand the epidemiology of an infectious disease, we consider an SIR model (Susceptible-Infected-Recovered) with non-linear incidence function. We choose the incidence term as proportions of people that infected. The new incidence term $\frac{(1-\rho)}{b}$, is used that shows the number of infected people in fraction. We will solve the model mathematically and numerically, the theoretical results will be justified by numerical results. First, we formulate the model with their parameters. Then, we will find the basic reproduction number R_o. After that, we prove the co-existence of disease free and endemic equilibrium with R_o. The model is locally asymptotically stable about disease free equilibrium E^o when $R_o \leq 1$. We find that the model is also globally stable about disease free equilibrium point $R_o \leq 1$. Then, we obtain, that the endemic equilibrium point is stable locally if $R_o > 1$. Finally, we find the numerical solution of the analytical results for the purpose of justifications.

KEY WORDS: Epidemic model, Reproduction number, Global Stability, Disease free equilibrium, Numerical simulations

1. INTRODUCTION

To under the epidemiology of the disease dynamics mathematical models act an important role[1, 2]. For different diseases, different SIR/SEIR models have been used to study their dynamical behavior[3-7].These models provide a quantitative description of the complicated, non-linear process of disease transmission and help us to obtain inside into the dynamics of the disease so we are able to make such decision for public health policy. Mathematical models of the type [3-5] have been used to represent the compartmental dynamics of human and vector class. The human population is divided into sub-classes S, I and R while the vector is in S and I.[8] considered a deterministic model for the transmission of leptospirosis disease with the information of present's number of leptospirosis disease in Thailand.[9]studied an epidemic model and sub-divide the population into two categories i.e., human and vector. The author considered the real data presented in [8] to study the dynamics of the disease and optimal control [10].[11] developed a mathematical model to study the behavior of an epidemic disease. A variety of non-linear incidence rate have been used in many epidemic models [12-14]. In [15], Altaf et al. presented an epidemic model of leptospirosis, where the human population is categorized as a SIR and vector population in SI. The solution obtained by using the homotpoy perturbation method. A Vector-host epidemic model has been used by [16], the results were obtained by the standard, homotopy perturbation method. In [17], an epidemic model of leptospirosis disease is considered with time delay and the main results are presented. In [20], the author studied the basic model for Chikungunya disease in which he classified the male and female in an SIR category. The basic results and the numerical results presented for the analytical results. In [21] the model for the transmission of Influenza Pandemic due to a New-strain of the H1N1 Influenza A Virus with the risk of infection in human has been studied and mathematical results are presented with numerical simulations. In this work, we consider a simple SIR model with the non-linear saturated incidence rate that predicts the number of individuals who got infection. First, we study the basic properties of the model, i.e. the local asymptotical stability of (DFE) disease free equilibrium and endemic equilibrium (EE). Then, we find that the disease free equilibrium is stable globally when the threshold quantity decreases than unity. When the threshold quantity exceeds than unity a stable endemic equilibrium exists and globally stable will prove. Finally, we obtain the numerical solution of the theoretical results. In references [19-21] the author used the standard analytical methods to obtain the solution for such non-linear

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models. So, we motivate the readers, which an alternative way that they can study the biological models by using the methods presented in [19-21]. For more relevant work see the references [24-28].
We organized the paper is as: In Section2, first, the model will be formulated and then the disease-free, endemic equilibrium, threshold quantity will be discussed. The main results will be derived in Section 3. In Section 4, we find the numerical results of the model with justifications of the analytical results with complete discussion.

2. MATERIAL AND METHODS

To formulate our problem, we divide the population into three sub-classes, into three subclasses, i.e. susceptible, infected and recovered or removed. The system of differential equation that govern the model is given by,

$$\begin{aligned}
\frac{dS(t)}{dt} &= b - f(S,I) - \gamma S, \\
\frac{dI(t)}{dt} &= f(S,I) - (\gamma + \delta) I(t), \\
\frac{dR(t)}{dt} &= \delta I(t) - \gamma R(t). \qquad (1)
\end{aligned}$$

Here, the function $f(S,I) = \frac{(1-\rho)}{b} S(t) I(t)$, in which ρ represents the fraction of individuals that to be infected. Further, we write (1),

$$\begin{aligned}
\frac{dS(t)}{dt} &= b - \frac{(1-\rho)}{b} S(t) I(t) - \gamma S, \\
\frac{dI(t)}{dt} &= \frac{(1-\rho)}{b} S(t) I(t) - (\gamma + \delta) I(t), \\
\frac{dR(t)}{dt} &= \delta I(t) - \gamma R(t), \qquad (2)
\end{aligned}$$

Subject to the non-negative initial conditions.

Here γ is the mortality rate, b shows the recruitment of the individuals and ρ is the fraction of individuals that are going to be infectious while δ represents the rate of recovery. N(t) is the total size of the population. The term $\frac{(1-\rho)}{b}$ shows the fraction of individuals that are to be infected. The total population size is denoted by N(t), and their total dynamics is given by,

$$\frac{dN(t)}{dt} = b - \gamma N(t). \qquad (3)$$

The feasible region for the system (2) is given by

$$\Omega = \left\{ (S,I,R) \in \Re_+^3 \mid 0 \le S + I + R \le \frac{b}{\gamma} \right\}.$$

Our focus will be on those solutions lies in Ω.

2.1Basic properties of the model

For system (2), with setting left side equal to zero, i.e.

$$b-\frac{(1-\rho)}{b}S(t)I(t)-\gamma S=0,$$
$$\frac{(1-\rho)}{b}S(t)I(t)-(\gamma+\delta)I(t)=0,$$
$$\delta I(t)-\gamma R(t)=0,$$

With the point $E_o=(S^o,0,0)$ we get the disease-free equilibrium is $S^o=\frac{b}{\gamma}$. The reproduction number or threshold quantity is defined as; the average number of infection that are entered into a purely susceptible population and produce secondary infection and the rest of the population assumed susceptible. For the model (2), the reproduction number is given by,

$$R_o=\frac{(1-\rho)}{\gamma(\gamma+\delta)}$$

For $R_o \leq 1$, the disease-free equilibrium is stable locally asymptotically and the endemic state is locally asymptotically stable when R_o >1.

2.2 Endemic Equilibrium Points

To obtain, the endemic equilibrium points of the system (2), with the equilibrium point $E^1=(S^*,I^*,R^*)$, just set the left side of the system (2), equating to zero, we obtain

$$S^*=\frac{(\gamma+\delta)b}{(1-\rho)},$$
$$I^*=\frac{b(1-\rho)-(\gamma+\delta)\gamma b}{(\gamma+\delta)(1-\rho)},$$
$$R^*=\frac{\delta I^*}{\gamma},$$

After some rearrangements, we obtain,

$$S^*=\frac{b}{\gamma R_o},$$
$$I^*=\frac{(\gamma b}{(1-\rho)}(R_o-1),$$
$$R^*=\frac{\delta b}{(1-\rho)}(R_o-1),$$

3. RESULTS

3.1**Stability Analysis of the model**

In current section, we show the stability of both the disease-free and endemic equilibrium. First, we show that the system (2) is locally asymptotically stable. For this, we use the disease-free equilibrium point E_o about system (2), the following Jacobean matrix J_o, is presented.

Theorem 3.1: For $R_o \leq 1$, the disease-free equilibrium of the system (2), about the equilibrium point E_o, is locally asymptotically stable and unstable whenever R_o >1.

Proof: To Show that the system (2) is locally asymptotically stable, we use the disease-free equilibrium point E_o of the system (2), we get the following Jacobean matrix J_o,

$$J_o = \begin{pmatrix} -\gamma - \frac{(1-\rho)}{b} S^o & 0 & 0 \\ 0 & \frac{(1-\rho)}{b} S^o - \gamma - \delta & 0 \\ 0 & \delta & -\gamma \end{pmatrix}.$$

The eigenvalues corresponding to the Jacobean matrix Jo are given in the following by reducing the matrix to echelon form by elementary row operation. $\lambda_1 = -\gamma < 0$ and the rest of the two eigenvalues are obtained by the Routh-Hurwitz Criteria. We have to show, $A_1>0$ and $A_2>0$ for the pair of eigenvalues.

$$\lambda^2 + \lambda(2(\gamma+\delta)) + (-\frac{(1-\rho)^2}{b^2} S^{o2} + (\gamma+\delta)^2) = 0, \qquad (4)$$

Where

$A_1 = (2(\gamma+\delta))$, and $A_2 = (-\frac{(1-\rho)^2}{b^2} S^{o2} + (\gamma+\delta)^2)$.

Here A_1 is clearly positive and for A_2 , with some arrangements, we get,

$A_2 = -(R_0+1)(R_0-1)(\gamma+\delta) > 0.$

So all the eigenvalues associated to system (2) are having negative parts (real). Thus we conclude that the system (2) is stable locally, when, $R_o \le 1$.

3.2Local Stability of Endemic Equilibrium

In this subsection, we find the local asymptotic stability of the system (2) about an endemic equilibrium point E^1.

Theorem 3.2.1: If Ro >1, then the system (2) about an endemic equilibrium point E^1 is locally asymptotically stable, and unstable when $R_o \le 1$.

Proof: For the system (2), the Jacobean matrix J^*,is given in the following,

$$J^* = \begin{pmatrix} -\frac{(1-\rho)}{b} I^* - \gamma & -\frac{(1-\rho)}{b} S^* & 0 \\ \frac{(1-\rho)}{b} I^* & \frac{(1-\rho)}{b} S^* - (\gamma+\delta) & 0 \\ 0 & \delta & -\gamma \end{pmatrix}.$$

The eigenvalues associated to the Jacobean matrix J^* are follows:

$\lambda_1 = -\frac{(1-\rho)}{b} I^* - \gamma,$

$\lambda_2 = -\gamma(R_o - 1),$

$\lambda_3 = -\gamma^2(R_o - 1).$

Thus, the eigenvalues depends upon $R_o > 1$, so system (2) is locally stable at endemic equilibrium point E^1.

3.3Global Stability

This section deals to study the global properties of the proposed system (2). Here, first we have to find the global stability of the disease free and then we used the geometric approach method to obtain the global stability of endemic equilibrium.

Theorem 3.3.1: For$R_o \leq 1$, the disease free equilibrium point (DFE) E_o of the proposed system (2), is globally asymptotically stable and instability of disease free equilibrium exists when $R_o > 1$.

Proof: To prove this result, we define the following functions,

$$V(t) = \frac{1}{2}(N - \frac{b}{\gamma})^2 \quad (5)$$

Taking the time derivative of the equation (5), we get

$$V'(t) = (N - \frac{b}{\gamma})\frac{dN}{dt}, \quad (6)$$

Using the value of $\frac{dN}{dt} = b - \gamma N$, we obtain

$$V'(t) = -(N - \frac{b}{\gamma})^2 \gamma \quad (7)$$

Equation (7)is clearly a decreasing function, i.e.$V'(t) \leq 0,\ for\ S = S^o, I = R = 0$ .Thus, the (DFE) point E_o is globally stable.

To prove the global stability of the endemic equilibrium point E^1 of the system (2), we first present the following lemma and then we will proceeds to prove the result.

Lemma: The system in the form of $\frac{dy}{dt} = k(\mathrm{y})$, and the function $\mathrm{k} : \mathrm{X}$ ™ R^n, there exists a unique equilibrium point y^* also the existing of a compact absorbing set, so, the point y^* is said to be stable globally asymptotically with the addition of function P(x) and ℓ (Lozinskii measure) such that

$limit_{t \text{™} \infty} sup\ sup_y \frac{1}{t}\int_0^t \ell(\mathrm{K}(\mathrm{y}(s, \mathrm{y})))ds < 0.$ [18]. In the Theorem 3.3.2 we define the symbols P, Kand ℓ will be defined.

Theorem:3.3.2. For $R_o > 1$,the endemic equilibrium point E_1 of the system (2) is globally asymptotically stable.

Proof: The second additive compound matrix $J^{[2]}$ corresponds to J(S, I, R)is,

$$J^{[2]} = \begin{pmatrix} -\frac{(1-\rho)}{b}I - \gamma + \frac{(1-\rho)}{b}S - (\gamma+\delta) & 0 & 0 \\ \delta & -\frac{(1-\rho)}{b}I - 2\gamma & -\frac{(1-\rho)}{b}S \\ 0 & \frac{(1-\rho)}{b}I & -\frac{(1-\rho)}{b}S - (\gamma+\delta) - \gamma \end{pmatrix}.$$

We consider the function, $P = P(S, I, R) = diag\left\{\frac{S}{I}, \frac{S}{I}, \frac{S}{I}\right\}$

And

$$P^{-1} = diag\left\{\frac{I}{S}, \frac{I}{S}, \frac{I}{S}\right\}$$

And

$$P_f = diag\left\{\frac{\dot{S}}{I} - \frac{S}{I^2}\dot{I}, \frac{\dot{S}}{I} - \frac{S}{I^2}\dot{I}, \frac{\dot{S}}{I} - \frac{S}{I^2}\dot{I}\right\}$$

Then, we get

$$P_f P^{-1} = diag\left\{\frac{\dot{S}}{S} - \frac{\dot{I}}{I}, \frac{\dot{S}}{S} - \frac{\dot{I}}{I}, \frac{\dot{S}}{S} - \frac{\dot{I}}{I}\right\}.$$

And,

$$P_f J^{[2]} P^{-1} = P_f \begin{pmatrix} -\frac{(1-\rho)}{b} I - \gamma + \frac{(1-\rho)}{b} S - (\gamma + \delta) & 0 & 0 \\ \delta & -\frac{(1-\rho)}{b} I - 2\gamma & -\frac{(1-\rho)}{b} S \\ 0 & \frac{(1-\rho)}{b} I & -\frac{(1-\rho)}{b} S - (\gamma + \delta) - \gamma \end{pmatrix} P^{-1}.$$

Therefore, we write,

$$K = P_f P^{-1} + P_f J^{[2]} P^{-1} = \begin{pmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{pmatrix}.$$

Where,

$$K_{11} = \frac{\dot{S}}{S} - \frac{\dot{I}}{I} - \frac{(1-\rho)}{b} I - \gamma - (\gamma + \delta) + \frac{(1-\rho)}{b} S,$$

$$K_{12} = 0, \quad K_{21} = \delta,$$

$$K_{22} = \begin{pmatrix} \frac{\dot{S}}{S} - \frac{\dot{I}}{I} - \frac{(1-\rho)}{b} I + \frac{(1-\rho)}{b} S - \gamma & \frac{(1-\rho)}{b} S \\ \frac{(1-\rho)}{b} I & \frac{\dot{S}}{S} - \frac{\dot{I}}{I} + \frac{(1-\rho)}{b} S - (\gamma + \delta) \end{pmatrix}$$

Considering the norm in R^3 as:

$|(a,b,c)| = \max\{|a|, |b| + |c|\},$

Here, (a, b, c) represents the vector of R^3 and ℓ is the Lozinskii measure associated to the norm defined above [18].

$$\ell(K) \le Sup\{h_1, h_2\},$$

$$= Sup\left\{\ell(K_{11}) + |K_{12}|, \ell(K_{22}) + |K_{21}|\right\}$$

$$|K_{12}| = 0, \ |K_{21}| = \delta.$$

The Lozinskii measure $\ell(K_{11})$ and $\ell(K_{22})$ are:

$$\ell(\mathrm{K}_{11}) = \frac{\dot{S}}{S} - \frac{\dot{I}}{I} - \frac{(1-\rho)}{b} I - \gamma - (\gamma + \delta) + \frac{(1-\rho)}{b} S,$$

$$\ell(\mathrm{K}_{22}) = max\left\{\left(\frac{\dot{S}}{S} - \frac{\dot{I}}{I} - \frac{(1-\rho)}{b} I + \frac{(1-\rho)}{b} S - \gamma\right)\left(\frac{\dot{S}}{S} - \frac{\dot{I}}{I} + \frac{(1-\rho)}{b} S - (\gamma + \delta)\right)\right\}$$

$$-min\left(\frac{(1-\rho)}{b} S \frac{(1-\rho)}{b} I\right)$$

And,

$$(n,k) = \left(\frac{(1-\rho)}{b} S \frac{(1-\rho)}{b} I\right)$$

Therefore, we write h_1 as:

$$g_1 = \ell(\mathrm{K}_{11}) + |\mathrm{K}_{12}|,$$

$$= \frac{\dot{S}}{S} - \frac{\dot{I}}{I} - \frac{(1-\rho)}{b} I - \gamma - (\gamma + \delta) + \frac{(1-\rho)}{b} S,$$

$$= \frac{\dot{S}}{S} - \frac{(1-\rho)}{b} I - \gamma, = \frac{\dot{S}}{S} - \gamma$$

By using the system (2), with

$$\frac{\dot{I}}{I} = \frac{(1-\rho)}{b} S - (\gamma + \delta),$$

h_2 is given by:

$$h_2 = \ell(\mathrm{K}_{22}) + |\mathrm{K}_{21}|,$$

$$= \frac{\dot{S}}{S} - \frac{\dot{I}}{I} + \delta - min\{n,k\}$$

Using,

$$\frac{\dot{I}}{I} = \frac{(1-\rho)}{b} S - (\gamma + \delta),$$

we get,

$$h_2 = \frac{\dot{S}}{S} - \frac{(1-\rho)}{b} S - \gamma - min\{n,k\},$$

$$\leq \frac{\dot{S}}{S} - \gamma$$

Further, we obtain,

$$\ell(\mathrm{K}) \leq \{\mathrm{h}_1, \mathrm{h}_2\}$$

$$\leq Sup\left\{\frac{\dot{S}}{S} - \frac{(1-\rho)}{b} I - \gamma, \frac{\dot{S}}{S} - \frac{(1-\rho)}{b} S - min(n,k) - \gamma\right\},$$

$$\leq \frac{\dot{S}}{S} - \gamma,$$

Then,

$$q = \frac{1}{t}\int_0^t \ell(B) ds \leq \frac{1}{t}\int_0^t \left(\frac{\dot{S}}{S} - \gamma\right) ds = \frac{1}{t} log \frac{S(t)}{S(0)} - \gamma .$$

This implies $q = -\frac{\gamma}{2} < 0$. This is based on [18]. Thus, the (EE) point is stable globally asymptotically at E_1 of the system (2).

4. DISCUSSION

Here, we find the numerical solution of the system (2), with the non-negative initial conditions for (S=20, I=10, R=5). The parameters values used in these simulations are presenting in **Table 1.** The numerical results for the system (2) are presented in Figures 1 to 5. In this paper, an SIR model with the new non-linear saturated incidence has been presented. The endemic equilibrium point is $E^1 = (S^*, I^*, R^*) = (0.1320,\ 0.9700,\ 0.0970)$. We found that the model stability depends upon the basic reproduction number R_o. The disease free or infection free equilibrium found to be stable when the reproduction number less than unity. Also, the disease free equilibrium for the system (2), found to be globally stable when the reproduction number less than 1. The reproduction number also called the threshold quantity gives an average number of secondary infections, when a single infection is introduced in a purely susceptible population. The stability of the system greatly influenced when the value of the reproduction number less than unity, equal to one or greater than 1. In such a case when the basic reproduction number less than unity for disease free equilibrium the disease dies out from the population and it may be control by some specific prevention or vaccination. When the reproduction number exceeds unity the local stability of endemic equilibrium exists. That's means that the disease may spread in the community and further infections produces. For, our model, we found that, when the reproduction number exceeds unity, the disease exists in the community and become epidemic. Similarly, the global stability of endemic equilibrium exists when R_o >1. The threshold quantity of the model examines the stability of the disease free and endemic equilibrium is proved. Further, we proved that the disease-free and endemic equilibrium is stable globally asymptotically. Finally, the numerical results of the model justified our theoretical results.

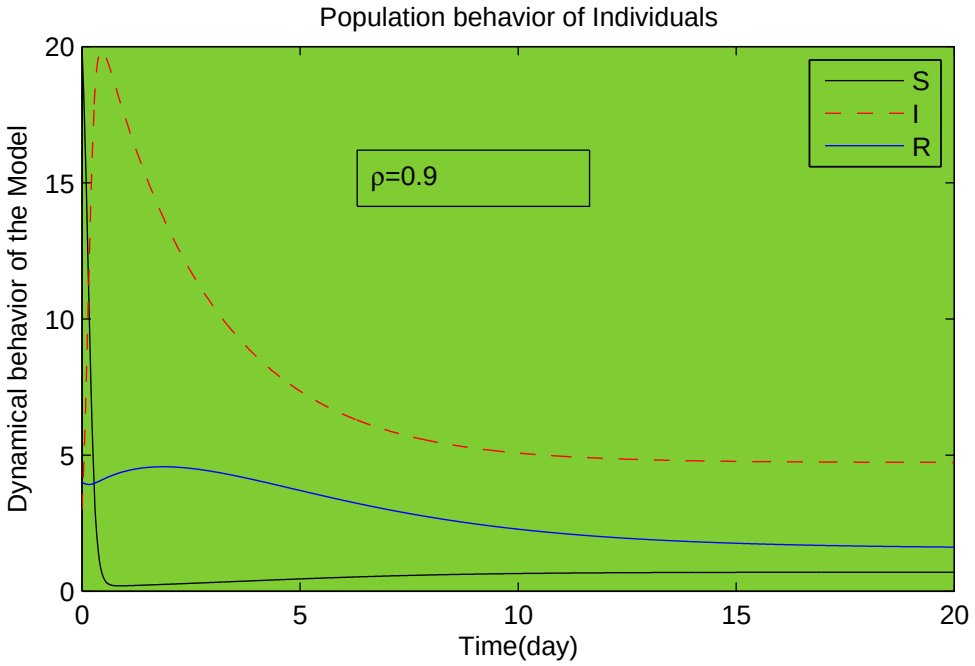

Figure 1: The Population behavior of individuals when $\rho = 0.9$.

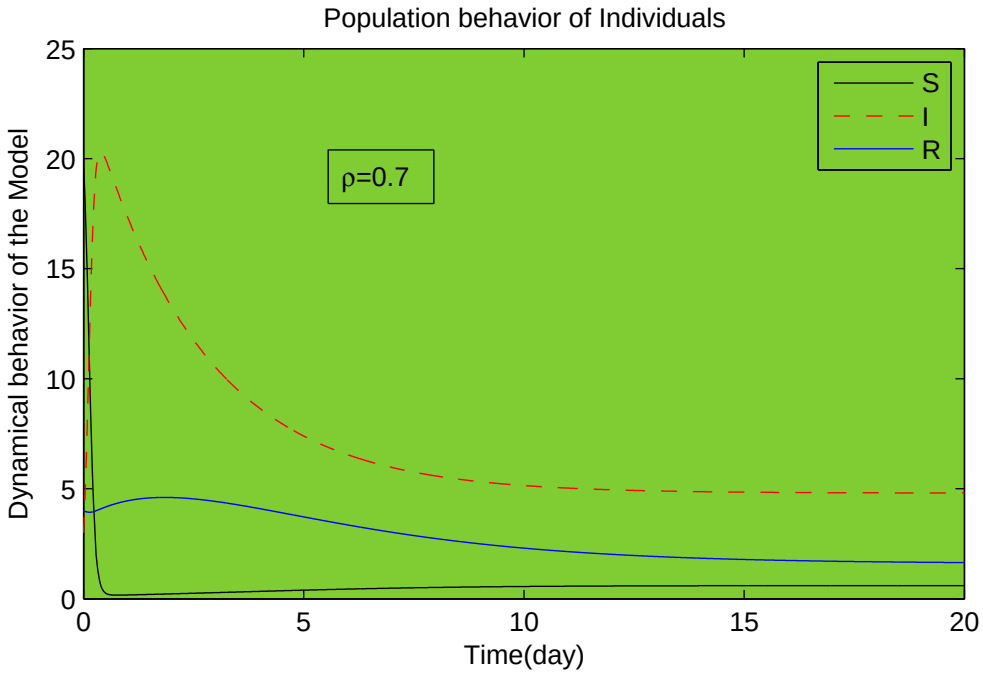

Figure 2: The Population behavior of individuals when $\rho = 0.7$.

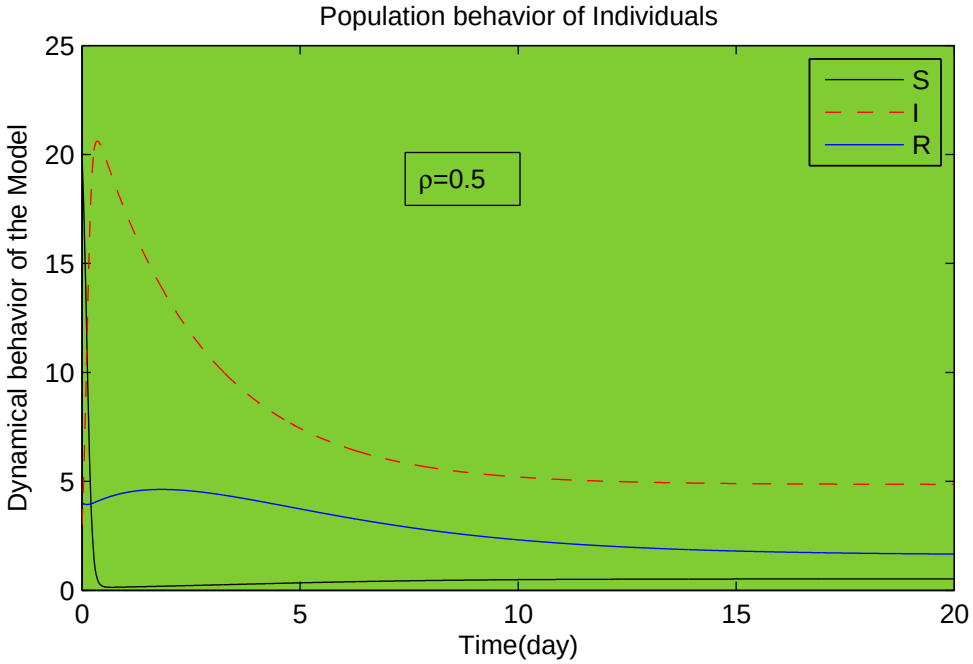

Figure 3: The Population behavior of individuals when $\rho = 0.5$.

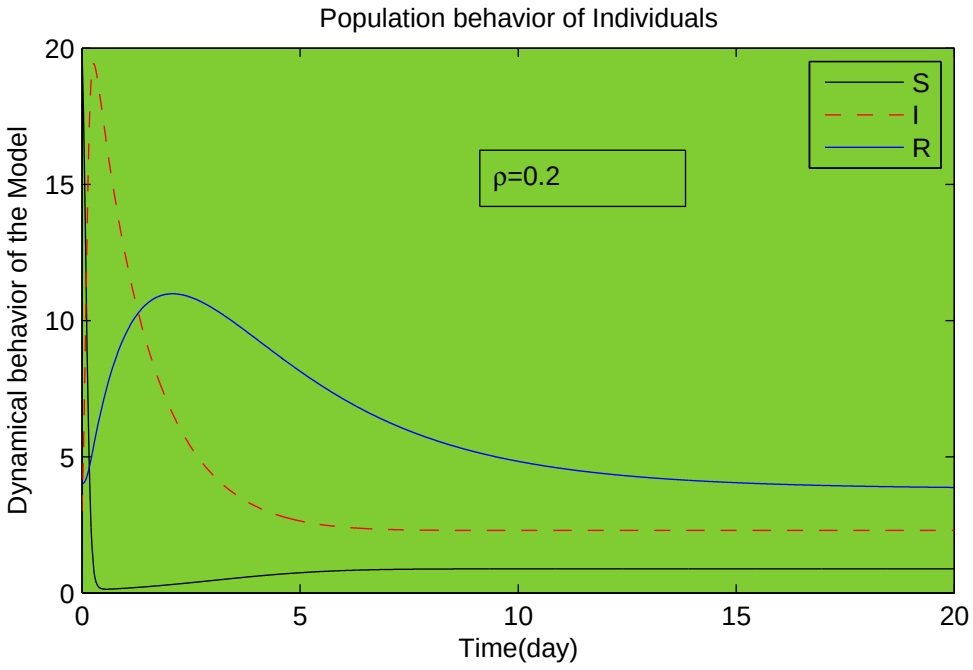

Figure 4: The Population behavior of individuals when $\rho = 0.2$.

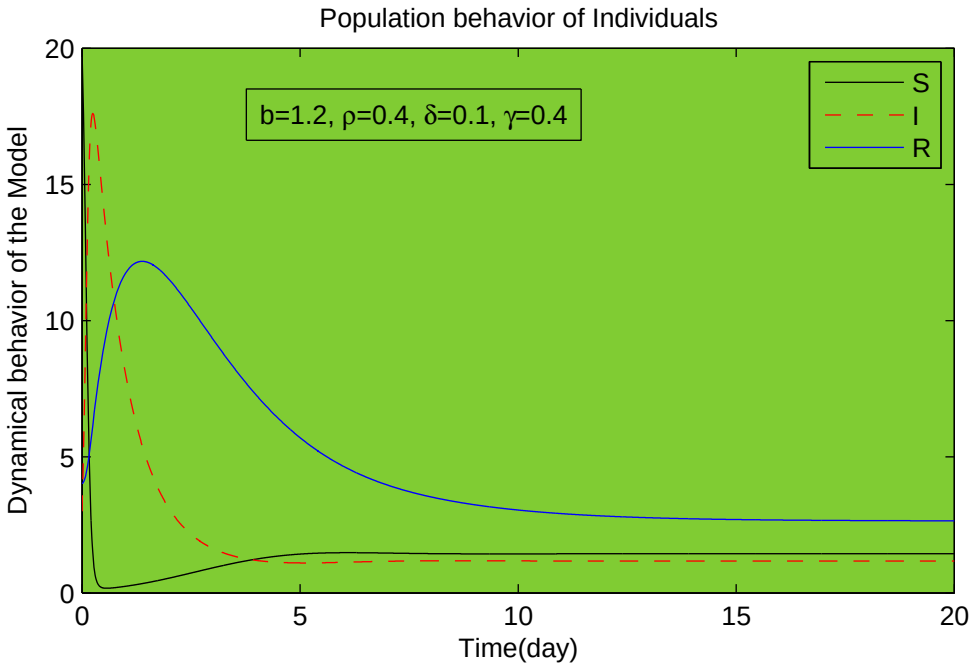

Figure 5: The Population behavior of individuals when $\rho = 0.4, b = 1.2, \delta = 0.1, \gamma = 0.4$.

Table 1: Parameters Values

Notation	Parameter description	Range	Source
γ	Natural death rate	0.03	Assumed
b	Represent the birth rate	1-5	Assumed
ρ	Fraction of infected individuals	$\frac{1}{20}-\frac{1}{60}$	Assumed
δ	Rate of recovery	0.003	Assumed

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