J. Basic. Appl. Sci. Res., 5(11)58-66, 2015
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ISSN 2090-4304
Journal of Basic and Applied Scientific Research
www.textroad.com

Blending Two Parametric Quadratic Bezier Curves

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Received: August 13, 2015
Accepted: October 28, 2015

ABSTRACT

The main objective of this project is to show on how to blend two parametric curves where the two parametric curves is blended to produce one new curve under certain conditions given. Normally, a Bezier curve is a parametric curve that is frequently used in Computer-Aided Design (CAD) and Computer-Aided Graphic Design (CAGD). Therefore, this kind of curve is selected to be used in this project that will focus on quadratic Bezier curve. Then, this idea on blending two parametric curves is expanded to blend the curves in plane. Furthermore, by this way, we can generate various shapes in three dimensional (3D).
KEYWORDS: Bezier curve, blending parametric curve.

INTRODUCTION

The concept and application of Bezier curves is used to model parametric curves and surfaces independently. Bezier curve is a base curve that is widely used in geometric design. There are many graphics packages which have been used Bezier curves in their CAGD system such as Adobe Illustrator, CorelDraw and generate fonts to PostScript. In 1998, Hui described that a curve can be blended into another curve. Farin (2002) stated that by blending bilinear curves can produced a patch Coons. In Piegl (1997), the term function is defined as mixture of basic function for the curve involved. The formula usually is created in an orderly and organized method so that the results of the curve which is blended qualify and correspond to each point on the curve bases. Many theories and methods have been introduced in the blending curves and surfaces involving Hermite interpolation, cubic ratio interpolation splints and Casteljau algorithm. There are several methods has been used to solve problems involving blending curves. For example, C.Hoffmann and J.Hopcroft (1986) suggest a potential method for blending surface automatically, but they provided that the surface to be blended should be quadric. Therefore, Jinsan Cheng (2002) discussed the continuity and smoothness in combining these two quadric surfaces in explicit formula. A curved base is built to connect the two axes of the surface to be blended. Through this method, the surface needs a normal vector at each point within the limits.

Bezier curve is defined by the endpoints of the curve and control points that will determine the degree of curvature. This curve does not provide a local control point. So, by changing the control points it will affect the overall shape of the curve. This project shows how two quadratic Bezier curves with parameter t are blended or mixed together to produce a new third curve according to certain conditions. In addition, this project also discusses the blending of free surface equation.

MATERIAL AND METHOD

Bezier Curve

Bezier curve with degree n is known as

$$B(t) = \sum_{i=0}^{n} \binom{n}{i}(1-t)^{n-i}t^{i}P_i$$

$$= (1-t)^n P_0 + \binom{n}{1}(1-t)^{n-1}\mathrm{t}P_1 + \cdots + \binom{n}{n-1}(1-\mathrm{t})t^{n-1}P_{n-1} + t^n P_n$$

where t is a parameter, $t \in [0,1]$ and $\binom{n}{i}$ is a binomial coefficient. $P_0, P_1, P_2, \ldots, P_n$ are the control points.

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Quadratic Bezier Curve

Given that three control points P_0, P_1 dan P_2, we divide the segment based on parameter t where $t \in [0,1]$.

- Let ${P_1}^{(1)}$ is a point for segment $\overline{P_0P_1}$ by using the following equation:

$$ {P_1}^{(1)} = (1-t)P_0 + tP_1 \tag{1.1} $$

- Let ${P_2}^{(1)}$ is a point for segment $\overline{P_1P_2}$ by using the following equation

$$ {P_2}^{(1)} = (1-t)P_1 + tP_2 \tag{1.2} $$

- Let ${P_2}^{(2)}$ is a point for segment $\overline{{P_1}^{(1)}{P_2}^{(1)}}$ by using the following equation

$$ {P_2}^{(2)} = (1-t){P_1}^{(1)} + t{P_2}^{(1)} \tag{1.3} $$

- Given

$$ P(t) = {P_2}^{(2)} $$

Based on the three equations above, each point ${P_1}^{(1)}$, ${P_2}^{(1)}$ dan ${P_2}^{(2)}$ is a function with parameter t. ${P_2}^{(2)}$ can be solved using $P(t)$. Therefore, $P(t)$ is determined as a function of Bezier curve.

$$ \begin{aligned} P(t) &= {P_2}^{(2)} \\ &= (1-t){P_1}^{(1)} + t{P_2}^{(1)} \end{aligned} \tag{1.4} $$

Equation (1.1) and (1.2) then substituted into equation (1.4), we get

$$ P(t) = (1-t)^2P_0 + 2(1-t)tP_1 + t^2P_2, \qquad t \in [0,1] \tag{1.5} $$

Equation (1.5) is a quadratic polynomial with parabolic shaped. Therefore, quadratic Bezier curve is a parabolic curve.

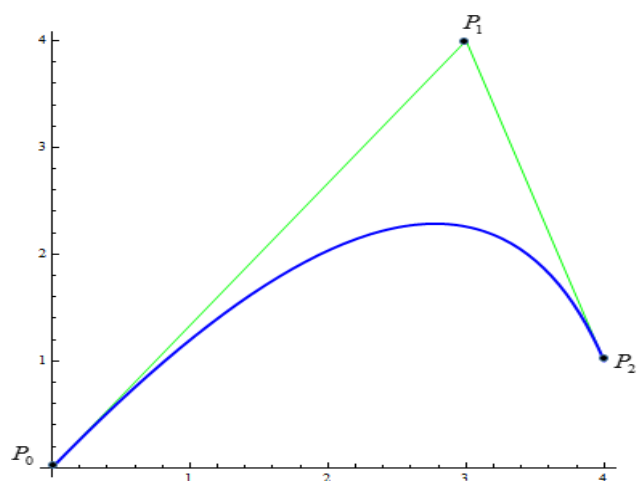


Figure 1
Quadratic Bezier curve

Blending curves

Let k_1 and k_2 are curves with parameter t where $t \in [0,1]$. $b(t)$ is required to be start with $t_0 = 0$ at point $k_1(0) = A_1$ for the curve k_1 and last at $t_1 = 1$ at the point $k_2(1) = B_2$ for the curve k_2. Therefore,

$$b(0) = k_1(0) = A_1 \quad , \quad k_1(1) = A_2$$
$$b(1) = k_2(1) = B_2 \quad , \quad k_2(0) = B_1$$

In addition, the curve b (t) should also have tangent vectors which are

$$b'(0) = k_1{}'(0) = D_0 \qquad\qquad b'(1) = k_2{}'(1) = D_e$$

The equation for the new curve by blending k_1 dan k_2 together is given by

$$b(t) = [1 - H(t)]k_1(t) + H(t)k_2(t) \qquad , 0 \le t \le 1$$

Based on the condition $b(0) = k_1(0)$ and $b(1) = k_2(1)$, it shows that H (0) = 0 and H (1) = 1. The lowest degree polynomial for $H(t)$ is one. Thus we obtain

$$b(t) = [1 - t]k_1(t) + tk_2(t) \qquad , 0 \le t \le 1$$

Blending two quadratic Bezier curve

Quadratic Bezier curve is generally given by the following equation

$$k_1\,(t) = (1-t)p_0 + 2t(1-t)p_1 + t^2 p_e$$
$$k_2\,(t) = (1-t)q_0 + 2t(1-t)q_1 + t^2 q_e \qquad (2.1)$$

The linear blend for quadratic Bezier curve is given by

$$b(t) = [1 - t]k_1(t) + tk_2(t) \qquad , 0 \le t \le 1$$

Therefore, equation (2.1) can be simplified as

$$b(t) = (1-t)((1-t)p_0 + 2t(1-t)p_1 + t^2 p_e) + t((1-t)^2 q_0 + 2(1-t)tq_1 + t^2 q_e) \qquad (2.2)$$

Equation (2.1) is defined as the following with $t = 0$ at start and end at $t = 1$.

$$k_1\,(0) = p_0 = (x_i, y_i)$$
$$k_2\,(0) = q_0 = (x_i, y_i)$$
$$k_1\,(1) = p_e = (x_{i+1}, y_{i+1}) \qquad \text{for } i = 0,1,2,\dots,k$$
$$k_2\,(1) = q_e = (x_{i+1}, y_{i+1})$$

For the curve $b(t)$, it must satisfy

$$b(0) = k_1(0) = p_0 \qquad\qquad b'(0) = k_1{}'(0) = D_0$$
$$b(1) = k_2(1) = p_e \qquad\qquad b'(1) = k_2{}'(1) = D_e$$

Tangent vector for equation (2.1) and (2.2) is determined by differentiate those three equations k_1, k_2 and b with respect to t and we obtained

$$k_1{}'\,(t) = -2(1-t)p_0 + 2(1-t)p_1 - 2tp_1 + 2tp_e$$
$$k_2{}'\,(t) = -2(1-t)q_0 + 2(1-t)q_1 - 2tq_1 + 2tq_e$$
$$b'(t) = -(1-t)^2 p_0 - 2(1-t)tp_1 - t^2 p_e + (1-t)(-2(1-t)p_0 + 2(1-t)p_1 - 2tp_1 + 2tp_e) + (1-t)^2 q_0 + 2(1-t)tq_1 + t^2 q_e + t(-2(1-t)q_0 + 2(1-t)q_1 - 2tq_1 + 2tq_e)$$

Then tangent vector for Equation (2.1) and (2.2) with $t = 0$ at start and end at $t = 1$ are as follows

$$\begin{aligned} k_1{}'(0) &= -2p_0 + 2p_1 \\ k_2{}'(0) &= -2q_0 + 2q_1 \\ k_1{}'(1) &= -2p_1 + 2p_e \\ k_2{}'(1) &= -2q_1 + 2q_e \\ b'(0) &= -3p_0 + 2p_1 + q_0 \\ b'(1) &= -p_e - 2q_1 + 3q_e \end{aligned} \tag{2.3}$$

From (2.3), we found that $b'(0) \neq k_1{}'(0)$ and $b'(1) \neq k_2{}'(1)$.

In order to qualify the requirement for curve $b(t)$, the starting point and the end point for the two curves $k_1$and k_2 should be the same. So $k_1(0) = k_2(0) = p_0$ and $k_1(1) = k_2(1) = p_e$.

Equation (2.1) and (2.2) become

$$\begin{aligned} k_1(t) &= (1-t)p_0 + 2t(1-t)p_1 + t^2 p_e \\ k_2(t) &= (1-t)p_0 + 2t(1-t)q_1 + t^2 p_e \\ b(t) &= (1-t)((1-t)p_0 + 2t(1-t)p_1 + t^2 p_e) \\ &\quad + t((1-t)^2 q_0 + 2(1-t)tq_1 + t^2 q_e) \end{aligned} \tag{2.4}$$

Then, the tangent vector for equation (2.4) with $t = 0$ at start and end at $t = 1$ are as follows

$$\begin{aligned} k_1{}'(0) &= -2p_0 + 2p_1 \\ k_2{}'(0) &= -2p_0 + 2q_1 \\ k_1{}'(1) &= -2p_1 + 2p_e \\ k_2{}'(1) &= -2q_1 + 2p_e \\ b'(0) &= -2p_0 + 2p_1 \\ b'(1) &= -2q_1 + 2p_e \end{aligned} \tag{2.5}$$

From (2.5), the tangent vector for $b(t)$ follows the condition where

$b'(0) = k_1{}'(0)$ $\qquad$ $b'(1) = k_2{}'(1)$

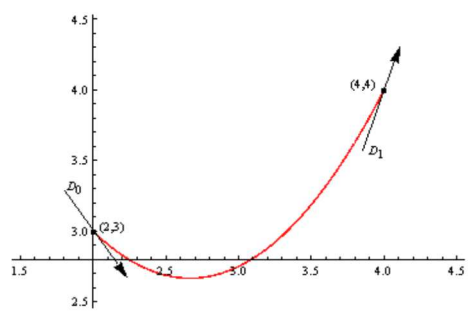


Figure 2
k_1 is plotted with coordinate $p_0 = (2,3), p_1 = (3,2)$,
$p_e = (4,4)$ and tangent vector $D_0 = (2,-2)$
and $D_1 = (2,4)$

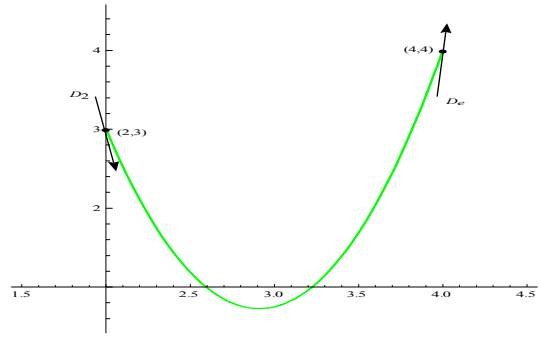


Figure 3
k_2 is plotted with coordinate $p_0 = (2,3)$, $q_1 = (3,-2)$,
$p_e = (4,4)$ and tangent vector $D_2 = (2,-10)$ and $D_e = (2,12)$

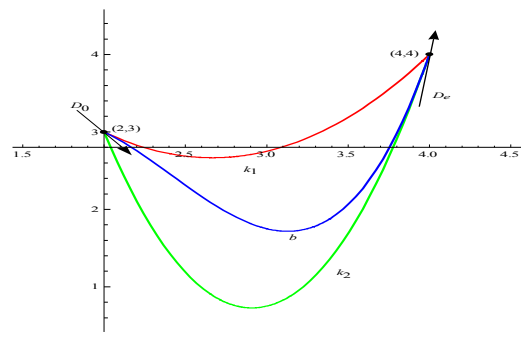


Figure 4
the new curve $b(t)$ formed by blending the curves k_1
and k_2.

Based on figure 4, it shows that the three curves k_1, k_2 and b are plotted on the same axis. The curve b that is generated follows the condition where it starts at the same point at $p_0 = (2,3)$ and ends at the same point at $p_e = (4,4)$. The vector tangent also meets the conditions where $b'(0) = k_1{}'(0) = (2,-2)$ and $b'(1) = k_2{}'(1) = (2,12)$.

Blending two quadratic Bezier curve in a plane

According to the equation 2.1 p_0, p_1, p_e, q_0, q_1 and q_e are points in plane (x, y, z). For this discussion, curve k_1 is plotted in $z - axis$ so that the coordinate is $(x, y, 0)$. For the curve k_2, it is plotted in $x - axis$ so that the coordinate is $(0, y, z)$. Therefore, the control points are given by

$$p_0 = (x_0, y_0, 0), \qquad p_1 = (x_1, y_1, 0), \qquad p_e = (x_2, y_2, 0)$$
$$q_0 = (0, y_0, z_0), \qquad q_1 = (0, y_1, z_1), \qquad q_e = (0, y_2, z_2)$$

We have

$$k_1 = ((1-t)^2 x_0 + 2(1-t)tx_1 + t^2 x_2, (1-t)^2 y_0 + 2(1-t)ty_1 + t^2 y_2, 0)$$
$$k_2 = (0, (1-t)^2 y_0 + 2(1-t)ty_1 + t^2 y_2, (1-t)^2 z_0 + 2(1-t)tz_1 + t^2 z_2)$$

(a) curve $k_1(t)$ with coordinate $(x, y, 0)$ (b) curve $k_2(t)$ with coordinate $(0, y, z)$

Figure 5
Quadratic Bezier curve, k_1 and k_2 in plane

Based on Figure 5, $k_1(t)$ is a parabolic curve formed along the z-plane. While $k_2(t)$ is a parabolic curve formed along the x-plane.

DISCUSSION

Linear Blending for Quadratic curve
Given a linear function with the following general equation

$$k_3(t) = (1-t)r_0 + tr_e \tag{3.1}$$

The three curves k_1, k_2 and k_3 will be blending together to form a new shaped based on the following surface equation.

$$S(u) = (1-u)^2 k_1(t) + 2u(1-u)k_3(t) + u^2 k_2(t) \tag{3.2}$$

where $t \in (0,1)$ and $u \in (0,1)$.

The coordinate for point r_0 and r_e should be controlled to ensure that the result is as expected. The coordinate for point r_0 and r_e are set as $r_0 = (x_0, y_a, z_0)$ and $r_e = (x_2, y_b, z_2)$. Therefore we have

$$k_3 = (1-t)x_0 + tx_2, (1-t)y_a + ty_b, (1-t)z_0 + tz_2$$

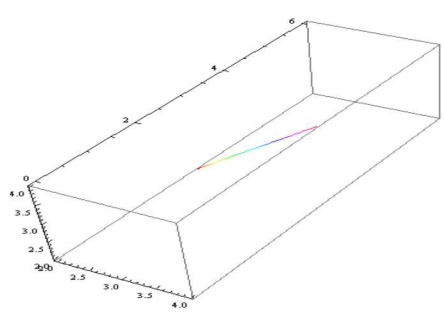
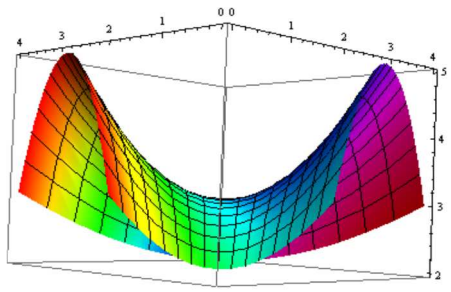
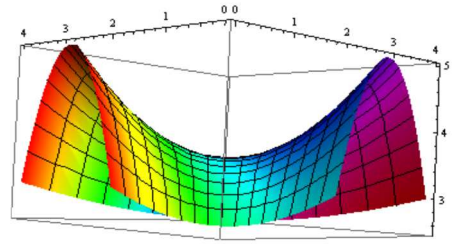


Figure 6
Linear curve $k_3(t)$ with coordinate $r_0 = (x, y_a, z)$and $r_e = (x, y_b, z)$

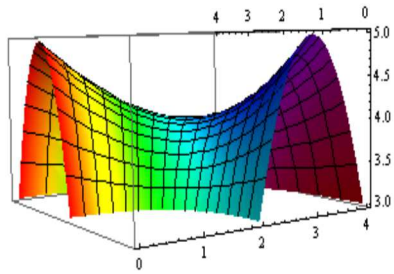
In a plane surface, $k_3(t)$ is a straight line along y-plane.



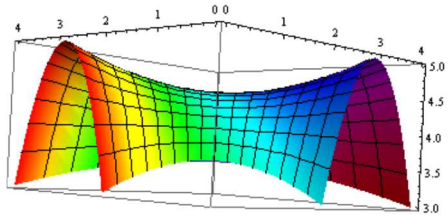
(a) Curve with coordinate
$r_0 = (x_0, 1, z_0)$ and $r_e = (x_2, 1, z_2)$



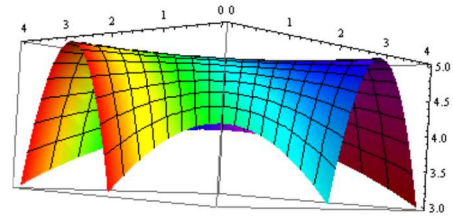
(b) Curve with coordinate
$r_0 = (x_0, 2, z_0)$ and $r_e = (x_2, 2, z_2)$



(c) Curve with coordinate $r_0 = (x_0, 3, z_0)$ and $r_e = (x_2, 3, z_2)$



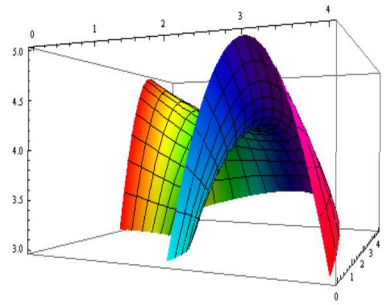
(d) Curve with coordinate
$r_0 = (x_0, 4, z_0)$ and $r_e = (x_2, 4, z_2)$



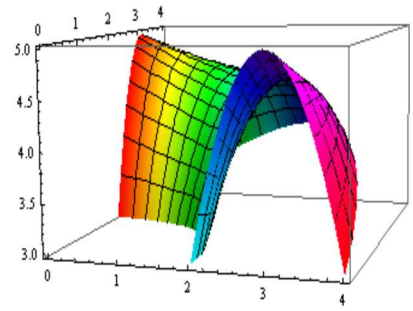
(e) Curve with coordinate
$r_0 = (x_0, 5, z_0)$ and $r_e = (x_2, 5, z_2)$

Figure 7
the blending linear curve $k_3(t)$ with quadratic Bezier curve
$k_1(t)$ and $k_2(t)$ with value $y_a = y_b$ but different r_0 and r_e.

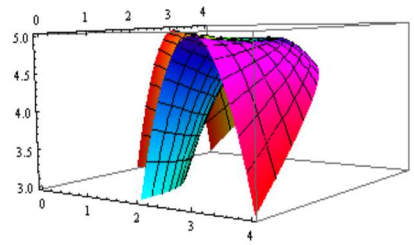
Based on Figure 7 (a), when $y_a = y_b = 1$, the curve $k_3(t)$ intercepts the y-axis at y = 1. Figure 7 (b) shows the curve $k_3(t)$ intercepts the y-axis at y = 2. By setting $y_a = y_b = y_2$, $k_3(t)$ will be in the same plane with the curve $k_1(t)$ and $k_2(t)$. When the value of y_a and y_b are greater than y_2, it can be seen that the interception between the three curves form a new curve between $k_1(t)$ and $k_2(t)$as in figure 7 (d) and (e).



(a) Curve with coordinate $r_0 = (x_0, 3, z_0)$ and $r_e = (x_2, 4, z_2)$



(b) Curve with coordinate $r_0 = (x_0, 3, z_0)$ and $r_e = (x_2, 5, z_2)$



(c) Curve with coordinate $r_0 = (x_0, 3, z_0)$ and $r_e = (x_2, 6, z_2)$

Figure 8
the blending of linear curve $k_3(t)$ with quadratic Bezier curve $k_1(t)$ and $k_2(t)$ with value $y_a \neq y_b$ with different coordinate for r_0 and r_e

Based on figure 8, the value of $y_a < y_b$. The curve $k_3(t)$crosses the curves $k_1(t)$ and $k_2(t)$ at y_a in the front and at y_b at the behind.

CONCLUSIONS

This whole project discuss about the two parametric curves that are blend together to generate a new curve by using some specific conditions and rules. If two curves for Bezier functions are blended, the most important thing is to control the control points so that the new third curve is as expected. The more points that need to be controlled, it is difficult to produce the curved as desired. The advantage of using Bezier curve is it is easily formed by controlling the control points.

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