J. Basic. Appl. Sci. Res., 4(6)68-76, 2014
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ISSN 2090-4304
Journal of Basic and Applied Scientific Research
www.textroad.com

A Mathematical Model for the Rods with Heat Generation and Convective Cooling

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Received: December 5, 2013
Accepted: January 10, 2014

ABSTRACT

In this paper, the problem of the temperature distribution and the heat transfer from the rods with variable thermal conductivity keeping steady conduction is solved analytically. Three cases of the heat transfer from the rod with constant temperature at ends of the rod, the heat transfer from the rod with insulated ends and the heat transfer from the rod with convection are discussed here.
KEY WORDS: Heat Transfer, Variable thermal conductivity, convection, homotopy Perturbation Method

1 INTRODUCTION

We know that most of the engineering problems of the heat transfer are nonlinear. These problems are difficult to solve. Some problems are solved using numerical techniques and some are solved analytically by using traditional perturbation methods. There are many different methods to solve nonlinear problems such as artificial small parameter method introduced by J. H. He [1, 2], the homotopy analysis method by Lio [3, 5], the variational iteration technique by J. H. He and the Homotopy Perturbation Method (HPM) introduced by J. H. He [6]-[10]. The Homotopy Perturbation Methods is a new analytical method to solve nonlinear problems and it is applicable in different fields of non linear problems, integro differential equation, laplace transforms, fluid mechanics and the heat transfer.

In the Recent years, the heat transfer researchers are trying to understand the heat transfers in various material systems, such as solidifying process and melting, surface thermal processing by lasers, temperature control of superconductors, laser surgery and freezing, metal cutting, welding, rolling and food processing. Other areas of heat transfers are in power production, chemical and metallurgical industry, heat and air conditioning of building, design of combustion engine and design of electrical machinery.

In [2], the problem of non-linear convective-radiative non-Fourier conduction heat transfer equation with variable specific heat coefficient has been studied by applying homotopy perturbation method which was proved to be more effective and convenient as compared with the results obtained by the fourth order Runge-Kutta method in order to verify the accuracy of the proposed method. The homotopy perturbation method also proved its reliability and efficiency in solving the temperature distribution in lumped system of combined convection–radiation. Also the proposed method was applied to a nonlinear equation of the steady conduction in a slab with variable thermal conductivity [1]. A comparison of traditional perturbation method and the homotopy perturbation method for the heat transfer problem of conduction heat transfer with the variable physical properties, and a comparison with the exact solution is also made in [3]. Also the numerical solution of the heat radiation and conduction equations of a fin in the steady state and in the free space is compared with the homotopy perturbation method which got a much higher accuracy [3].

One of the interesting applications of heat transfer is the model of lumped and distributed systems with heat generation due to harmonic. The numerical results are presented for a long, thin, visco-elastic rod with hysteretic damping [11]. Another parametric study of the influence of the variable heat transfer coefficient, the fin geometry and the surface curvature on fin efficiency was studied for various boundary layer conditions and a generalized analytical solution which computes the loss of heat loss extended surface based on these parameters [13]. We have also given a comparison of different perturation techniques with illustrative examples in [14-19].

One of the main applications of the heat transfer study is to increase the rate of heat transfer from the rods with variable thermal conductivity. The rods are in cylindrical form or tapered rods. Rods are widely used in car radiator, heat exchangers, air cooled engines. Most of rods problems are solved analytically and numerically keeping constant thermal conductivity.

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In this paper, our aim is the study of the problem of the temperature distribution and the heat transfer from the rods with variable thermal conductivity keeping steady conduction.
We will emphasize on the three cases of the heat transfer;

- The heat transfer from the rod with constant temperature at ends of the rod
- The heat transfer from the rod with insulated ends
- The heat transfer from the rod with convection

We have found the analytical solutions using the homotopy perturbation technique. Like the other traditional perturbation techniques, this techniques does not require the assumption of small or large parameter in the equation at all. Instead, a homotopy is constructed with an embedding parameter which is considered to be a small parameter. The comparisons of this technique with other traditional perturbation methods shows that the approximations obtained by the this method are uniformly valid not only for small parameters but also for large parameters. The solution which is considered as the sum of an infinite series converges rapidly to accurate solutions. For the basic ideal of homotopy perturbation method, we will refer the readers to [6]-[10].

2 Problem Formulation

We consider the steady conduction in a constant cross-section rod with area A, perimeter P, radius a and length L. The thermal conductivity k is variable and the strength of the distributed heat sources is constant. There is convective cooling at the lateral boundary through a constant heat transfer coefficient h into a constant fluid at temperature T. The steady state heat balance equation for the rods with heat generation and variable thermal conductivity is given by the following equation

$$-\frac{dq_x}{dx}=\frac{ph}{bw}(T-T_a)-S, \tag{1}$$

By Fourier law

$$q_x=-k\frac{dT}{dx}. \tag{2}$$

By substitution equation (2) in equation (1) we get

$$\frac{d}{dx}\left[k\frac{dT}{dx}\right]=\frac{ph}{A}(T-T_a)-S, \tag{3}$$

where

$$k=K_a[1+\beta(T-T_a)]. \tag{4}$$

Let dimensionless parameters are

$$\theta=\frac{T-T_a}{T_w-T_a}, \qquad x^*=\frac{x}{L}, \tag{5}$$

and

$$k=K_a[1+\beta\theta]. \tag{6}$$

The dimensionless equation becomes

$$\frac{d}{dx^*}\left[\left(1+\beta\theta\right)\frac{d\theta}{dx^*}\right]=M^2\theta-A^2, \tag{7}$$

where

$$\frac{L^2ph}{k_aA}=m^2L^2=M^2, \tag{8}$$

and

$$\frac{L^2S}{k_a(T_w-T_a)}=A^2, \tag{9}$$

Thus the dimensionless governing equation is

$$\frac{d}{dx}\left[\left(1+\beta\theta\right)\frac{d\theta}{dx}\right]=M^2\theta-A^2. \tag{10}$$

2.1 Heat Transfer for the Rods with Variable Thermal Conductivity with Constant Temperature at Ends

As rod is very large and the temperature at the ends is the temperature of the surroundings. The dimensionless BVP is

$$\frac{d}{dx}\left[(1+\beta\theta)\frac{d\theta}{dx}\right]=M^2\theta-A^2, \tag{11}$$

subject to

$$\theta(0)=1, \tag{12}$$

$$\theta(1)=0. \tag{13}$$

2.2 Solution by HPM

For basic idea of HPM, we refer [6]-[10]

$$L[\theta]=\frac{d^2\theta}{dx^2}-M^2,\theta \tag{14}$$

$$N[\theta]=\beta\theta\frac{d^2\theta}{dx^2}+\beta\left[\frac{d\theta}{dx}\right]^2, \tag{15}$$

$$F(x)=-A^2. \tag{16}$$

First we will find the initial guess for which we will equate linear part of the equation to $F(x)$.

$$\frac{d^2\theta}{dx^2}-M^2\theta=-A^2, \tag{17}$$

With BC's

$$at \quad x=0, \quad \theta=1, \tag{18}$$

$$at \quad x=1, \quad \theta=0. \tag{19}$$

Now solving equation (17) and considering equation (18) and equation (19), we get the solution as

$$C_1e^{Mx}+C_2e^{-Mx}+\frac{A^2}{M^2}, \tag{20}$$

which is the required initial guess.

Now we construct a Homotopy

$$\theta(x,q):\quad \Omega\times[0,1]\to R, \tag{21}$$

for equation (12) which satisfies the equation

$$(1-q)[\ L(\theta)-L(u_0)\ ]+q\left[\beta\theta\frac{d^2\theta}{dx^2}+\beta\left[\frac{d\theta}{dx}\right]^2+A^2\right]=0, \tag{22}$$

where $q\in[0,1]$ is an embedding parameter, u_0 is the initial approximation. Therefore equation (22) can be expressed as

$$L[\theta]-L[u_0]+q[u_0]+q\left[\beta\theta\frac{d^2\theta}{dx^2}+\beta\left[\frac{d\theta}{dx}\right]^2+A^2\right]=0, \tag{23}$$

with BC's

$$at \quad x=0, \quad \theta=1, \tag{24}$$

$$at \quad x=1, \quad \theta=0. \tag{25}$$

We assume that equation (12) has a solution of the form

$$\theta(x,q)=\theta_0+q\theta_1+q^2\theta_2+\cdots, \tag{26}$$

By substituting equation (26) in equation (23) and equating the power of q we obtain the following system of ODE's.

Zero Order Problem

The differential equation of Zero Order is

$$L[\theta_0]-L[u_0]=0, \tag{27}$$

with BC's

$$at \quad x=0, \quad \theta_0=1, \tag{28}$$

$$at \quad x=1, \quad \theta_0=0. \tag{29}$$

Ist Order Problem

The differential equation of first Order is

$$L[\theta_1]+L[u_0]+\beta\theta_0\frac{d^2\theta_0}{dx^2}+\beta\left[\frac{d\theta_0}{dx}\right]^2+A^2=0, \tag{30}$$

with BC's

$$at \quad x=0, \quad \theta_1=0, \tag{31}$$

$$at \quad x=1, \quad \theta_1=0. \tag{32}$$

Second Order Problem

The differential equation of second Order is

$$L[\theta_2]+\beta\theta_0\frac{d^2\theta_1}{dx^2}+\beta\theta_1\frac{d^2\theta_0}{dx^2}+2\beta\left(\frac{d\theta_1}{dx}\right)\left(\frac{d\theta_0}{dx}\right)=0 \tag{33}$$

with BC's

$$at \quad x=0, \quad \theta_2=0, \tag{34}$$

$$as \quad x=1, \quad \theta_2=0. \tag{35}$$

Now the solution of Zero and First Order problems are

$$\theta_0(x)=C_1e^{Mx}+C_2e^{-Mx}+\frac{A^2}{M^2}, \tag{36}$$

$$\theta_1(x)=C_3e^{Mx}+C_4e^{-Mx}$$
$$-\beta\left\{\frac{2}{3}(C_1^2e^{2Mx}+C_2^2e^{-2Mx})+\frac{A^2}{2M}(C_1e^{Mx}-C_2e^{-Mx}).\right\} \tag{37}$$

Where

$$C1=1-C_2-\frac{A^2}{M^2},$$

$$C2=\frac{1}{e^{-M}-e^{M}}\left\{\frac{A^2}{M^2}(e^M-1)-e^M)\right\},$$

$$C3=\frac{2}{3}\beta(C_1^2+C_2^2)-C_4,$$

$$C4=\frac{1}{e^M-e^{-M}}\left\{\frac{2}{3}\beta e^M(C1^2+C2^2)\right\}$$
$$-\frac{\beta}{e^M-e^{-M}}\left\{\frac{2}{3}(C1^2e^{2M}+C2^2e^{-2M})+\frac{A^2}{2M}(C1e^M-C2^2e^{-M})\right\}.$$

Hence the temperature distribution is

$$\theta=\lim_{q\to1}\left(\theta_0+q\theta_1+q^2\theta_2+\cdots\right). \tag{38}$$

Now using the values of θ_0 and θ_1 and in equation (38) we get the required temperature distribution. Graphically, it is shown that as the length increases, temperature approaches to ambient temperature.

2.3 Heat Transfer from Finite Rod Having Insulated Tip with Variable Thermal Conductivity

As the rod is long and the heat is neglected at the tip, so boundary conditions at the $x = L$ becomes

$$\frac{d}{dx}\left[(1+\beta\theta)\frac{d\theta}{dx}\right]=M^2\theta-A^2, \tag{39}$$

subject to

$$\theta(0)=1, \tag{40}$$

$$\frac{d\theta(1)}{dx}=0. \tag{41}$$

First of all we will find initial guess

$$\frac{d^2\theta}{dx^2}-M^2\theta=-A^2, \tag{42}$$

With BC's

$$at \quad x=0, \quad \theta=1, \tag{43}$$

$$at \quad x=1, \frac{d\theta_0}{dx}=0. \tag{44}$$

Now solving equation (42) considering equation(43) and equation (44) we get the solution as

$$u_0=C_1e^{Mx}+C_2e^{-Mx}+\frac{A^2}{M^2}. \tag{45}$$

is the required initial guess.

Now by applying the same procedure as in the first case, we get the dimensionless solution of Zero and First Order problems as:

$$\theta_0(x)=C_1e^{Mx}+C_2e^{-Mx}+\frac{A^2}{M^2}, \tag{46}$$

$$\theta_1(x)=C3e^{Mx}+C4e^{-Mx}-\beta\left\{\frac{2}{3}C_1^2e^{2Mx}+\frac{2}{3}C_2^2e^{-2Mx}\right\}$$

$$-\beta\left\{\frac{A^2}{2M}C_1xe^{Mx}-\frac{A^2}{2M}C_2xe^{-Mx}\right\}. \tag{47}$$

where

$$C_1=C_2e^{-2M},$$

$$C_2=\frac{1}{1+e^{-2M}}\left\{1-\frac{A^2}{M^2}\right\},$$

$$C_3=\frac{2}{3}\beta(C_1^2+C_2^2)-C_4,$$

$$C_4=\frac{1}{M(e^M+e^{-M})}\left\{\frac{2}{3}\beta Me^M(C_1^2+C_2^2)\right\}$$

$$-\frac{\beta}{M(e^M+e^{-M})}\left\{\frac{4M}{3}(C_1^2e^{2M}-C_2^2e^{-2M})+\frac{A^2}{2M}C_1e^M(M+1)+\frac{A^2}{2M}C_2e^{-M}(M-1).\right\}$$

Hence the temperature distribution is

$$\theta=\lim_{q\to 1}\left(\theta_0+q\theta_1+q^2\theta_2+\cdots\right) \tag{48}$$

Now putting the values of θ_0 and θ_1 in equation (48) we get the required temperature distribution.

2.4 Heat Transfer from Finite Rod with Convection Along with Variable Thermal Conductivity

In this case nondimensional equation and the nondimensional boundary conditions are

$$\frac{d}{dx}\left[(1+\beta\theta)\frac{d\theta}{dx}\right]=M^2\theta-A^2 \tag{49}$$

Subject to

$$\theta(0)=1 \tag{50}$$

$$\frac{d\theta(1)}{dx}=-B_i\theta \tag{51}$$

First of all we will find initial guess

$$\frac{d^2\theta}{dx^2} - M^2\theta = -A^2 \tag{52}$$

with BC's

$$at \quad x = 0, \quad \theta = 1 \tag{53}$$

$$at \quad x = 1, \frac{d\theta_0}{dx} = -B_i\theta \tag{54}$$

Now solving equation (52) and considering equation (53) and equation (54) we get the solution as

$$u_0 = C_1 e^{Mx} + C_2 e^{-Mx} + \frac{A^2}{M^2} \tag{55}$$

is the required initial guess.

Now solution by HPM is

2.5 Zero Order Problem

The differential equation of Zero Order is

$$L[\theta_0] - L[u_0] = 0 \tag{56}$$

With BC's

$$at \quad x = 0, \theta_0 = 1 \tag{57}$$

$$at \quad x = 1, \frac{d\theta_0}{dx} = -B_i\theta \tag{58}$$

2.6 Ist Order Problem

$$L[\theta_1] + L[u_0] + \beta\theta_0 \frac{d^2\theta_0}{dx^2} + \beta\left[\frac{d\theta_0}{dx}\right]^2 = 0 \tag{59}$$

With BC's

$$at \quad x = 0, \theta_1 = 0 \tag{60}$$

$$at \quad x = 1, \frac{d\theta_1}{dx} = -B_i\theta \tag{61}$$

Now the dimensionless solutions of Zero, and First Order problems are

$$\theta_0(x) = C_1 e^{Mx} + C_2 e^{-Mx} + \frac{A^2}{M^2} \tag{62}$$

$$\theta_1(x) = C_3 e^{Mx} + C_4 e^{-Mx} - \beta\left\{\frac{2}{3}C_1^2 e^{2Mx} + \frac{2}{3}C_2^2 e^{-2Mx}\right\}$$

$$-\beta\left\{\frac{A^2}{2M}C_1 x e^{Mx} - \frac{A^2}{2M}C_2 x e^{-Mx}\right\} \tag{63}$$

Where

$$C1 = 1 - C_2 - \frac{A^2}{M^2}$$

$$C2 = \frac{1}{-M(e^M + e^{-M}) + B_i(e^{-M} - e^M)}\left\{-Me^M(1 - A^2/M^2)\right\}$$

$$-\frac{B_i}{-M(e^M + e^{-M}) + B_i(e^{-M} - e^M)}\left\{(1 - \frac{A^2}{M^2})e^M + \frac{A^2}{M^2}\right\}$$

$$C3 = \frac{2}{3}\beta(C_1^2 + C_2^2) - C_4$$

$$C4 = \frac{1}{-M(e^M + e^{-M}) + B_i(e^{-M} - e^M)}\left\{\frac{-2M}{3}\beta e^M(C1^2 + C2^2)\right\}$$

$$-\frac{\beta}{-M(e^{M}+e^{-M})+B_i(e^{-M}-e^{M})}\left\{\frac{4M}{3}(C_1^2e^{2M}-C_2^2e^{-2M})+\frac{A^2}{2M}C1e^{M}(M+1)+\frac{A^2}{2M}C_2e^{-M}(M-1)\right\}$$

$$-\beta B_i\frac{1}{-M(e^{M}+e^{-M})+B_i(e^{-M}-e^{M})}\left\{\frac{2}{3}e^{M}(C1^2+C2^2)+\frac{2}{3}(C_1^2e^{2M}+C_2^2e^{-2M})+\frac{A^2}{2M}(C_1e^{M}-C_2e^{-M})\right\}$$

Hence the temperature distribution is

$$\theta=\lim_{q\to 1}\left(\theta_0+q\theta_1+q^2\theta_2+\cdots\right) \tag{64}$$

Now using the values of θ_0 and θ_1 in equation (64) and putting q equal to 1 we get the required temperature distribution.

3 RESULTS AND DISCUSSION

Fig. 1 represents the temperature distribution for the rod with constant temperature at ends with different values of thermal expansion or contraction parameter β and for fixed value of M. Also fig 1. shows that for zero value of β, the temperature approaches to ambient temperature. We have given three different values to β as 0. 1 and -1. The results are displayed in the graph.

In Fig. 2, the temperature distribution for rod with insulated ends with different values of thermal expansion or contraction parameter β is represented and value of M is fixed. Three different graphs are representes for different values of the conduction papameter β.

Fig 3 represents the temperature distribution for rod with convection with different values of thermal expansion or contraction parameter β and for fixed value of M. Fig. 2 and fig. 3 represent that solution temperature approaches to ambient temperature for positive value of β, whereas this is not in the case with the rods with the constat temerature at the ends.

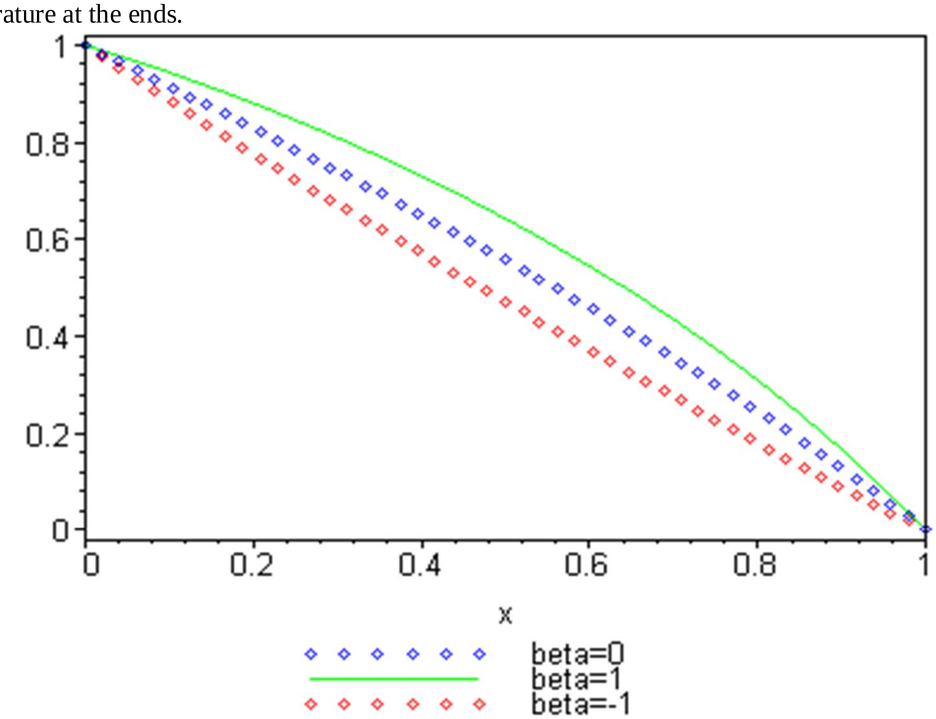

Figure 1: Temperature distribution for rod having constant temperature at ends with different values of β keeping M fixed

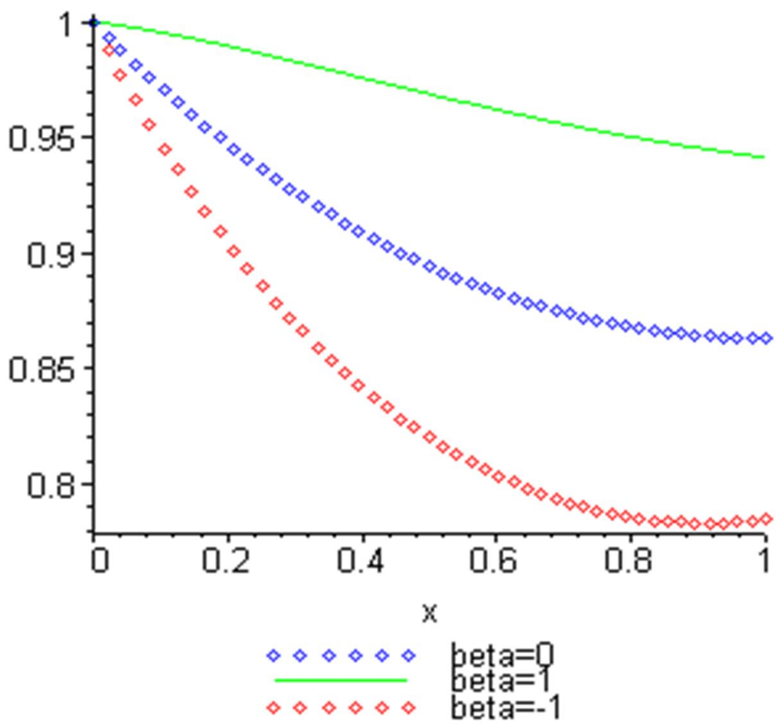

Figure 2: Temperature distribution for rod with insulated end with different values of β keeping M fixed

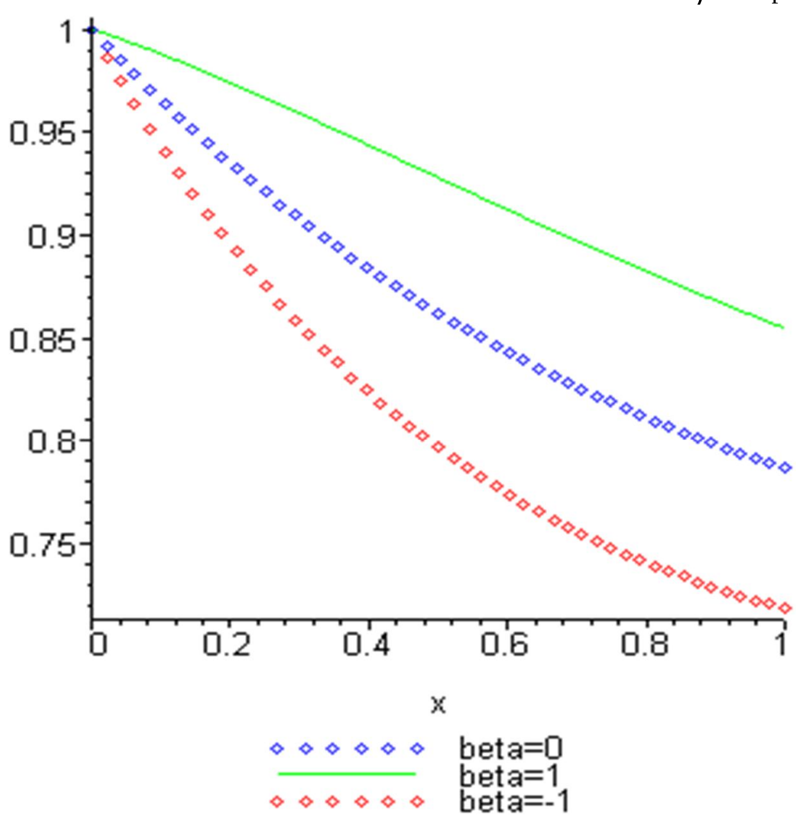

Figure 3: Temperature distribution for rods with convection with different values of β and fixed value of M

4 SUMMARY AND CONCLUSIONS

The Homotopy perturbation method is a powerful mathematical tool to solve nonlinear equations so this method is used to solve the well known energy equation for the rods. Three types of boundary conditions are used to solve the governing equations using HPM. The HPM supplies reliable results in the form of an analytical approximation converging very rapidly . This method provides simply an approximate solution without any assumptions of linearization. This character is very important for systems with strong nonlinearities which could be extremely sensitive to small changes in parameter. It would be useful to apply this method to a variety of nonlinear heat conduction problems and helpful for engineer to analyze nonlinear system . In our work, we have seen that we get very reliable results as compared to other analytical and numerical techniques. The solution series is convergent and gives better approximations as shown in the figures. It would be useful to apply this method to a variety of nonlinear heat conduction problems, and helpful for engineer to analyze highly nonlinear system.

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