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An Alternative Solution for Inversion of Laplace, Comparison with Conventional ILT Method

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ABSTRACT

In this paper a method for solving inverse Laplace transform based on the concept of limit is presented. The inverse problems are difficult to evaluate because it involves evaluation of complex integrals. The focus of this paper is to avoid the use of extensive properties and tables used for solving the problems. The main goal is to find inverse Laplace transform using the proposed technique. After comparing results of both proposed and conventional technique, result shows that the proposed technique is more flexible and adaptive, reduced complexity and takes less computational time as compared to the conventional IL method.

KEY WORDS: Inverse Laplace transform, Fourier transform, Fractional derivatives, Complex domain.

INTRODUCTION

As it is known that numerical inversion of Laplace has a wide range of applications .Numerous mathematical problems can be solved using Laplace transformation methods. Laplace transformation is more reliable than other methods.

Using Laplace inversion method as compare to other numerical method

Interpreting Laplace transformation we can say that it is the extended version of Fourier transform. In every field there is a little bit involvement of this method, so we can't ignore the importance of Laplace transform [20]. Many researchers have used the Laplace transformation technique and Fourier transformation technique in a different form to interpret various effects and to solve a numbers of problems [21, 22].

Extensive tables are currently used to calculate to solve Laplace transform problems, so a universal method is required to solve these particular problems. Various researches tried to solve inverse transform problems but unsuccessful to produce a generic formula/ equation to solve all type of problems.

A method is needed to solve all kind of laplace related problems so,

Let,

F(s) = $\int_{-\infty}^{\infty} e^{-st}F(t)dt$ Eq. (1)

In this case *s* is subset of *ω* of the complex plane. It is observed that *F (s)* is often an analytic function on a half plane {Re*s*> *α*} for a suitable real number *α*. commonly, the image of a Laplace transform is known only on a subset *ω* of the right half plane {Re*s*>*α*}. Depending on the set *ω*, we shall have appropriate methods to construct the function *F* from the values in the set{. *LF (s):* s €*ω*}

Hence, there are no universal methods of inversion of the Laplace transform [1].

If the data *k(s)* is given as a function on a line *(−i∞+a,+i∞+a)* (i.e., *ω* = {*s*: *s*=*a*+*iy y* €**R**}) on the complex plane then we can use the Bromwich inversion formula [2] to find the function *F (t)*.

If *ω* € {*s*€**R**: *s*>0} then we have the problem of real inverse Laplace transform. The right-hand side is known only on *(0,∞)* or a subset of *(0,∞)*. In this case, the use of the Bromwich formula is therefore not feasible. The researches have been made in both theoretical and computational aspects where Bromwich formula fails. (see, e.g., [1,3,4,5]). If F(s) is properly defined then we have many inversion formulas to evaluate problems.(see, e.g., [6,7,8].).Some authors have used the method of Fourier for inversion of Laplace transform [9].

Laplace transform is actually the extension of Fourier transform. In Fourier transform only imaginary part is used while in Laplace transform both real and imaginary parts are used. Laplace transform and inverse transform formula's are given below.

F(s)=$\int_{-\infty}^{\infty} e^{-st}F(t)dt$ Eq. (2)

Laplace inverse

F(t) =$\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{st}F(s)ds$ Eq.(3)

Where, s= a+jω

Fourier transform formula is

F(jω)= $\int_{-\infty}^{\infty} e^{-jwt}F(t)dt$ Eq.(4)

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Fourier inverse transform formula is

$F(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{jwt}F(jw)dw$ Eq. (5)

If we compare equation2 & 4 they are almost same, only real part is missing in eq.4, similarly equation 3 & 5 are similar but in eq.5 real part is missing.

Now we define Fourier integral method that is most closely related to inverse Laplace transform.

Laplace inverse transform is given below,

$F(t) =\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{st}F(s)ds$ Eq. (6) Taking, s= a+jω

$F(t)=\frac{e^{at}}{2\pi}\int_{-\infty}^{\infty} e^{jwt}F(a+jw)dw$ Eq. (7)

Taking, F(t) e^{-at} = x(t) and F(a+jω)= x(ω),

We get,

$G(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{jwt}G(\omega)d\omega$ Eq. (8)

So we can say that inverse Laplace transformation may be performed by inverse Fourier transform [10].

First Abate and Whitt method, Second Abate and Whitt method(see, e.g., [11,12, 13].) and Fourier sine series expansion, Numerical quadrature method [10], are also used for numerical inversion of Laplace transform.

Proposed Method for Laplace Inversion:

We are very well familiar with the concepts of derivatives .The most common notations used for derivatives are

$\frac{df(x)}{dx}$ or $Df(x)$

But what is the meaning of fractional derivative like

$\frac{d^{1/2}f(x)}{dx^{1/2}}$ or $D^{1/2}f(x)$

A vast literature exists on fractional derivatives called "fractional calculus". At graduate level two text books were found on the subject [14], [15]. On fractional derivatives the paper delivered at conferences are [16] and [17]. Wheeler [18] has prepared seminar notes on the subject of fractional calculus, but these have not been published.

Fractional derivatives of exponential functions can be written as

$D^{\alpha}e^{ax}=a^{\alpha}e^{ax}$ (a)

From eq.(a) fractional derivative of e^{x} is

$D^{\alpha}e^{x}=e^{x}$ (b)

But from Taylor series,

$$e^{x} = \sum_{n=0}^{\infty}\frac{1}{n!}x^{n}$$

This series gives,

$$D^{\alpha} e^{x} = \sum_{n=0}^{\infty}\frac{x^{n-a}}{(n-\alpha+1)} \qquad (C)$$

Eq. (a) and Eq. (b) have same results when α is a whole number but when $\alpha$is not a whole number, the results are different. This mystery is solved by introducing a new concept that a fractional derivative involves limits [19]. The reason of contradiction of eq.(b) and eq.(c) is that there were two different limits of integration being used. A type of fractional derivative in which the lower limit is –∞ is called Weyl fractional derivative.

$_{-\infty}D_{x}^{\alpha}e^{x} = a^{\alpha}e^{ax}$

In this paper we have used the Weyl fractional derivative in a gentle manner.

The Laplace inverse transform is an ill-posed problem, it is difficult to find the signal in time domain using inverse Laplace transform formula. So my area of interest is to provide an efficient method and a simple numerical technique to transform the signal in time domainand to find the limit of the signals in both time-domain and s-domain by using simple mathematical techniques. We have used the concept of limit in the formula. Our main goal is to find the value of "s" (for the signal) and the value of "t" (for the limit of signal in time domain). The time domain and s domain are both related to each other, we have to extract all the information from the given problem. The formula is,

$F(t) = {}_{Ds^{v}{}_{\frac{-}{\infty}}{}^{1}} * \text{Limit}_{s\to n} e^{s(t+t0)} \frac{u(t+t0)}{\Gamma(v)}$ Where $D = \frac{d}{ds}$

Before describing the main part of the paper, we define some rules that are used in solving the problem.

FORMULA:

$F(t) = {}_{Ds^{v}-1} * \text{Limit}_{s->} n e^{s(t+t0)} \frac{u(t+t0)}{\Gamma(v)}$

Where D = $\frac{-d}{ds}$

u(t) in the formula indicates the range in which the signal will affect the system.
The above mentioned formula is applicable to the problem's given below

$$\frac{1}{(s+a)^{V}}$$

Where a is constant it may be integer or a decimal number.To obtain this form we have to apply some mathematical techniques like quadratic formula and partial fraction.

RULE 1: If the degree of denominator is 1. i.e. V=1.then the formula will reduce to Limit $_{s->n} e^{s(t+t0)}$ u(t+t_0). Find the value of 's' and put it into the formula.

RULE 2: If the degree of denominator is greater than 1,i.e V>1 then put the value of 'V' in the formula it will shows the no. of derivative of the formula, take the derivatives of formula and then put the value of 's' in the formula. e.g.{1/$(s+6)^2$ in this case V=2, put the value of V in the formula, now take one derivative of the formula and then by simply putting the value of 's' in the formula the required result is obtained.

A kind of problems like that $\frac{1}{(s2+2)2}$are not solved directly, first apply quadratic formula, find the roots, use partial fraction technique now we have four terms, now by applying the formula the required result is obtained. In all kinds of problems, first use simple mathematical techniques and then apply the formula according to the rules.

RULE 3: Ifthe value of "t"is given in the formula then it means that it will show the limit of a signal in time domain. E.g.{$e^{-5s}/(s+2)^3$ in this case the value of " t " is -5.so the limit of signal in time domain becomes u(t-5).}. in every problem t=t+t_0. It can-not be applied for complex exponential functions.

SIMULATION RESULT

Here we present some examples to compare our method with the inverse Laplace transform.

PROBLEM 1:

1) $\frac{1}{s+2}$

In this case V=1. so the formula reduce to

Limit $_{s->n} e^{s(t+t0)}$u(t+t_0)

Where, s+2=0

S= -2; t_0= 0;

Limit $_{s->-2} e^{s(t+t0)}$u(t+0)

e^{-2t}u(t).

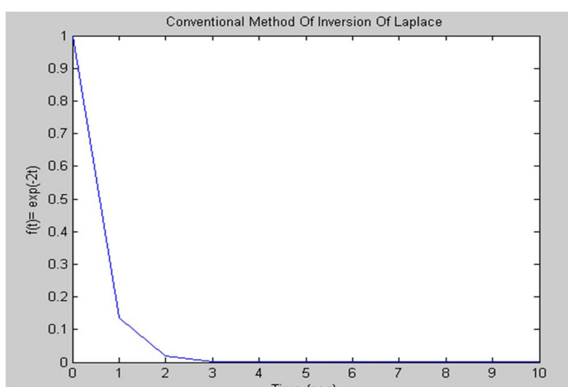

Figure 1: Conventional Method of Inversion of Laplace

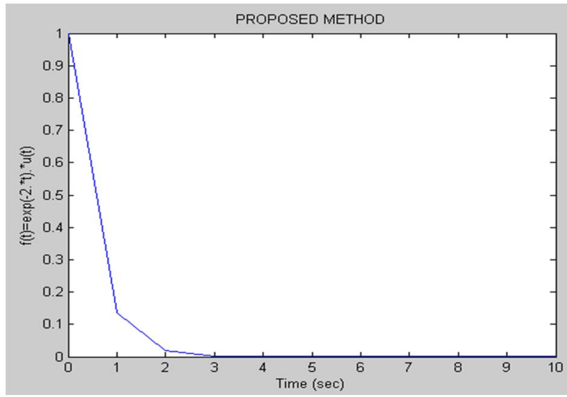

Figure 2: Proposed Method

PROBLEM 2:

2) $\frac{1}{(s+6)^6}$

The value of V>1 means we have to take derivatives of formula.

V=6.

Formula is

$$F(t) = {Ds^{v}_{-\infty}}^{1} * \boldsymbol{Limit}_{s\to n}\, \mathbf{e}^{s(t+t0)}\, \frac{\mathbf{u(t+t0)}}{\boldsymbol{\Gamma}(\mathbf{v})}$$

$$F(t) = {Ds^{v}_{-\infty}}^{1} * \boldsymbol{Limit}_{s\to n}\, \mathbf{e}^{s(t+t0)}\, \frac{\mathbf{u(t+t0)}}{\boldsymbol{\Gamma}(\mathbf{v})}$$

F(t)= $(t)^{5} \frac{\{Limits \to n e^{st} u(t+t0)\}}{5!}$

Now, s+6=0, S=-6

$t^{5} \frac{\{Limits \to -6 e^{st} u(t)\}}{5!}$

$t^{5} e^{-6t} \frac{u(t)}{5!}$

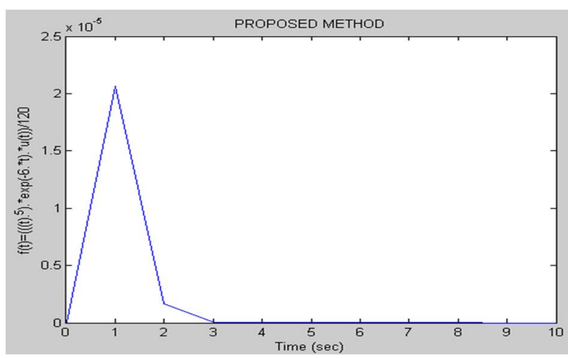

Figure 3:Proposed method

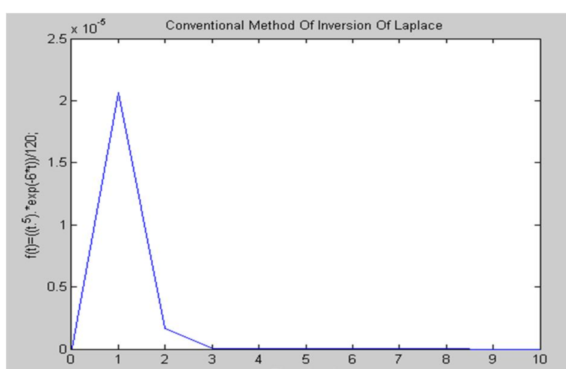

Figure 4:Conventional Method of Inversion of Laplace.

PROBLEM3:

This is an example for simple exponential. We have solved this expression by using our proposed method and also get the result by conventional method for ILT. In the end we get the similar results with fewer complexities. So the method we proposed is best fit for the function involving simple exponential.

$$\frac{e^{-5s}}{(s+2)^3}$$

in numerator e^{-5s} ,as the formula is e^{st} so it show the value of t in time domain.

so $u(t+t_0)$, where t_0=-5.

u(t-5)

V=3

Formula is

$$F(t) = D{s^{v}}_{-\infty}^{\,1} * \boldsymbol{Limit}_{s\to n}\, e^{s(t+t0)}\, \frac{u(t+t0)}{\Gamma(v)}$$

$$F(t) = D{s^{3}}_{-\infty}^{\,1} * \frac{\boldsymbol{Limit}_{s\to n}\, e^{s(t+t0)} \frac{u(t+t0)}{\Gamma(v)}}{\Gamma(3)}$$

$$F(t)= \frac{d^2}{ds^2}\frac{\{Limits\to ne^{st}u(t+t0)\}}{2!}$$

$$F(t)= t^2\frac{\{Limits\to -2e^{st}u(t)\}}{2!}$$

S+2=0, S=-2

Where t= t-5.

$$\frac{[(t-5)^2 e^{-2(t-5)} u(t-5)]}{2!}$$

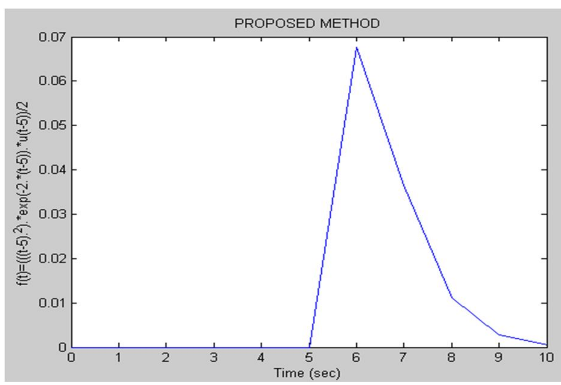

Figure 5: Proposed method.

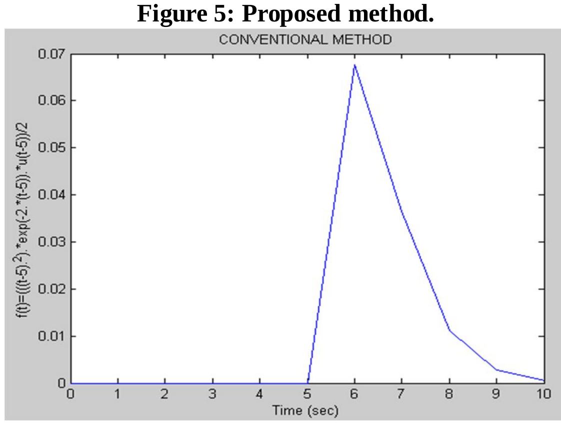

Figure 6: Conventional Method of Inversion of Laplace.

PROBLEM 4:

In this problem we have tried to solve the function by using quadratic formula by the both ways .i.e. proposed method and the conventional method. Figure 7 represent the graphical representation of the function by using proposed method while in figure 8 represents the solution by the conventional method.

$$\frac{1}{(s^2+9)}$$

Using quadratic formula

a=1, b=0,c=9

it becomes

s= 3j,   s= -3j.

$$\frac{1}{(s+3j)(s-3j)}$$

Using partial fraction

$\frac{1}{6j}\frac{1}{(s+3j)}$ - $\frac{1}{6j}\frac{1}{(s+3j)}$

Now applying formula

F(t)= $_{-\infty}D_s{}^{V-1}$Limit $_{s\to n}e^{s(t+t0)}\frac{u(t+t0)}{\Gamma(v)}$

s= 3j, s= -3j

V=1.

$\frac{1}{6j}e^{3jt}$ - $\frac{1}{6j}e^{-3jt}$

$$\frac{1}{3}\{\left(\frac{e^{3jt}-e^{-3jt}}{2j}\right)\}$$

$$\frac{\{sin(3t)\}u(t)}{3}$$

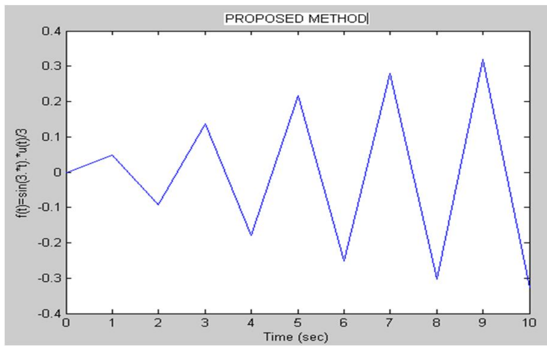

Figure 7: Proposed method

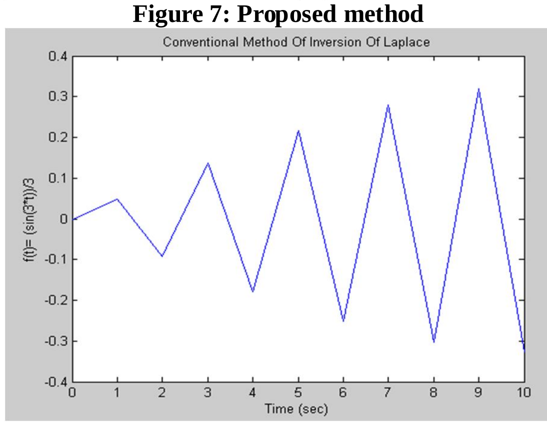

Figure 8: Conventional Method of Inversion of Laplace

PROBLEM 5:

$$\frac{1}{(s+6)^{0.5}}$$

V= 0.5

Formula is

$F(t)\;=\;{}_{Ds^{v}_{-\infty}{}^{-1}}*\boldsymbol{Limit}_{s\to n}\mathbf{e}^{s(t+t0)}\frac{\mathbf{u(t+t0)}}{\Gamma(\mathbf{v})}$

s+6=0, s= -6

$F(t)\;=\;{}_{Ds^{0.5}_{-\infty}{}^{-1}}*\frac{\text{Limit}s\text{–>}n\mathbf{e}^{s(t+t0)}\frac{\mathbf{u(t+t0)}}{\Gamma(\mathbf{v})}}{\Gamma(\mathbf{0.5})}$

F(t)= $Ds^{0.5}_{-\infty}{}^{-1}\frac{\{Limits\to-6e^{st}u(t+t0)\}}{\sqrt{\pi}}$

$$e^{-6t}\frac{u(t)}{(\sqrt{\pi})\sqrt{t}}$$

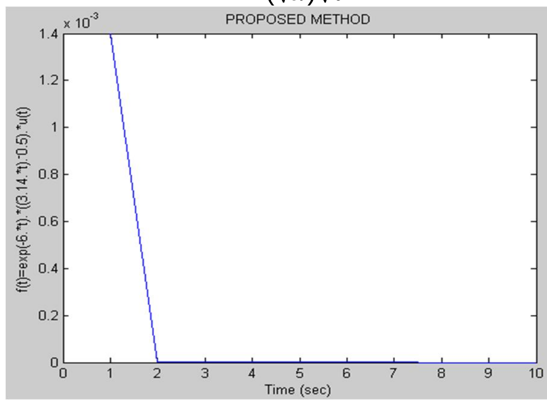

Figure 9: Proposed method

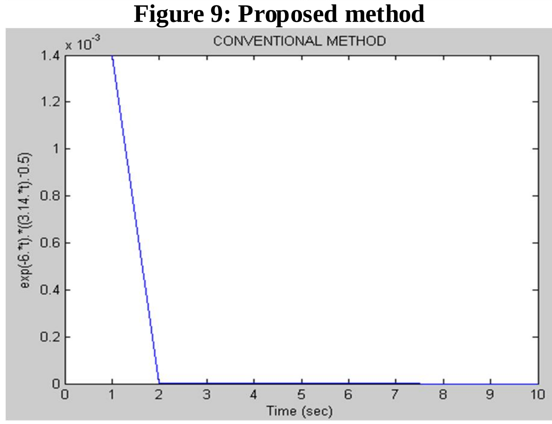

Figure 10: Conventional method

PROBLEM 6:

$$\frac{1}{(s+3)^{2.5}}$$

V= 2.5

Formula is

$$F(t) = Ds^{0.5}_{-\infty}{}^{-1} * \text{Limits}{-}{>}\ n\mathbf{e}^{\mathbf{s(t+t0)}}\frac{\mathbf{u(t+t0)}}{\mathbf{\Gamma(v)}}$$

s+3=0, s= -3

$$F(t) = Ds^{v}_{-\infty}{}^{-1} * \frac{\text{Limits}{-}{>}n\mathbf{e}^{\mathbf{s(t+t0)}}\frac{\mathbf{u(t+t0)}}{\mathbf{\Gamma(v)}}}{\mathbf{\Gamma(2.5)}}$$

F(t)= $Ds^{2.5}_{-\infty}{}^{-1}\frac{\{Limits\to -3e^{st}u(t+t0)\}}{1.32934}$

$$e^{-3t}\frac{u(t)}{(1.32934)}t^{1.5}$$

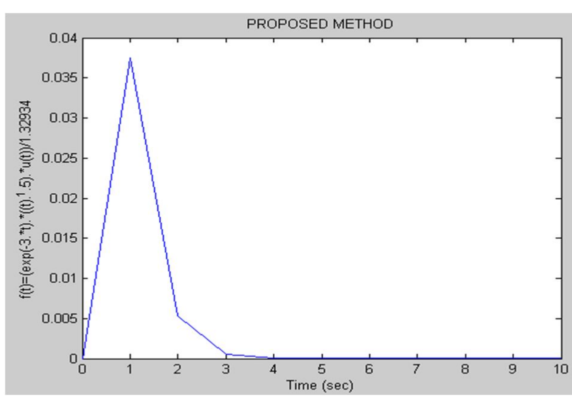

Figure 11: Proposed method.

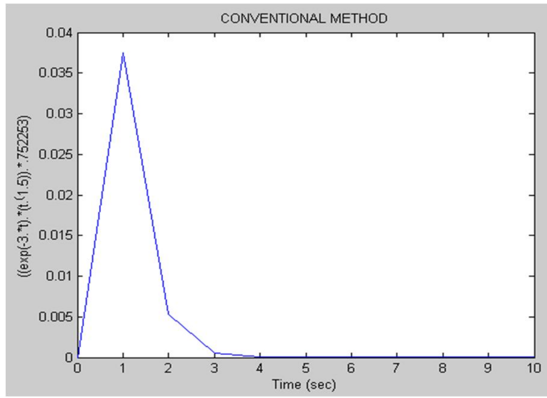

Figure 12: Conventional method

The results obtained by our method in all the four examples shown in **Figs. 2, 3, 5, 7, 9,11** are perfect to compare with the
results obtained by inverse Laplace method shown in **Figs. 1, 4, 6, 8, 10,12.**
Here we present a table 1 to compare the results obtained by proposed method to the results obtained by the conventional method of Laplace transform.

Table 1: Comparison of results

Problems	Conventional Method Results	Proposed Method Results
1) $\frac{1}{s+2}$	e^{-2t} .	e^{-2t}u(t).
2). $\frac{1}{(s+6)6}$	$t^5 e^{-6t}\frac{1}{5!}$	$t^5 e^{-6t}\frac{u(t)}{5!}$
3) $.\frac{e^{-5s}}{(s+2)3}$	$(t-5)2 * \frac{[e^{-2(t-5)}]}{2!} * u(t-5)$	$(t-5)2 * \frac{[e^{-2(t-5)}]}{2!} * u(t-5)$
4). $\frac{1}{(s2+9)}$	$\frac{\{sin(3t)\}}{3}$	$\frac{\{sin(3t)\}u(t)}{3}$
5) $\frac{1}{(s+6)^{0.5}}$	$e^{-6t}\frac{1}{(\sqrt{\pi})\sqrt{t}}$	$e^{-6t}\frac{u(t)}{(\sqrt{\pi})\sqrt{t}}$
6) $\frac{1}{(s+3)^{2.5}}$	$e^{-3t}\frac{1}{(1.32934)}t^{1.5}$	$e^{-3t}\frac{u(t)}{(1.32934)}t^{1.5}$

CONCLUSION:

A lot of research has been focused for evaluation of inverse Laplace transform. But proposed methods failed on many occasions. No universal method has been
found in correspondence with inverse Laplace transformation. The method we have presented is perfect to compare with inverse Laplace transformation. The accuracy obtained by the method is perfect. There are few limitations in applying the method. It cannot be applied to complex signals signal involving error functions, complementary error functions and complex exponential functions.

FUTURE WORK:

Future work is to solve complex signals
Like

$$\frac{1}{(s+a)V(s+b)V'}$$

In this case V or V'or both are decimal number's. To solve these kinds of problems we have expanded the proposed method, the expanded formula is able to solve the complex problems. Complex exponential signals are also solved by the expanded formula. The expansion of the proposed formula will discuss in next paper.

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