J. Appl. Environ. Biol. Sci., 2(4)159-171, 2012

ISSN 2090-4274
Journal of Applied Environmental and Biological Sciences
www.textroad.com

Solutions of Predator-Prey populations From Fractional Integral Operator Lotka-Volterra Equations

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ABSTRACT

This paper presents the solutions of fractional Lotka-Volterra equation by applying fractional variational iteration method , HPM and Sinc-Nyström method. Moreover, we discuss the convergence analysis and stability of the fractional variational iteration method and HPM by using the Banach fixed point theory, which could provide a good iteration algorithm. Finally, we also give some numerical illustrations to the obtained results.

KEYWORDS: Fractional integral operator, Nyström method, mathematical ecology, fractional Lotka - Volterra equations.

Mathematics Subject Classification (2000): 45P05, 92A17, 92D25, 34D23

1. INTRODUCTION

The Lotka-Volterra model was constructed to enable ecologists to predict the possible outcome of competion between two species for the same resources. Essentially, the model attempts to account for the effect of the presence of one species on the population growth of the other species, in relation to the competitive effect of two members of the same species on each other. This model modifies predator-prey interactions introduced independently by Lotka [27] and Volterra [35].

In this paper we consider the following generalized Lotka-Volterra model as a fractional ordinary differential system:

$$\begin{cases} D_t^{\alpha,\beta} u(t) = F(u(t)), \\ u(0) = u_0, \end{cases} \quad (1)$$

where $0 < \alpha, \beta \leq 1$, $u(t) = (x(t), y(t))^T$, $u_0 = (\delta, \gamma)^T$,

$$D_t^{\alpha,\beta} u(t) = \begin{pmatrix} D_t^{\alpha} x(t) & D_t^{\beta} y(t) \end{pmatrix}^T,$$

and

$$F(u(t)) = \begin{pmatrix} F_1(x(t), y(t)) = a(t)x(t) - b(t)x(t)y(t) \\ \\ F_2(x(t), y(t)) = c(t)x(t)y(t) - d(t)y(t) \end{pmatrix}.$$

Also, $x(t)$, $y(t)$ are unknown and we define the following concepts:
$x(t)$ is the density of a prey,
$y(t)$ is the density of predators,
$a(t)$ is the intrinsic rate of a prey population increase,
$b(t)$ is the predation rate coefficient,
$c(t)$ is the reproduction rate of predators per 1 prey eaten, and
$d(t)$ is the predator mortality rate.

The outcome of competition between two species over ecological time described by the fractional Lotka-Volterra competition model. Based on concept of ecological time, we know that the competitively-inferior species may increase the range of food types wich it eats for survival and one species can competitively leave out another species. Therefore, we can conclude that the reaction of species to inter-specific competition in evolutionary time is often the opposite of what occurs. We return to earlier work in S. Das n, P.K. Gupta J. of Theoretical Biology (2011) [8], concerning fractional Lotka–Volterra equations by HPM. In the present work, we emphasize the need for convergence analysis and stability on this method.

There are many texts on theory of Lotka -Volterra equations. Among them, we note May (1972) [28], Albrecht et al. (1973) [1], Freedman (1980) [10], Arditi and Ginzburg (1989) [2], Berreta and Kuang (1998) [4], Jost et al.(1999) [24], Hsu et al.(2001) [21], Berezovskaya et al.(2001) [5], Xiao and Ruan (2001) [38], Maiti et al. (2007)

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[30], Boudjellaba and Sari (2009) [6]. However the above problem of Lotka–Volterra equation with convergence analysis and stability for fractional time derivatives and with a, b, c and d as functions of time t have not yet been solved by any scientists and researchers.

For the concept of fractional derivative we will adopt Caputo's definition [29, 31, 32] and Li, He (2011)[39, 14, 15] which is a modification of the Riemann- Liouville definition and has the advantage of dealing properly with initial value problems in which the initial conditions are given in terms of the field variables and their integer order which is the case in most physical processes [32].

Definition 1.1 *Jumarie is defined the fractional derivative Jumarie(2009) [22, 23] as the following limit from*

$$f^{(\alpha)} = \lim_{h\to 0}\frac{\Delta^{\alpha}[f(t)-f(0)]}{h^{\alpha}}.$$

This definition is close to the standard definition of derivative, and as a direct result, the derivative of a constant, $0 < \alpha < 1$ is zero.

Definition 1.2 *A real function $f(t)$, $t > 0$, is said to be in the space C_{μ}, $\mu \in \mathbb{R}$, if there exists a real number $p > \mu$, such that $f(t) = t^{p} f_1(t)$, where $f_1(t) \in C(0,\infty)$ and it is said to be in the space C_{μ}^{n} iff $f^{(n)}(x) \in C_{\mu}$, $n \in \mathbb{N}$.*

Definition 1.3 *The Riemann-Liouville fractional integral operator of (J^{α}) order $\alpha > 0$, of a function $f(t) \in C_{\mu}$, $\mu \geq -1$ is defined as:*

$$J^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-\xi)^{(\alpha-1)} f(\xi)d\xi, \qquad (2)$$

where $\alpha > 0$, $t > 0$ and $J^{0} f(t) = f(t)$.

Properties of the operator J^{α} can be found in [7, 22, 23, 29, 31], we mention the following properties:

For $f(t) \in C_{\mu}$, $\mu \geq -1$, $\alpha, \beta \geq 0$ and $m \geq -1$ we have

$$J^{\alpha} J^{\beta} f(t) = J^{\alpha+\beta} f(t) = J^{\beta} J^{\alpha} f(t), \qquad (3)$$

also, we can write

$$J^{\alpha} t^{m} = \frac{\Gamma(m+1)}{\Gamma(\alpha+m+1)} t^{\alpha+m}. \qquad (4)$$

Definition 1.4 *The fractional derivative of $f(t)$ in the Caputo sense is defined as:*

$$D_*^{\alpha} f(t) = J^{n-\alpha} D^{n} f(t) = \frac{1}{\Gamma(n-\alpha)}\int_0^t (t-\xi)^{(n-\alpha-1)} f^{(n)}(\xi)d\xi,$$

for $n-1 < \alpha \leq n$, $n \in \mathbb{N}$, $t > 0$, $f \in C_{-1}^{n}$.

In the following, the properties of the Caputo's fractional derivative are given:
If $n-1 < \alpha \leq n$, $n \in \mathbb{N}$, $f \in C_{\mu}^{n}$, and $\mu \geq -1$, then $D_*^{\alpha} J^{\alpha} f(t) = f(t)$,
also, we can write

$$J^{\alpha} D_*^{\alpha} f(t) = f(t) - \sum_{k=0}^{n-1} f^{(k)}(0^{+})\frac{t^{k}}{k!}, \quad t > 0. \qquad (5)$$

Definition 1.5 *The integral with respect to $(dt)^{\alpha}$ is defined as the solution of the fractional differential equation Jumarie(2009) [22, 23]*

$$dy \cong f(t)(dt)^{\alpha}, \qquad (6)$$

where $t \geq 0$, $y(0) = 0$ and $0 < \alpha < 1$.

Lemma 1.6 *Let $f(t)$ denote a continuous function Jumarie(2009) [22, 23] then the solution of the Eq.6 is defined as*

$$y = \int_0^t f(\xi)(d\xi)^{\alpha} = \alpha\int_0^t (t-\xi)^{(\alpha-1)} f(\xi)d\xi, \quad 0 < \alpha \leq 1. \qquad (7)$$

For example $f(t) = t^{m}$ in Eq.7 one obtains

$$\int_0^t \xi^{m}(d\xi)^{\alpha} = \frac{\Gamma(\alpha+1)\Gamma(m+1)}{\Gamma(\alpha+m+1)} t^{\alpha+m},$$

for $0 < \alpha \leq 1$.

In this paper, there's an emphasis on assessing the convergence analysis for fractional variational iteration method on the above problem. Therefore, this paper is organized as follows:

In section 2, we recall HPM for Eq.(1). Also, we introduce fractional variational iteration method for Eq.(1) and we consider particular cases in section 3. In section 4, we give a new Sinc-Nyström method for solving Eq.(1).

We verify the contraction mapping in HPM and fractional variational iteration method for this problem in sections 5 and 6. The stability and numerical solutions of the problem are obtained by using this method in section 7 and 8 respectively. Also, we compare the results of numerical solution between FVIM, HPM and Sinc- Nyström in this section. Finally, section 9 is devoted to the concluding remarks.

2 The Homotopy Perturbation Method (HPM)

In this section, we introduce the homotopy perturbation method proposed by He [16, 17, 18, 19, 20] to solve the prey and predator problem, and discuss the convergence analysis [26]. To illustrate the basic ideas of the homotopy perturbation method, we consider the following nonlinear differential equation:

$$D_t^{\alpha,\beta}u - F(t) = 0, \quad t \in \Omega, \tag{8}$$

with the boundary conditions

$$\mathcal{B}(u, \frac{\partial u}{\partial n}) = 0, \quad t \in \Gamma, \tag{9}$$

where $D_t^{\alpha,\beta}$ is a general differential operator, $\mathcal{B}$ is a boundary operator, $F(r)$ is a known analytical function and Γ is the boundary of the domain Ω. Generally speaking, the operator $D_t^{\alpha,\beta}$ can be divided into two parts which are L and N, where L is linear, but N is nonlinear. Therefore Eq.(8) can be rewritten as follows:

$$L(u) - N(u) - F(t) = 0. \tag{10}$$

By the homotopy perturbation technique, we construct a homotopy $v(t,\theta): \Omega \times [0,1] \to \mathbb{R}^2$ with satisfies

$$H(v,\theta) = (1-\theta)[L(v) - L(u_0)] + \theta\big[D_t^{\alpha,\beta}v - F(t)\big] = 0, \quad \theta \in [0,1], t \in \Omega, \tag{11}$$

or

$$H(v,\theta) = L(v) - L(u_0) + \theta L(u_0) + \theta[N(v) - F(t)] = 0, \tag{12}$$

where $\theta \in [0,1]$ is an embedding parameter and u_0 is an initial approximation of Eq.(8). Obviously, from these definitions we will have

$$H(v,0) = L(v) - L(u_0) = 0, \quad H(v,1) = D_t^{\alpha,\beta}v - F(t) = 0.$$

The process of changing θ from zero to unity is just that of $v(t,\theta)$ from $u_0(t)$ to $u(t)$. In topology, this is called deformation, $L(v) - L(u_0)$ and $D_t^{\alpha,\beta}v - F(t)$ are called homotopies. According to the HPM, we can first use the embedding parameter θ as a small paramter, and assume that the solution of Eq.(11) and Eq.(12) can be written as a power series in θ:

$$v = v(t,\theta) = \begin{pmatrix} x(t) = x_0(t) + \theta x_1(t) + \theta^2 x_2(t) + \cdots \\ y(t) = y_0(t) + \theta y_1(t) + \theta^2 y_2(t) + \cdots \end{pmatrix}. \tag{13}$$

Setting $\theta = 1$ results in the approximate solution of Eq.(13):

$$u = \lim_{\theta \to 1} v = \begin{pmatrix} x(t) = x_0(t) + x_1(t) + x_2(t) + \cdots \\ y(t) = y_0(t) + y_1(t) + y_2(t) + \cdots \end{pmatrix}.$$

3 Fractional Variational Iteration Method (FVIM)

Recently, the variational iteration method [12, 13] has been widely applied to analytically solve fractional differential equations [25, 37], where the term with the fractional derivative was considered as a restricted variation, making the identification of the Lagrange multiplier very inaccurate to overcome this problem. We investigate the local behavior of fractional differential equations and determine the Lagrange multiplier in a more accurate way with the fractional variation iteration method [7]. In this article the approximate analytical solutions of the fractional prey-predator model Lotka-Volterra equation are obtained with the help of the powerful, easy to use and effective mathematical tool-FVIM. Then a corrected functional for Eq.(1) can be constructed as follows:

$$\begin{cases} x_{j+1}(t) = x_j(t) + \frac{1}{\Gamma(1+\alpha)}\int_0^t \lambda_1(\xi)[\frac{d^\alpha x_j(\xi)}{d\xi^\alpha} - a(\xi)\tilde{x}_j(\xi) + b(\xi)\tilde{x}_j(\xi)\tilde{y}_j(\xi)](d\xi)^\alpha, \\ \\ y_{j+1}(t) = y_j(t) + \frac{1}{\Gamma(1+\beta)}\int_0^t \lambda_2(\xi)[\frac{d^\beta y_j(\xi)}{d\xi^\beta} - c(\xi)\tilde{x}_j(\xi)\tilde{y}_j(\xi) + d(\xi)\tilde{y}_j(\xi)](d\xi)^\beta. \end{cases} \tag{14}$$

Where $\lambda_1(\xi)$and$\lambda_2(\xi)$ are general Lagrange multipliers and $\delta\tilde{x}_j$ and $\delta\tilde{y}_j$ denote restricted variations, i.e. $\delta\tilde{x}_j = 0$ and $\delta\tilde{y}_j = 0$ making the above correction functionals stationary, we can obtain the following stationary conditions:

$$\delta x_j: \begin{cases} 1 + \lambda_1(\xi)|_{(\xi=t)} = 0 \\ \lambda_1^\alpha(\xi)|_{(\xi=t)} = 0, \end{cases}$$

and

$$\delta y_j: \begin{cases} 1 + \lambda_2(\xi)|_{(\xi=t)} = 0 \\ \lambda_2^\beta(\xi)|_{(\xi=t)} = 0. \end{cases}$$

The Lagrange multipliers can be identified as $\lambda_1 = \lambda_2 = -1$, and the following fractional variational iteration formula can be obtained by:

$$v_{j+1} = B_j^{\alpha,\beta}(v_j), \qquad (15)$$

where $v_{j+1} = \begin{pmatrix} x_{j+1}(t) \\ y_{j+1}(t) \end{pmatrix}$ and

$$B_j^{\alpha,\beta}(v_j) = \begin{pmatrix} x_j(t) - \frac{1}{\Gamma(1+\alpha)} \int_0^t [\frac{d^\alpha x_j(\xi)}{d\xi^\alpha} - a(\xi)x_j(\xi) + b(\xi)x_j(\xi)y_j(\xi)](d\xi)^\alpha \\ \\ y_j(t) - \frac{1}{\Gamma(1+\beta)} \int_0^t [\frac{d^\beta y_j(\xi)}{d\xi^\beta} - c(\xi)x_j(\xi)y_j(\xi) + d(\xi)y_j(\xi)](d\xi)^\beta \end{pmatrix}.$$

Now we begin with an arbitrary initial approximation $x(0) = \delta$ and $y(0) = \gamma$ and in the following, we consider the different particular cases:

Case 1. When $a(t) = t,\ \ b(t) = 1,\ \ c(t) = 1,\ \ d(t) = t$

$$\begin{cases} x_{j+1}(t) = x_j(t) - \frac{1}{\Gamma(1+\alpha)} \int_0^t [\frac{d^\alpha x_j(\xi)}{d\xi^\alpha} - \xi x_j(\xi) + x_j(\xi)y_j(\xi)](d\xi)^\alpha \\ \\ y_{j+1}(t) = y_j(t) - \frac{1}{\Gamma(1+\beta)} \int_0^t [\frac{d^\beta y_j(\xi)}{d\xi^\beta} - x_j(\xi)y_j(\xi) + \xi y_j(\xi)](d\xi)^\beta, \end{cases} \qquad (16)$$

therefore, we get

$$x_1(t) = \frac{\delta t^{\alpha+1}}{\Gamma(\alpha+2)} - \frac{\alpha\gamma\delta t^\alpha}{\Gamma(\alpha+2)} - \frac{\gamma\delta t^\alpha}{\Gamma(\alpha+2)} + \delta,$$

and

$$y_1(t) = \frac{\gamma t^{\beta+1}}{\Gamma(\beta+2)} + \frac{\beta\gamma\delta t^\beta}{\Gamma(\beta+2)} + \frac{\gamma\delta t^\beta}{\Gamma(\beta+2)} + \gamma.$$

By the same manipulation, we have

$$\begin{aligned} x_2(t) = &-\frac{\beta\gamma\delta^2\Gamma(\beta+1)t^{\alpha+\beta}}{\Gamma(\beta+2)\Gamma(\alpha+\beta+1)} - \frac{\gamma\delta^2\Gamma(\beta+1)t^{\alpha+\beta}}{\Gamma(\beta+2)\Gamma(\alpha+\beta+1)} - \frac{\gamma\delta t^{\alpha+\beta+1}}{\Gamma(\alpha+\beta+2)} + \frac{\delta t^{\alpha+1}}{\Gamma(\alpha+2)} \\ &+ \frac{\alpha\gamma^2\delta^2 t^{2\alpha+\beta}\Gamma(\alpha+\beta+1)}{\Gamma(\alpha+2)\Gamma(\beta+2)\Gamma(2\alpha+\beta+1)} + \frac{\alpha\beta\gamma^2\delta^2 t^{2\alpha+\beta}\Gamma(\alpha+\beta+1)}{\Gamma(\alpha+2)\Gamma(\beta+2)\Gamma(2\alpha+\beta+1)} \\ &+ \frac{\beta\gamma^2\delta^2 t^{2\alpha+\beta}\Gamma(\alpha+\beta+1)}{\Gamma(\alpha+2)\Gamma(\beta+2)\Gamma(2\alpha+\beta+1)} + \frac{\gamma^2\delta^2 t^{2\alpha+\beta}\Gamma(\alpha+\beta+1)}{\Gamma(\alpha+2)\Gamma(\beta+2)\Gamma(2\alpha+\beta+1)} \\ &+ \frac{\alpha\gamma^2\delta t^{2\alpha+\beta+1}\Gamma(\alpha+\beta+2)}{\Gamma(\alpha+2)\Gamma(\beta+2)\Gamma(2\alpha+\beta+2)} + \frac{\gamma^2\delta t^{2\alpha+\beta+1}\Gamma(\alpha+\beta+2)}{\Gamma(\alpha+2)\Gamma(\beta+2)\Gamma(2\alpha+\beta+2)} \\ &- \frac{\beta\gamma\delta^2 t^{2\alpha+\beta+1}\Gamma(\alpha+\beta+2)}{\Gamma(\alpha+2)\Gamma(\beta+2)\Gamma(2\alpha+\beta+2)} - \frac{\gamma\delta^2 t^{2\alpha+\beta+1}\Gamma(\alpha+\beta+2)}{\Gamma(\alpha+2)\Gamma(\beta+2)\Gamma(2\alpha+\beta+2)} \\ &- \frac{\gamma\delta t^{2\alpha+\beta+2}\Gamma(\alpha+\beta+3)}{\Gamma(\alpha+2)\Gamma(\beta+2)\Gamma(2\alpha+\beta+3)} - \frac{\alpha\gamma\delta t^{2\alpha+1}}{\Gamma(2\alpha+2)} - \frac{2\gamma\delta t^{2\alpha+1}}{\Gamma(2\alpha+2)} \\ &+ \frac{\delta t^{2\alpha+2}\Gamma(\alpha+3)}{\Gamma(\alpha+2)\Gamma(2\alpha+3)} + \frac{\sqrt{\pi}4^{-\alpha}\alpha\gamma^2\delta t^{2\alpha}}{\Gamma\left(\alpha+\frac{1}{2}\right)\Gamma(\alpha+2)} + \frac{\sqrt{\pi}4^{-\alpha}\gamma^2\delta t^{2\alpha}}{\Gamma\left(\alpha+\frac{1}{2}\right)\Gamma(\alpha+2)} - \frac{\gamma\delta t^\alpha}{\Gamma(\alpha+1)} + \delta, \end{aligned}$$

and

$$\begin{aligned} y_2(t) = &-\frac{\gamma t^{\beta+1}}{\Gamma(\beta+2)} - \frac{\alpha\gamma^2\delta^2 t^{\alpha+2\beta}\Gamma(\alpha+\beta+1)}{\Gamma(\alpha+2)\Gamma(\beta+2)\Gamma(\alpha+2\beta+1)} - \frac{\alpha\beta\gamma^2\delta^2 t^{\alpha+2\beta}\Gamma(\alpha+\beta+1)}{\Gamma(\alpha+2)\Gamma(\beta+2)\Gamma(\alpha+2\beta+1)} \\ &- \frac{\beta\gamma^2\delta^2 t^{\alpha+2\beta}\Gamma(\alpha+\beta+1)}{\Gamma(\alpha+2)\Gamma(\beta+2)\Gamma(\alpha+2\beta+1)} - \frac{\gamma^2\delta^2 t^{\alpha+2\beta}\Gamma(\alpha+\beta+1)}{\Gamma(\alpha+2)\Gamma(\beta+2)\Gamma(\alpha+2\beta+1)} \\ &- \frac{\alpha\gamma^2\delta\Gamma(\alpha+1)t^{\alpha+\beta}}{\Gamma(\alpha+2)\Gamma(\alpha+\beta+1)} - \frac{\gamma^2\delta\Gamma(\alpha+1)t^{\alpha+\beta}}{\Gamma(\alpha+2)\Gamma(\alpha+\beta+1)} - \frac{\alpha\gamma^2\delta t^{\alpha+2\beta+1}\Gamma(\alpha+\beta+2)}{\Gamma(\alpha+2)\Gamma(\beta+2)\Gamma(\alpha+2\beta+2)} \\ &- \frac{\gamma^2\delta t^{\alpha+2\beta+1}\Gamma(\alpha+\beta+2)}{\Gamma(\alpha+2)\Gamma(\beta+2)\Gamma(\alpha+2\beta+2)} + \frac{\beta\gamma\delta^2 t^{\alpha+2\beta+1}\Gamma(\alpha+\beta+2)}{\Gamma(\alpha+2)\Gamma(\beta+2)\Gamma(\alpha+2\beta+2)} \\ &+ \frac{\gamma\delta^2 t^{\alpha+2\beta+1}\Gamma(\alpha+\beta+2)}{\Gamma(\alpha+2)\Gamma(\beta+2)\Gamma(\alpha+2\beta+2)} + \frac{\gamma\delta t^{\alpha+\beta+1}}{\Gamma(\alpha+\beta+2)} + \frac{\gamma\delta t^{\alpha+2\beta+2}\Gamma(\alpha+\beta+3)}{\Gamma(\alpha+2)\Gamma(\beta+2)\Gamma(\alpha+2\beta+3)} \end{aligned}$$

$$-\frac{\beta\gamma\delta t^{2\beta+1}}{\Gamma(2\beta+2)}-\frac{\gamma t^{2\beta+2}\Gamma(\beta+3)}{\Gamma(\beta+2)\Gamma(2\beta+3)}+\frac{\sqrt{\pi}4^{-\beta}\beta\gamma\delta^2 t^{2\beta}}{\Gamma\left(\beta+\frac{1}{2}\right)\Gamma(\beta+2)}+\frac{\sqrt{\pi}4^{-\beta}\gamma\delta^2 t^{2\beta}}{\Gamma\left(\beta+\frac{1}{2}\right)\Gamma(\beta+2)}+\frac{\gamma\delta t^{\beta}}{\Gamma(\beta+1)}$$
$$+\gamma.$$

In similar manner, the rest of the components can be obtained. Further, we get the approximate solutions of $x(t)$ and $y(t)$,

$$\begin{cases} x(t)=\lim_{j\to\infty} x_j(t) \\ y(t)=\lim_{j\to\infty} y_j(t). \end{cases}$$

In section 6, we will prove that the above two series converge after few terms are required to get the approximate solutions.

Case 2. When $a(t)$, $b(t)$, $c(t)$, $d(t)$ are constant with an arbitrary initial approximation x_0, y_0 then we have

$$\begin{cases} x_{j+1}(t)=x_j(t)-\frac{1}{\Gamma(1+\alpha)}\int_0^t \left[\frac{d^{\alpha}x_j(\xi)}{d\xi^{\alpha}}-ax_j(\xi)+bx_j(\xi)y_j(\xi)\right](d\xi)^{\alpha} \\ y_{j+1}(t)=y_j(t)-\frac{1}{\Gamma(1+\beta)}\int_0^t \left[\frac{d^{\beta}y_j(\xi)}{d\xi^{\beta}}-cx_j(\xi)y_j(\xi)+dy_j(\xi)\right](d\xi)^{\beta}. \end{cases} \quad (17)$$

Therefore, we can write

$$x_1(t)=\frac{ax_0t^{\alpha}}{\alpha\Gamma(\alpha)}-\frac{bx_0y_0t^{\alpha}}{\alpha\Gamma(\alpha)}+x_0,$$

$$y_1(t)=\frac{cx_0y_0t^{\beta}}{\beta\Gamma(\beta)}-\frac{dy_0t^{\beta}}{\beta\Gamma(\beta)}+y_0.$$

By the same manipulation, we have

$$x_2(t)=\frac{\sqrt{\pi}a^2x_0 2^{-2\alpha}t^{2\alpha}}{\alpha\Gamma(\alpha)\Gamma\left(\alpha+\frac{1}{2}\right)}-\frac{abcx_0^2y_0t^{2\alpha+\beta}\Gamma(\alpha+\beta+1)}{\alpha\beta\Gamma(\alpha)\Gamma(\beta)\Gamma(2\alpha+\beta+1)}+\frac{abdx_0y_0t^{2\alpha+\beta}\Gamma(\alpha+\beta+1)}{\alpha\beta\Gamma(\alpha)\Gamma(\beta)\Gamma(2\alpha+\beta+1)}$$
$$-\frac{\sqrt{\pi}abx_0y_0 2^{1-2\alpha}t^{2\alpha}}{\alpha\Gamma(\alpha)\Gamma\left(\alpha+\frac{1}{2}\right)}+\frac{ax_0t^{\alpha}}{\Gamma(\alpha+1)}+\frac{b^2cx_0^2y_0^2t^{2\alpha+\beta}\Gamma(\alpha+\beta+1)}{\alpha\beta\Gamma(\alpha)\Gamma(\beta)\Gamma(2\alpha+\beta+1)}$$
$$-\frac{b^2dx_0y_0^2t^{2\alpha+\beta}\Gamma(\alpha+\beta+1)}{\alpha\beta\Gamma(\alpha)\Gamma(\beta)\Gamma(2\alpha+\beta+1)}+\frac{\sqrt{\pi}b^2x_0y_0^2 2^{-2\alpha}t^{2\alpha}}{\alpha\Gamma(\alpha)\Gamma\left(\alpha+\frac{1}{2}\right)}-\frac{bcx_0^2y_0\Gamma(\beta+1)t^{\alpha+\beta}}{\beta\Gamma(\beta)\Gamma(\alpha+\beta+1)}$$
$$+\frac{bdx_0y_0\Gamma(\beta+1)t^{\alpha+\beta}}{\beta\Gamma(\beta)\Gamma(\alpha+\beta+1)}-\frac{bx_0y_0t^{\alpha}}{\Gamma(\alpha+1)}+x_0,$$

and

$$y_2(t)=\frac{ac^2x_0^2y_0t^{\alpha+2\beta}\Gamma(\alpha+\beta+1)}{\alpha\beta\Gamma(\alpha)\Gamma(\beta)\Gamma(\alpha+2\beta+1)}-\frac{acdx_0y_0t^{\alpha+2\beta}\Gamma(\alpha+\beta+1)}{\alpha\beta\Gamma(\alpha)\Gamma(\beta)\Gamma(\alpha+2\beta+1)}+\frac{acx_0y_0\Gamma(\alpha+1)t^{\alpha+\beta}}{\alpha\Gamma(\alpha)\Gamma(\alpha+\beta+1)}$$
$$-\frac{bc^2x_0^2y_0^2t^{\alpha+2\beta}\Gamma(\alpha+\beta+1)}{\alpha\beta\Gamma(\alpha)\Gamma(\beta)\Gamma(\alpha+2\beta+1)}+\frac{bcdx_0y_0^2t^{\alpha+2\beta}\Gamma(\alpha+\beta+1)}{\alpha\beta\Gamma(\alpha)\Gamma(\beta)\Gamma(\alpha+2\beta+1)}-\frac{bcx_0y_0^2\Gamma(\alpha+1)t^{\alpha+\beta}}{\alpha\Gamma(\alpha)\Gamma(\alpha+\beta+1)}$$
$$+\frac{\sqrt{\pi}c^2x_0^2y_0 2^{-2\beta}t^{2\beta}}{\beta\Gamma(\beta)\Gamma\left(\beta+\frac{1}{2}\right)}-\frac{\sqrt{\pi}cdx_0y_0 2^{1-2\beta}t^{2\beta}}{\beta\Gamma(\beta)\Gamma\left(\beta+\frac{1}{2}\right)}+\frac{cx_0y_0t^{\beta}}{\Gamma(\beta+1)}+\frac{\sqrt{\pi}d^2y_0 2^{-2\beta}t^{2\beta}}{\beta\Gamma(\beta)\Gamma\left(\beta+\frac{1}{2}\right)}$$
$$-\frac{dy_0t^{\beta}}{\Gamma(\beta+1)}+y_0.$$

In similar manner, the rest of the components can be obtained. We also get the approximate solutions of $x(t)$ and $y(t)$ such that

$$\begin{cases} x(t)=\lim_{n\to\infty} x_j(t), \\ y(t)=\lim_{j\to\infty} y_j(t). \end{cases}$$

In section 6, we will prove that the above two series converge after few terms are required to get the approximate solutions.

4 Sinc-Nyström method

Sinc-Nyström method we defined by Haber [11] and Okayama et al.[34]. In this section, we extend the Sinc-Nyström method for solving the predator and prey problem.

Let a numerical intgration scheme be given:

$$D_t^{\gamma}x(t) = \frac{1}{\Gamma(1-\gamma)}\int_0^t k(t-s)x^{(1)}(s)ds$$
$$\simeq \frac{1}{h\Gamma(1-\gamma)}\sum_{\substack{j=-N\\ i\neq j}}^{N}\{k(t,\psi(jh))(x(\psi(jh))$$
$$-x(\psi((j-1)h)))\{\psi\}\ (jh)J(j,h)(\{\psi\}\ (t))\}, \qquad (18)$$

where $k(t,s) = (t-s)^{-\gamma}$ and the mesh size h is selcted by the following formula $h = \frac{\log(2dN/\bar{r})}{N}$ where $0 < d < \frac{\pi}{2}, N$ is positive integer and $\bar{r} > 0$. Also, $J(j,h)(x) = h\left\{\frac{1}{2}+\frac{1}{n}Si\left\{\pi(\frac{x}{h}-j)\right\}\right\}$ such that $Si(x)$ is the so called Sinc integral function, whose routine is available in some numerical librarie (see NAG, IMSL). On the other hand, we define the following function.

$$\psi(\sigma) = \frac{b}{2}\tanh\left(\frac{\sigma}{2}\right)+\frac{b}{2},$$

as the single exponential transformation which maps $\sigma \in \mathbb{R}$ onto $s \in (0,b)$. Hence, we can obtain the inverse function as follows:

$$\sigma = \{\psi\}^{-1}(s) = \log\left(\frac{s}{b-s}\right).$$

We call the above approximation the improve of single exponential Sinc integration.

Using the above quadrature scheme, approximate the integral in Eq.(1), obtaining the following equation:

$$\left\{\begin{matrix}\widetilde{D}_{t_j}^{\alpha,\beta}U_N(t_j) = F(U_N(t_j))\\ i,j = -N,\dots,0,\dots,N\end{matrix}\right\}, \qquad (19)$$

where $$\widetilde{D}_t^{\alpha,\beta}U_N(t) = \left(\widetilde{D}_t^{\alpha}x_N(t), \widetilde{D}_t^{\beta}y_N(t)\right).$$

We write this as an exact equation with a new unknow function $U_N(t)$. To find the solution at the node point, let trun through the quadrature node point t_j. Therefore, the equation (19) is a linear system of order $2N+1$. We call the Eq.(19) the Sinc-Nyström method for solving Eq.(1).

5 Contraction mapping in HPM

The simplicity of contraction mapping in HPM for Eq.(1) is the fact that it is un-complicated. Therefore, in this section, we apply the homotopy perturbation method for Eq.(1). After that we present an example to show the efficiency and high accuracy of the described method for solving Eq.(1).

In order to solve Eq.(1) by means of homotopy perturbation method, according to Eq.(11), and substituting Eq.(13) by equating the coefficient of like powers of θ yield:

$$\theta^j:\begin{cases}D_t^{\alpha}x_j(t) = M_{j-1}(t)\\ \\ D_t^{\beta}y_j(t) = N_{j-1}(t),\end{cases} \qquad (20)$$

for $j = 1,2,3,\dots,n$ where $\theta^0:\begin{cases}D_t^{\alpha}x_0(t) = 0\\ D_t^{\beta}y_0(t) = 0\end{cases}$,

$$M_{j-1}(t) = a(t)x_{j-1}(t) - b(t)\sum_{i=0}^{j-1}x_i(t)y_{(j-1)-i}(t),$$

and

$$N_{j-1}(t) = -d(t)y_{j-1}(t) + c(t)\sum_{i=0}^{j-1}x_i(t)y_{(j-1)-i}(t).$$

Then starting with an initial approximation and solving the above equations, we get the nth approximation. On the other hand, for investigating convergence analysis, we rearrange Eq.(20) to the following iterative method by Eq.(4):

$$v_j = \begin{pmatrix} x_j = \frac{\Gamma(2)}{\Gamma(\alpha+2)}\left(a(t)x_{j-1}(t) - b(t)\sum_{i=0}^{j-1} x_i(t)y_{(j-1)-i}(t)\right)^{(\alpha+1)} \\ \\ y_j = \frac{\Gamma(2)}{\Gamma(\beta+2)}\left(-d(t)y_{j-1}(t) + c(t)\sum_{i=0}^{j-1} x_i(t)y_{(j-1)-i}(t)\right)^{(\beta+1)} \end{pmatrix} = A_j^{\alpha,\beta}(v_{j-1}), \qquad (21)$$

where $j = 1,2,3,\ldots,n$. We show that the above iterative method is a contraction.

The following definition is imposed (see [3]), then convergence of the homotopy perturbation method is discussed.

Definition 5.1 *Let k be a non-negative integer, $r \in [1,\infty)$. Then Sobolev space $W^{k,r}(\Omega)$ is the set of all the functions $v \in L^r(\Omega)$ such that for each multi-index α with $|\alpha| \le k$, the α^{th} weak derivative $\partial^\alpha v$ exists and $\partial^\alpha v \in L^r(\Omega)$. The norm in the space $W^{k,r}(\Omega)$ is defined as*

$$\| v \|_{W^{k,r}(\Omega)} = \begin{cases} \left[\sum_{|\alpha|\le k} \| \partial^\alpha v \|_{L^r(\Omega)}^r\right]^{1/r}, & 1 \le r < \infty, \\ \max_{|\alpha|\le k} \| \partial^\alpha v \|_{L^\infty(\Omega)}, & p = \infty. \end{cases}$$

Theorem 5.2 *In Eq. (21) we define, $A_j^{\alpha,\beta}: W^{k,r} \to W^{k,r}$ and if $(A_j^{\alpha,\beta})'$, $(A_j^{\alpha,\beta})''$ are bounded in some neighborhood. Then (i) $A_j^{\alpha,\beta}$ is a contraction mapping, that is*

$$\forall v,w \in W^{k,r}; \quad \| A_j^{\alpha,\beta}(v) - A_j^{\alpha,\beta}(w) \|_{W^{k,r}} \le \eta \| v - w \|_{W^{k,r}}, \quad 0 < \alpha, \qquad \beta \le 1, \quad 0 < \eta < 1.$$

On the other hand, according to Banach's fixed point theorem, having the fixed point u, that is $u = A_j^{\alpha,\beta}(u)$. Assume that the sequence generated by homotopy perturbation method can be written as

$$V_n = A_j^{\alpha,\beta}(V_{n-1}), \quad V_{n-1} = \sum_{i=0}^{n-1} u_i, \quad n = 1,2,3,\ldots,$$

and suppose that $V_0 = v_0 = u_0 = \binom{x_0}{y_0}$,then,

(ii)The sequence of $\{V_n\}_{n=1}^{\infty}$ is convergence, i.e.,

$$\exists u \in W^{k,r}, \quad \lim_{n\to\infty} V_n = u = \binom{x}{y},$$

(iii) u *in* (ii) *is the exact solution.*

Proof. (i) By using definition 5.3, lemmas 5.4 and 5.5 and Theorem 5.6 we can conclude this claim.

(ii) It is enough to show that, $\{V_n\}_{n=1}^{\infty}$ is a Cauchy sequence in the Sobolev space. For this reason, consider, for every $n,m \in \mathbb{N}$, $n \ge m$, we have

$$\| V_n - V_m \|_{W^{k,r}} = \| (V_n - V_{n-1}) + (V_{n-1} - V_{n-2}) + \cdots + (V_{m+1} - V_m) \|_{W^{k,r}}$$

$$\le \| V_n - V_{n-1} \|_{W^{k,r}} + \| V_{n-1} - V_{n-2} \|_{W^{k,r}} + \cdots + \| V_{m+1} - V_m \|_{W^{k,r}}$$

$$= \| A_j^{\alpha,\beta}(V_{n-1}) - A_j^{\alpha,\beta}(V_{n-2}) \|_{W^{k,r}} + \| A_j^{\alpha,\beta}(V_{n-2}) - A_j^{\alpha,\beta}(V_{n-3}) \|_{W^{k,r}}$$

$$+ \cdots + \| A_j^{\alpha,\beta}(V_m) - A_j^{\alpha,\beta}(V_{m-1}) \|_{W^{k,r}}$$

$$\le \eta^{n-1} \| V_1 - v_0 \|_{W^{k,r}} + \eta^{n-2} \| V_1 - v_0 \|_{W^{k,r}} + \cdots + \eta^m \| V_1 - v_0 \|_{W^{k,r}}$$

$$\le (\eta^m + \eta^{m+1} + \cdots) \| V_1 - v_0 \|_{W^{k,r}} = \frac{\eta^m}{1-\eta} \| V_1 - v_0 \|_{W^{k,r}}.$$

Since $\| V_1 - v_0 \|_{W^{k,r}} < \infty$, hence, $\lim_{n.m\to+\infty} \| V_n - V_m \|_{W^{k,r}} = 0$, i.e., $\{V_n\}_{n=1}^{\infty}$ is a Cauchy sequence in the Sobolev space $W^{k,r}$ and it implies that

$$\exists u \in W^{k,r}, \quad \lim_{n\to+\infty} V_n = u.$$

(iii)Using (ii), we have

$$A_j^{\alpha,\beta}(u) = A_j^{\alpha,\beta}(\lim_{n\to+\infty} V_n) = \lim_{n\to+\infty} A_j^{\alpha,\beta}(V_n) = \lim_{n\to+\infty} V_{n+1} = u,$$

i.e., u is a solution of Eq.(1).

We can split the operator of $A_j^{\alpha,\beta}$ to the following linear and nonlinear operators:

$$A_j^{\alpha,\beta}(u) = A_{L,j}^{\alpha,\beta}(u) + A_{N,j}^{\alpha,\beta}(u),$$

where $A_{L,j}^{\alpha,\beta}$, $A_{N,j}^{\alpha,\beta}$ are linear and nonlinear operators, respectively. Moreover, we use the idea of the derivative of an operator, by generalizing the definition for the derivative of a function of one variable. Hence, this generalization can be made in different ways. We are led to several possible definitions for the derivative of an operator but in this paper we will use the definition of *Fréchet derivative*. In the following we recall it:

Definition 5.3 *Let* $A_j^{\alpha,\beta}: W^{k,r} \to W^{k,r}$ *and* $\overline{u} \in W^{k,r}$ *and there exists a bounded linear operator* $(A_j^{\alpha,\beta})'(\overline{u})$ *such that for all* $\| \Delta u \|_{W^{k,r}} \to 0$

$$\lim_{\|\Delta u\|_{W^{k,r}} \to 0} \frac{\| A_j^{\alpha,\beta}(\overline{u} + \Delta u) - A_j^{\alpha,\beta}(\overline{u}) - (A_j^{\alpha,\beta})'(\overline{u})\Delta u \|_{W^{k,r}}}{\| \Delta u \|_{W^{k,r}}} = 0,$$

then we say that $A_j^{\alpha,\beta}$ is strongly differentiable at $\overline{u}$. The operator $(A_j^{\alpha,\beta})'(\overline{u})$ is called the strong or *Fréchet derivative*.

We can also define the higher order of differential it (see [3]). In elementary numerical analysis Taylor's theorem is frequently used in analyzing algorithms. For the operator of $A_j^{\alpha,\beta}$ there exists an extension of Taylor's Theorem which is equally used. In the following we highlight some lemmas about it.

Lemma 5.4 *The second Fréchet derivative of* $A_{L,j}^{\alpha,\beta}$ *is zero that is for every* $\overline{u} \in W^{k,r}$ *we have* $(A_{L,j}^{\alpha,\beta})'(\overline{u}) = A_{L,j}^{\alpha,\beta}$.

Proof. It is obvious from the above definition. Of course, we note that this does not say that $(A_{L,j}^{\alpha,\beta})'$ and $A_{L,j}^{\alpha,\beta}$ are identical, but that $(A_{L,j}^{\alpha,\beta})'$ has the same value $A_L^{\alpha,\beta}$ at all points $\overline{u} \in W^{k,r}$. The observation is analogous to the fact that the derivative of linear operator is a constant and we have $(A_{L,j}^{\alpha,\beta})'' = 0$.

Lemma 5.5 *The third Fréchet derivative of* $A_{N,j}^{\alpha,\beta}$ *is zero.*

Proof. Consider the operator of $A_{N,0}^{\alpha,\beta}(u)$, we can write

$$(A_{N,0}^{\alpha,\beta})'''(\overline{u}) = 0,$$

as is easily verified. We can repeat the above proof for every $A_{N,j}^{\alpha,\beta}$, $j = 1,2,\dots,n-1$ by induction.

Theorem 5.6 *The operator of* $A_j^{\alpha,\beta}$ *in Eq.(21) is a contraction mapping if* $\eta < 1$, $(A_j^{\alpha,\beta})'$ *and* $(A_j^{\alpha,\beta})''$ *are bounded in some neighborhood.*

Proof. we can write:

$$\| A_j^{\alpha,\beta}(u) - A_j^{\alpha,\beta}(\overline{u}) \|_{W^{k,r}} = \| A_j^{\alpha,\beta}(u) - A_j^{\alpha,\beta}(\overline{u}) - (A_j^{\alpha,\beta})'(u)(u-\overline{u}) + (A_j^{\alpha,\beta})'(u)(u-\overline{u}) \|_{W^{k,r}} \leq$$

$$\leq \| A_j^{\alpha,\beta}(u) - A_j^{\alpha,\beta}(\overline{u}) - (A_j^{\alpha,\beta})'(u)(u-\overline{u}) \|_{W^{k,r}} + \| (A_j^{\alpha,\beta})'(u)(u-\overline{u}) \|_{W^{k,r}}.$$

According to the theorem of generalized Taylor (see [3]) and the above lemmas, we can write the following inequality, where $l_j(\overline{u}, u)$ is the line segment between $\overline{u}$, u :

$$\| A_j^{\alpha,\beta}(u) - A_j^{\alpha,\beta}(\overline{u}) \|_{W^{k,r}} \leq \sup_{u \in l_j(\overline{u},u)} \| (A_{N,j}^{\alpha,\beta})''(\overline{u}) \|_{W^{k,r}} \frac{\| u - \overline{u} \|_{W^{k,r}}^2}{2}$$

$$+ \| (A_j^{\alpha,\beta})'(\overline{u}) \|_{W^{k,r}} \| u - \overline{u} \|_{W^{k,r}} \leq \eta \| u - \overline{u} \|_{W^{k,r}},$$

where $\eta = \sup_{u \in l_j(\overline{u},u)} \| (A_{N,j}^{\alpha,\beta})''(\overline{u}) \|_{W^{k,r}} \frac{\|u-\overline{u}\|_{W^{k,r}}}{2} + \| (A_j^{\alpha,\beta})'(\overline{u}) \|_{W^{k,r}}$.

6 Contraction mapping in FVIM

The simplicity of contraction mapping in FVIM for Eq.(1) is the fact that it is un-complicated with applying the above section.

Theorem 6.1 *The operator of* $B_j^{\alpha,\beta}$ *in Eq.(15) is a contraction mapping if* $\eta < 1$, $(B_j^{\alpha,\beta})'$ *and* $(B_j^{\alpha,\beta})''$ *are* *bounded in some neighborhood.*

Proof. We can split the operator of $B_j^{\alpha,\beta}$ to the following linear and nonlinear operators:

$$B_j^{\alpha,\beta}(u) = B_{L,j}^{\alpha,\beta}(u) + B_{N,j}^{\alpha,\beta}(u),$$

where $B_{L,j}^{\alpha,\beta}$, $B_{N,j}^{\alpha,\beta}$ are linear and nonlinear operators respectively. Consider the operator, of $B_{N,j}^{\alpha,\beta}(u)$, we can obtain the following results:

$$(B_{L,j}^{\alpha,\beta})''(\overline{u}) = 0,$$

$$(B_{N,j}^{\alpha,\beta})'''(\overline{u}) = 0,$$

as is easily verified. Also, we can write:

$$\| B_j^{\alpha,\beta}(u) - B_j^{\alpha,\beta}(\overline{u}) \|_{W^{k,r}} = \| B_j^{\alpha,\beta}(u) - B_j^{\alpha,\beta}(\overline{u}) - (B_j^{\alpha,\beta})'(u)(u-\overline{u}) + (B_j^{\alpha,\beta})'(u)(u-\overline{u}) \|_{W^{k,r}}$$

$$\leq \| B_j^{\alpha,\beta}(u) - B_j^{\alpha,\beta}(\overline{u}) - (B_j^{\alpha,\beta})'(u)(u-\overline{u}) \|_{W^{k,r}} + \| (B_j^{\alpha,\beta})'(u)(u-\overline{u}) \|_{W^{k,r}}.$$

According to the theorem of generalized Taylor (see [12]) and the above lemmas, we can write the following inequality, where $l_j(\overline{u},u)$ is the line segment between $\overline{u}$, u :

$$\| B_j^{\alpha,\beta}(u) - B_j^{\alpha,\beta}(\overline{u}) \|_{W^{k,r}} \leq \sup_{u\in l_j(\overline{u},u)} \| (B_{N,j}^{\alpha,\beta})''(\overline{u}) \|_{W^{k,r}} \frac{\| u-\overline{u} \|_{W^{k,r}}^2}{2}$$

$$+ \| (B_j^{\alpha,\beta})'(\overline{u}) \|_{W^{k,r}} \| u-\overline{u} \|_{W^{k,r}} \leq \eta \| u-\overline{u} \|_{W^{k,r}},$$

where $\eta = \sup_{u\in l_j(\overline{u},u)} \| (B_{N,j}^{\alpha,\beta})''(\overline{u}) \|_{W^{k,r}} \frac{\|u-\overline{u}\|_{W^{k,r}}}{2} + \| (B_j^{\alpha,\beta})'(\overline{u}) \|_{W^{k,r}}$.

7 Stability of fractional Lotka-Volterra System

The Stability of fractional Lotka-Volterra model of the fixed point (x^*, y^*) at the origin $0^T = (0,0)$ can be determined by performing a nonlinear using partial derivatives, while the other fixed point requires a slightly more sophisticated method. In this section, Eq.(1), is a perturbation in the structure of the model (see[36]). We can create the Jacobian matrix at the fixed point for the predator-prey describe by the Eq.(1).

$$J^* = J(x^*,y^*) = \begin{pmatrix} \frac{\partial F_1(x^*,y^*)}{\partial x} & \frac{\partial F_1(x^*,y^*)}{\partial y} \\ \frac{\partial F_2(x^*,y^*)}{\partial x} & \frac{\partial F_2(x^*,y^*)}{\partial y} \end{pmatrix} = \begin{pmatrix} a(t)-b(t)y^* & -b(t)x^* \\ c(t)y^* & c(t)x^*-d(t) \end{pmatrix}. \quad (22)$$

The constant coefficients, J^*, of Jacobian matrix is identified with the above matrix. Near a fixed point (x^*, y^*), of the nonlinear system are similar (see [33]) to the following of the linear system associated with the J^* matrix, provided the eigenvalues of the J^* matrix have non-zero real parts i.e.

$$D_t^{\alpha,\beta}u(t) = J^*u(t). \quad (23)$$

The origin, is an equilibrium fixed point of Eq. (21) because $J^*0 = 0$. We can discuss the behaviour of stability in the solution vector near the origin, by finding the eigenvalues and eigenvectors of J^*.

Let

$$P = trace(J^*) = a(t) - b(t)y^* + c(t)x^* - d(t),$$

and

$$q = det(J^*) = c(t)x^*a(t) - d(t)a(t) + d(t)b(t)y^*,$$

are the trace and determinent for J^* respectivly. Then we have the characteristic equations:

$$f(k) = k^2 - pk + q.$$

We conclude that the eigenvalues from above equation, k_1 and k_2, are as follows:

$$k_1 = \frac{p+\sqrt{p^2-4q}}{2},$$

and

$$k_2 = \frac{p-\sqrt{p^2-4q}}{2}.$$

Therefore we can write $p = k_1 + k_2$ and $q = k_1k_2$. Then we have:

$$d = disc(J^*) = p^2 - 4q. \quad (24)$$

Where $a(t), b(t), c(t), d(t)$ are parameters that represent the interaction between the entities of the model (see Fig 1). These fixed points are $(x_1^*, y_1^*) = (0,0)$, and $(x_2^*, y_2^*) = (\frac{c(t)}{d(t)}, \frac{a(t)}{b(t)})$.

In the trivial fixed point, we have $q = det(J^*) = -a(t)d(t)$ and at $(x_2^*, y_2^*) = (\frac{c(t)}{d(t)}, \frac{a(t)}{b(t)})$, the Jacobian has $p = trace(J^*) = 0$, and $q = det(J^*) = a(t)d(t)$. Consequently, we have topological classification of fixed point for this system in Table 1 (see [33, 36]):

Table 1. Topological classification of fixed point.

Hyperbolic Fixed Points	Repellers (Sources) Unstable	$p > 0, \quad q > 0$	$Re(\kappa_1), \quad Re(\kappa_2) > 0$
	Attractors (Sinks) Stable	$p < 0, \quad q > 0$	$Re(\kappa_1), \quad Re(\kappa_2) > 0$
	Saddle Points Unstable	$q > 0$	$Re(\kappa_1) < 0, Re(\kappa_2) > 0$
Non-hyperbolic Fixed Points	Marginal Case 1	$p = 0, \quad q > 0$	$Re(\kappa_1), \quad Re(\kappa_2) = 0$
	Marginal Case 2	$p \neq 0, \quad q = 0$	$k_1 = 0,$ or $\kappa_2 = 0$
	Marginal Case 3	$q = p = 0$	$k_1 = 0,$ and $\kappa_2 = 0$

In the next section,we give some numerical stable and unstable results.

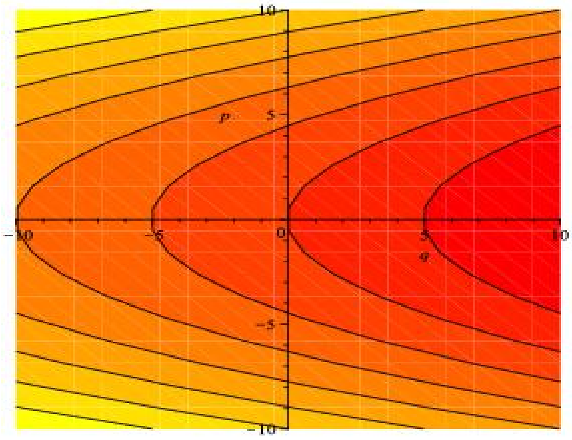

***Fig*. 1**. Contour plot of the characteristic equation.

8 Numerical results

This section graphically shows numerical results of the approximation solution in the prey and predator population in case 1, for Eq.(16), by using FVIM for different fractional Brownian motion such as $\alpha = \beta = 0.2$, 0.3, 0.5, 0.7 and for the standard motion $\alpha = \beta = 1$ are calculated for various values of time t and for constant initial condition $\delta = 1.3$, $\gamma = 0.6$. From the Fig. 2, it is clear to see the time evolution of prey-predator population density and we also know that the numerical solution of fractional prey-predator population model is continuous with the parameter α and β. In Fig. 3, which graphically represents, (a) prey decreases and (b) predator increases with time. It is observed that it takes more time for meeting prey-predator populations as the fractional time derivatives increases and finally takes, the maximum time for the standard motion i.e. $\alpha = \beta = 1$, and so the path intersection of the curves prey and predator is linear.

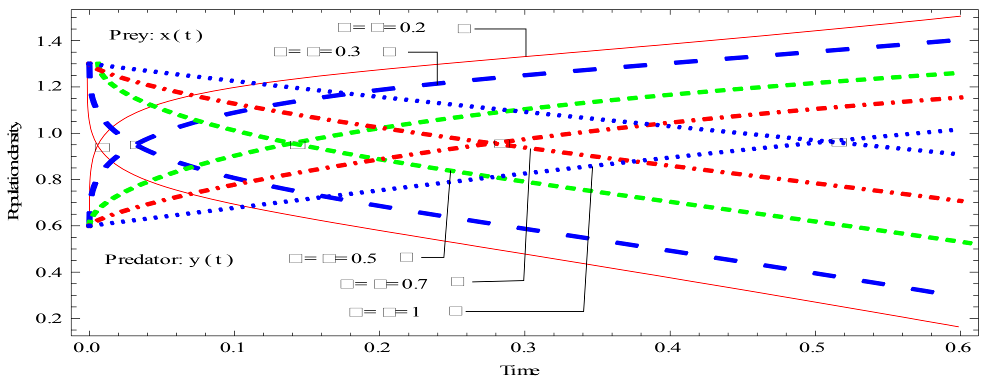

***Fig*. 2** . Plot the rate of prey and the predators population versus the time for different values of α, β.

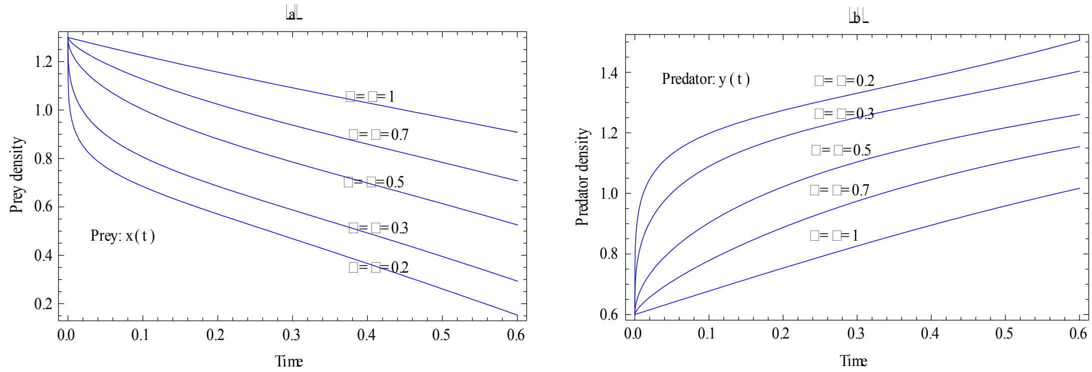

***Fig*. 3.** (a) Plot the rate of prey population versus the time for different values of α, β,
(b) Plot the rate of predator population versus the time for different values of α, β.

Table 2 shows the approximate solutions of prey-predator system for Eq.(16) with the constant initial condition $\delta = 1.3, \ \gamma = 0.6$ by fractional variational iteration method, using homotopy perturbation method and Sinc-Nyström method when parameters $\alpha = \beta = 0.5, 1$. It is noted that only with two iterations for FVIM versus of the third-order of the HPM were used. From the comparison of the numerical values with FVIM, HPM and Sinc-Nyström method, we observe that the time t and the parameters α and β increase, so the error for the three methods grows.

Table 2. Comparison of FVIM, HPM and Sinc-Nyström method with initial condition
$\delta = 1.3, \ \gamma = 0.6, \ N = 10, \ d = \frac{\pi}{3} \ and \ \bar{r} = 1.$

t	$\alpha = \beta$	Numerical value (x, y) of FVIM	Numerical value (x, y) of HPM	Numerical value (x, y) of Sinc-Nyström
0.1	0.5	(1.00237, 0.90287)	(1.00059, 0.90308)	(1.00139, 0.90296)
0.1	1	(1.22546, 0.67752)	(1.22565, 0.67733)	(1.22564, 0.67733)
0.2	0.5	(0.88071, 1.02192)	(0.88325, 1.01172)	(0.88189, 1.01691)
0.2	1	(1.15668, 0.75329)	(1.15815, 0.75185)	(1.15750, 0.75248)
0.3	0.5	(0.78523, 1.10387)	(0.79980, 1.07099)	(0.79242, 1.08752)
0.3	1	(1.09203, 0.82613)	(1.09682, 0.82146)	(1.09451, 0.82360)
0.4	0.5	(0.69859, 1.16608)	(0.73338, 1.09896)	(0.71589, 1.13261)
0.4	1	(1.03003, 0.89492)	(1.04099, 0.88437)	(1.03560, 0.88955)
0.5	0.5	(0.61345, 1.21668)	(0.67735, 1.10501)	(0.64531, 1.16093)
0.5	1	(0.96927, 0.95869)	(0.98996, 0.93907)	(0.97970, 0.94879)
0.6	0.5	(0.52570, 1.26082)	(0.62898, 1.09571)	(0.57725, 1.17835)
0.6	1	(0.90838, 1.01663)	(0.94303, 0.98441)	(0.92580, 1.00043)
0.7	0.5	(0.43270, 1.30244)	(0.58777, 1.07632)	(0.51014, 1.18947)
0.7	1	(0.84600, 1.06814)	(0.89951, 1.01959)	(0.87284, 1.04377)
0.8	0.5	(0.33250, 1.34482)	(0.55487, 1.05125)	(0.44359, 1.19812)
0.8	1	(0.78069, 1.11290)	(0.85875, 1.04420)	(0.81981, 1.07846)
0.9	0.5	(0.22357, 1.39095)	(0.53293, 1.02434)	(0.37816, 1.20773)
0.9	1	(0.71094, 1.15089)	(0.82024, 1.05819)	(0.76568, 1.10445)
1	0.5	(0.10459, 1.44365)	(0.52594, 0.99886)	(0.31517, 1.22134)
1	1	(0.63505, 1.18245)	(0.78368, 1.06190)	(0.70945, 1.12208)

In the Fig. 4 we use the results of above table graphically. The plots are the rate of prey and predator population versus the time with constant initial condition $\delta = 1.3, \ \gamma = 0.6$, for different values of α, β. So FVIM shows by solid line, HPM by dashed line and Sinc-Nyström method by dotdashed line.

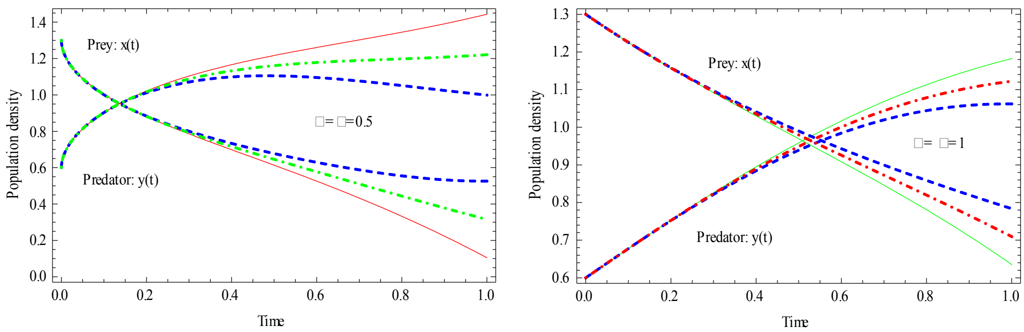

***Fig*. 4**. Plot the rate of prey and the predator population versus the time:
solid line by FVIM, dashed line by HPM and dotdashed line by Sinc-Nyström method.

9 Conclusion

This problem was solved numerically by the homotopy perturbation method (see S. Das and P.K. Gupta [9]). In our opinion, in some sense, the research results will be of theoretical significance and practical value for constructing the exact as well as approximate solution of nonlinear evolution equations. We observe that neither HPM nor FVIM tell us anything about the possibility of convergence analysis. But in this paper, one of the shortcoming in these methods is removed. Moreover, we will discuss in future about convergence analysis and stability of Sinc-Nyström method in other paper.

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