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Multigrid Method for 2D Helmholtz Equation using Higher Order Finite Difference Scheme Accelerated by Krylov Subspace

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ABSTRACT

A sixth-order compact difference scheme is applied with uniform mesh sizes in different coordinate directions to discretize a two dimensional Helmholtz equation. Multigrid method is designed to solve the resulting sparse linear systems. Numerical results are conducted to test the accuracy and performance of the sixth-order compact difference scheme with multigrid method and to compare it with the standard second-order difference scheme and fourth-order compact difference scheme. The errors norms L_2 is used to establish efficiency and accuracy of the proposed scheme with multigrid method.

KEYWORDS: Helmholtz equation, Compact iterative schemes, Multigrid method, Krylov subspace.

1 INTRODUCTION

The struggle for computing accurate solution using different grid sizes have increased researchers curiosity for developing high order difference schemes. Compact finite difference scheme is widely used in vast area of computational problems such as, the Helmholtz equations and other elliptic equations [7,1]. We seek high-accuracy numerical solution of the two-dimensional Helmholtz equation as

$$u_{xx}+u_{yy}+k^2u=f(x,y),(x,y)\in\Omega. \tag{1}$$

where Ω is a rectangular domain. Equation (1) is an elliptic partial differential equation (PDE). The above equation has broad application in physical phenomena, such as elasticity, electromagnetic waves, acoustic wave scattering, electromagnetic fields, water wave propagation, noise reduction in silencers, radar scattering and membrane vibration are governed in frequency domain. The forcing function $f(x,y)$ and the solution $u(x,y)$ are assumed to have the required continuous partial derivatives up to specific orders and sufficiently smooth. In this study we analyze finite difference approximation on uniform step size $\Delta x=\Delta y=h$, both in x and y directions. In this study we use a constant value of k to obtain a scheme with sixth order of accuracy[8]. The discretized form of eq. (1) is

$$\delta_x^2\, u_{i,j} + \delta_y^2\, u_{i,j+k^2u_{i,j}=f_{i,j}+O(h^2)} \tag{2}$$

Equation (1) has been numerically solved by different techniques and many approaches are developed such as the finite difference method (FDM) [6], finite element method (FEM) [3], the spectral-element method [4], compact finite-difference method [5].

In finite-difference methods, the number of mesh points will be enlarged to increase the accuracy but this is not desirable. Helmholtz equation is extensively solved by FEM, but the limitation of this method is the high computational cost. In spectral-element method, it needs fever mesh points per wavelength as compared to the FEM [4]. But the matrix is less sparse compared to the resulting finite-element method therefore computational time of both methods remain the same [4]. Many iterative techniques for the Helmholtz equation supper due to their slow convergence, when high frequencies are required. The investigation for fast iterative methods to acheive high-frequency Helmholtz equations is the interest area of research now a days.

In general to obtain more accuracy in solution by increasing nodes, which needs more computational time and space for storage. In this regards Turkel and Singer done a noticeable work [6]. They have developed compact finite-difference method using fourth order with Dirichlet and/or Neumann boundary conditions. Later on M. K. Siddique et al [5] developed a sixth-order method (6th) for equation (1) with Neumann boundary conditions in two-dimension (2D) as well as in one-dimension (1D). In this work the basic issue discussed is to develop the multigrid method (MGM) for solving Helmholtz equation with Dirichlet and Neumann boudary conditions. In order to accelerate multigrid method

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we are using Krylov subspace. However MGM alone is also convergent rapidly, but in some cases it converges slowly. In this work the basic issue discussed is to develop a 6th-order compact finite-difference method (6th-CFDM) for solving two-dimensional Helmholtz equation with MGM. This study is based on 6th-order compact finite diffeence scheme and the designing of specialized MGM using Krylov subspace as an accelerated multigrid.

This paper is organized as follows :

Section (2) describes the higher order compact scheme, section (3) shows the boundary conditions, in section (4) we have discuss multigrid method, section (5) describes Krylov subspace method, section (6) shows some numerical results and section (7) is the conclusion of the work.

2 Higher Order Compact Scheme

The two-dimensional Helmholtz equation

$$u_{xx}(x,y)+u_{yy}(x,y)+k^2u(x,y)=f(x,y),(x,y)\in\Omega. \tag{3}$$

Taylor series expansion is performed for appropriate description of the descritized field where the grid spacing $\Delta x=\Delta y=h$.

$$u_{i+1,j}=u_{i,j}+h\partial_x u_{i,j}+\frac{h^2}{2}\partial_x^2 u_{i,j}+\frac{h^3}{6}\partial_x^3 u_{i,j}+\frac{h^4}{24}\partial_x^4 u_{i,j}+\frac{h^5}{120}\partial_x^5 u_{i,j}+\frac{h^6}{720}\partial_x^6 u_{i,j}+O(h^7), \tag{4}$$

$$u_{i-1,j}=u_{i,j}-h\partial_x u_{i,j}+\frac{h^2}{2}\partial_x^2 u_{i,j}-\frac{h^3}{6}\partial_x^3 u_{i,j}+\frac{h^4}{24}\partial_x^4 u_{i,j}-\frac{h^5}{120}\partial_x^5 u_{i,j}+\frac{h^6}{720}\partial_x^6 u_{i,j}+O(h^7), \tag{5}$$

adding the above expressions and solving for the second derivative which gives :

$$\partial_x^2\, u_{i,j}=\frac{u_{i+1,j}-2u_{i,j}+u_{i-1,j}}{h^2}-\frac{h^2}{12}\partial_x^4\, u_{i,j}-\frac{h^4}{360}\partial_x^6\, u_{i,j}+O(h^6), \tag{6}$$

using equation (6), we have

$$\delta_x^2 u_{i,j}=\partial_x^2 u_{i,j}+\frac{h^2}{12}\partial_x^4 u_{i,j}+\frac{h^4}{360}\partial_x^6 u_{i,j}+O(h^6), \tag{7}$$

similarly we can find the approximation for the variable y. Therefore the central difference scheme for Helmholtz equation can be written as :

$$\delta_x^2 u_{i,j}+\delta_y^2 u_{i,j}+k^2(u_{i,j})+\alpha_{i,j}=f_{i,j}+O(h^6), \tag{8}$$

where

$$\alpha_{i,j}=-\frac{h^2}{12}\left[\frac{\partial^4 u_{i,j}}{\partial x^4}+\frac{\partial^4 u_{i,j}}{\partial x^4}\right]_{i,j}-\frac{h^4}{360}\left[\frac{\partial^6 u_{i,j}}{\partial x^6}+\frac{\partial^6 u_{i,j}}{\partial x^6}\right]_{i,j}+O(h^6). \tag{9}$$

The derivatives of higher order is expressed by differentiating equation (3), this process also applied on the forcing function $f(x,y)$. Applying appropriate derivatives of equation (3), we get

$$\left(\frac{\partial^4 u}{\partial x^4}\right)_{i,j}=\left(\frac{\partial^2 f}{\partial x^2}-k^2\frac{\partial^2 u}{\partial x^2}-\frac{\partial^4 u}{\partial x^2\partial y^2}\right)_{i,j},\quad \left(\frac{\partial^4 u}{\partial y^4}\right)_{i,j}=\left(\frac{\partial^2 f}{\partial y^2}-k^2\frac{\partial^2 u}{\partial y^2}-\frac{\partial^4 u}{\partial y^2\partial x^2}\right)_{i,j} \tag{10}$$

putting above equation in equation (9), we get

$$\alpha_{i,j}=-\frac{h^2}{12}\left[\left(\frac{\partial^2 f}{\partial x^2}+\frac{\partial^2 f}{\partial y^2}\right)_{i,j}-k^2\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)_{i,j}-2\left[\frac{\partial^4 u}{\partial x^2\partial y^2}\right]_{i,j}\right]$$
$$-\frac{h^4}{360}\left[\frac{\partial^6 u}{\partial x^6}+\frac{\partial^6 u}{\partial x^6}\right]_{i,j}+O(h^6). \tag{11}$$

Now in order to find the fourth-order approximation of $\left[\frac{\partial^4 u}{\partial x^2\partial y^2}\right]_{i,j}$ in above equation, which can be obtained from Taylor series expansion such that

$$\left[\frac{\partial^4 u}{\partial^2 x\partial^2 y}\right]_{i,j}=\delta_x^2\delta_y^2 u_{i,j}-\frac{h^2}{12}\left(\frac{\partial^6 u}{\partial x^4\partial y^2}+\frac{\partial^6 u}{\partial x^2\partial y^4}\right)_{i,j}+O(h^4), \tag{12}$$

now putting equation (12) in equation (11), we get

$$\alpha_{i,j}=\frac{h^2}{12}\left(-\nabla^2 f_{i,j}+2\delta_x^2\delta_y^2 u_{i,j}+k^2 f_{i,j}-k^4 u_{i,j}\right)-$$

$$\frac{h^4}{360}\left[\frac{\partial^6 u}{\partial x^6}+\frac{\partial^6 u}{\partial y^6}+5\frac{\partial^6 u}{\partial x^4\partial y^2}+5\frac{\partial^6 u}{\partial x^2\partial y^4}\right]_{i,j}+O(h^6). \tag{13}$$

For sixth-order scheme we need expression sixth order derivatives in equation (13), differentiating equation (3), we have

$$\left(\frac{\partial^6 u}{\partial x^4\partial y^2}+\frac{\partial^6 u}{\partial x^2\partial y^4}+k^2\frac{\partial^4 u}{\partial x^2\partial y^2}\right)_{i,j}=\frac{\partial^4 f_{i,j}}{\partial x^2\partial y^2}, \tag{14}$$

$$\left(\frac{\partial^6 u}{\partial x^6}+\frac{\partial^6 u}{\partial y^6}\right)_{i,j}=\nabla^4 f_{i,j}-k^2\left(\frac{\partial^4 u}{\partial x^4}+\frac{\partial^4 u}{\partial y^4}\right)_{i,j}-\left(\frac{\partial^6 u}{\partial x^4\partial y^2}+\frac{\partial^6 u}{\partial x^2\partial y^4}\right)_{i,j}, \tag{15}$$

putting equation (10) and equation (14) in equation (15), we get

$$\left(\frac{\partial^6 u}{\partial x^6}+\frac{\partial^6 u}{\partial y^6}\right)_{i,j}=\nabla^4 f_{i,j}-\left[\frac{\partial^4 f}{\partial x^2\partial y^2}\right]_{i,j}+3k^2\left(\frac{\partial^4 u}{\partial x^2\partial y^2}-k^2\nabla^2 f+k^2(-k^2u+f)\right)_{i,j}, \tag{16}$$

using equation (14) and equation (16), also using all different derivative values of $u_{i,j}$ used in equation (13), that is

$$\alpha_{i,j}=\frac{h^2}{12}(-\nabla^2 f_{i,j}+2\delta_x^2\delta_y^2 u_{i,j}+k^2 f_{i,j}-k^4 u_{i,j})$$
$$-\frac{h^4}{360}(\nabla^4 f_{i,j}-k^2\nabla^2 f_{i,j}+4\left[\frac{\partial^4 f}{\partial x^2\partial y^2}\right]_{i,j}-2k^2\delta_x^2\delta_y^2 u_{i,j}-k^6 u_{i,j}+k^4 f_{i,j})+O(h^6). \tag{17}$$

Consequently, the compact higher-order approximation of the 2D Helmholtz equation be written in modified form which leads to :

$$\frac{h^2}{6}\left(1+\frac{k^2h^2}{30}\right)(\delta_x^2\delta_y^2)v_{i,j}+(\delta_x^2+\delta_y^2)v_{i,j}+k^2\left(1-\frac{k^2h^2}{12}+\frac{k^4h^4}{360}\right)v_{i,j}=$$
$$\left(1-\frac{k^2h^2}{12}+\frac{k^4h^4}{360}\right)f_{i,j}+\left(\frac{h^2}{12}(1-\frac{k^2h^2}{30})\right)\nabla^2 f_{i,j}+\frac{h^4}{360}\nabla^4 f_{i,j}+\frac{h^4}{90}\left[\frac{\partial^4 f}{\partial x^2\partial y2}\right]_{i,j}, \tag{18}$$

$$L.H.S=R.H.S$$

$u_{i,j}$ is approximated by $v_{i,j}$ which satisfies the original equation (3), and $v_{i,j}=u_{i,j}+O(h^6)$.

$$(\delta_x^2\delta_y^2)v_{i,j}=\frac{1}{h^4}[(v_{i+1,j+1}+v_{i+1,j-1}+v_{i-1,j+1}+v_{i-1,j-1})-2(v_{i,j+1}+v_{i,j-1}+v_{i+1,j}+v_{i-1,j})$$
$$+4(v_{i,j})], \tag{19}$$

using equation (19), the left hand side of equation (18) can be written as :

$$\frac{h^2}{6}\left(1+\frac{k^2h^2}{30}\right)(\delta_x^2\delta_y^2)v_{i,j}+(\delta_x^2+\delta_y^2)v_{i,j}+k^2\left(1-\frac{k^2h^2}{12}+\frac{k^4h^4}{360}\right)v_{i,j}=a_1v_{i,j}+a_2L_{i,j}^1+a_3L_{i,j}^2, \tag{20}$$

where

$$a_1=-\frac{10}{3}+k^2h^2\left(\frac{46}{45}-\frac{k^2h^2}{12}+\frac{k^4h^4}{360}\right),\quad a_2=\frac{2}{3}-\frac{k^2h^2}{90},\quad a_3=\frac{1}{6}+\frac{k^2h^2}{180},$$

and

$$L_{i,j}^1=v_{i+1,j}+v_{i-1,j}+v_{i,j+1}+v_{i,j-1},$$

$$L_{i,j}^2=v_{i+1,j+1}+v_{i+1,j-1}+v_{i-1,j+1}+v_{i-1,j-1}.$$

Also the right hand side of equation (18) can be express as:

$$R.H.S = b_{11}f_{i,j}+b_{12}\nabla^2 f_{i,j}+b_{13}\nabla^4 f_{i,j}+b_{14}\left[\frac{\partial^4 f}{\partial x^2\partial y2}\right]_{i,j},$$
$$= b_1 f_{i,j}+\mathrm{b}_2\mathrm{R}_{\mathrm{i,j}}^1+\mathrm{b}_3\mathrm{R}_{\mathrm{i,j}}^2 \tag{21}$$

where ∇^2 is the laplace operator and ∇^4 is the bi-harmonic operator. where

$$b_1=\frac{h^2}{90}\left(63-\frac{13h^2k^2}{2}+\frac{h^4k^4}{4}\right),\quad b_2=\frac{h^2}{360}(23-h^2k^2),\quad b_3=\frac{h^2}{90},$$

and

$$R^1_{i,j}=f_{i+1,j}+f_{i-1,j}+f_{i,j+1}+f_{i,j-1},$$

$$R^2_{i,j}=f_{i+1,j+1}+f_{i+1,j-1}+f_{i-1,j+1}+f_{i-1,j-1}.$$

Thus the compact-higher approximation of 2D Helmholtz equation can be written in more general form

$$a_1v_{i,j}+a_2L^1_{i,j}+a_3L^2_{i,j}=b_1f_{i,j}+b_2R^1_{i,j}+b_3R^2_{i,j}. \tag{22}$$

In equation (22), we assume that the derivative of the right hand side function $f_{i,j}$ can be determined numerically. In case where $f_{i,j}$ is not analytically known, then it can be approximated with high order finite differences. In this case one need only a 4th-order accurate approximation of $\nabla^2 f_{i,j}$ and a 2nd-order accurate approximation of $\nabla^4 f_{i,j}$ and $\left[\frac{\partial^4 f}{\partial x^2\partial y2}\right]_{i,j}$. Fourth-order accurate approximation to $\nabla^2 f_{i,j}$ is obtained by using $(1-\frac{h^2}{12}\delta^2)^{-1}\ \delta^2 f_{i,j}$, that can be further express as

$$\frac{1}{12h^2}[-f_{i+1,j}-f_{i-1,j}+16f_{i+\frac{1}{2},j}+16f_{i-\frac{1}{2},j}-60f_{i,j}+16f_{i,j+\frac{1}{2}}+16f_{i,j-\frac{1}{2}}-f_{i,j+1}-f_{i,j-1}]. \tag{23}$$

Similarly the second-order accurate approximation to $\nabla^4 f_{i,j}$ can be express as

$$\nabla^4 f_{i,j}=[\frac{\partial^4 f}{\partial x^4}+\frac{\partial^4 f}{\partial y^4}]=\frac{1}{h^2}[f''''_{i+1,j}+f''''_{i-1,j}+f''''_{i+1,j}+f''''_{i-1,j}-4f''''_{i,j}]. \tag{24}$$

And the second-order accurate approximation of $[\frac{\partial^4 f}{\partial x^2\partial y^2}]$ is

$$[\frac{\partial^4 f}{\partial x^2\partial y^2}]=(\delta_x^2\delta_y^2)f_{j,k}=\frac{1}{h^2}[f''_{i+1,j}+f''_{i-1,j}+f''_{i+1,j}+f''_{i-1,j}-4f''_{i,j}]. \tag{25}$$

Using the usual natural ordering on equation (18), leads to a system of equation of the form $Au=b$, where A is a sparse and symmetric matrix.

3 Boundary Condition

When a Dirichlet boundary condition is applied , then the formula in equation (18) can be used for all interior points. In case of Neumann boundary condition, we develop a sixth order accurate method for two dimensional case, that is $u_x=g(y)$. Similar formula hold in the other directions. Setting i=0, in equation (18) and introducing a obsess point i=−1. We specify Helmholtz equation and Neumann boundary condition at the boundary i=0. Consider the equation (3), using equation (4) and (5) we have,

$$\delta_{x0}u_i=\partial_x u_{i,j}+\frac{h^2}{6}\partial_x^3 u_{i,j}+\frac{h^4}{120}\partial_x^5 u_{i,j}+O(h^6). \tag{26}$$

From equation (3), we have

$$(\frac{\partial^3 u}{\partial x^3})_{i,j}=[\left(\frac{\partial f}{\partial x}-\frac{\partial^3 u}{\partial x\partial y^2}\right)-k^2\left(\frac{\partial u}{\partial x}\right)]_{i,j},\quad (\frac{\partial^5 u}{\partial x^5})_{i,j}=[(\frac{\partial^3 f}{\partial x^3}-\frac{\partial^5 u}{\partial x^3\partial y^2})-k^2(\frac{\partial^3 u}{\partial x^3})]_{i,j}, \tag{27}$$

in above equation (27), we need fourth order approximation of $(\frac{\partial^3 u}{\partial x\partial y^2})$ that is

$$(\frac{\partial^3 u}{\partial x\partial y^2})_{i,j}=\delta_x\delta_y^2 u_{i,j}-\frac{h^2}{6}(\frac{\partial^5 u}{\partial x\partial y^4}+2\frac{\partial^5 u}{\partial x^3\partial y^2})_{i,j}+O(h^4), \tag{28}$$

making use of equation (28) in equation (27) we have

$$(\frac{\partial^3 u}{\partial x^3})_{i,j}=\frac{h^2}{12}(\frac{\partial^5 u}{\partial x\partial y^4}+2\frac{\partial^5 u}{\partial x^3\partial y^2})_{i,j}-\delta_x\delta_y^2 u_{i,j}-k^2(\frac{\partial u}{\partial x})_{i,j}+(\frac{\partial^3 f}{\partial x^3})_{i,j}+O(h^4). \tag{29}$$

Now the second order approximation of $(\frac{\partial^3 u}{\partial x^3})$ is

$$(\frac{\partial^3 u}{\partial x^3})_{i,j}=(\frac{\partial f}{\partial x})_{i,j}-\delta_x\delta_y^2 u_{i,j}-k^2(\frac{\partial u}{\partial x})_{i,j}+O(h^2), \tag{30}$$

also the second order approximation of $(\frac{\partial^5 u}{\partial x^5})$ is

$$(\frac{\partial^5 u}{\partial x^5})_{i,j}=(\frac{\partial^3 f}{\partial x^3})_{i,j}-k^2(\frac{\partial f}{\partial x})_{i,j}+k^4(\frac{\partial u}{\partial x})_{i,j}-k^2\delta_x\delta_y^2 u_{i,j}-(\frac{\partial^5 u}{\partial x^3\partial y^2})_{i,j}+O(h^2). \tag{31}$$

Using the derivatives of equation (3), we have

$$(\frac{\partial^5 u}{\partial x^3\partial y^2}+\frac{\partial^5 u}{\partial x\partial y^4})_{i,j}=(\frac{\partial^3 f}{\partial x^3}-k^2(\frac{\partial^3 u}{\partial x\partial y^2}))_{i,j}. \tag{32}$$

Using equation (29), (31) and (32) in equation (26) we have

$$\begin{aligned}\delta_{x0}u_i-\frac{k^4h^4}{120}(\frac{\partial u}{\partial x})_{i,j}+\frac{h^2}{6}(1+\frac{k^2h^2}{30})\delta_x\delta_y^2 u_{i,j} &= (1-\frac{k^2h^2}{6}(\frac{\partial u}{\partial x})_{i,j})+\frac{h^2}{6}(1-\frac{k^2h^2}{20})(\frac{\partial f}{\partial x})_{i,j}\\ &+ \frac{h^4}{120}(\frac{\partial^3 u}{\partial x^3})_{i,j}+\frac{h^4}{72}(\frac{\partial^3 f}{\partial x\partial y})_{i,j}+O(h^6),\end{aligned} \tag{33}$$

for $(\frac{\partial u}{\partial x})$ in equation (33) the approximation used is

$$(\frac{\partial u}{\partial x})_{i,j}=\delta_{x0}u_i+\beta h^2\delta_x\delta_y^2 u_{i,j}, \tag{34}$$

where β is arbitrary constant, using equation (34) in equation (33), we have

$$\begin{aligned}L_1^*(v_{1,j}-v_{-1,j})+L_2^*(v_{1,j+1}+v_{1,j-1}+v_{-1,j+1}+v_{-1,j-1}) &= [(1-\frac{k^2h^2}{6})g+\frac{h^2}{6}(1-\frac{k^2h^2}{20})(\frac{\partial f}{\partial x})\\ &+ \frac{h^4}{120}(\frac{\partial^3 u}{\partial x^3})+\frac{h^4}{72}(\frac{\partial^3 f}{\partial x\partial y})]_{i,j},\end{aligned} \tag{35}$$

where $g=g(y)$ and $L_1^*=1-\frac{k^4h^4}{120}(1-2\beta)-\frac{1}{6h}(1+\frac{k^2h^2}{30})$ and $L_2^*=-\frac{k^4h^4}{120}(-\beta)-\frac{1}{12h}(1+\frac{k^2h^2}{30})$. Putting $i=0$ in equation(22), we have

$$\begin{aligned}&a_1v_{0,j}+a_2(v_{1,j}+v_{-1,j}+v_{0,j+1}+v_{0,j-1})+a_3(v_{1,j+1}+v_{1,j-1}+v_{-1,j+1}+v_{-1,j-1})\\ &=b_{11}f_{0,j}+b_{12}\nabla^2 f_{0,j}+b_{13}\nabla^4 f_{0,j}+b_{14}\left[\frac{\partial^4 f}{\partial x^2\partial y2}\right]_{0,j}.\end{aligned} \tag{36}$$

Multiplying equation (35) by β and add with equation (36), we get formula for boundary nodes

$$\begin{aligned}&a_1v_{0,j}+a_2(2v_{1,j}+v_{0,j+1}+v_{0,j-1})+a_3(v_{1,j+1}+v_{1,j-1})\\ &= b_{11}f_{0,j}+b_{12}\nabla^2 f_{0,j}+b_{13}\nabla^4 f_{0,j}+b_{14}\left[\frac{\partial^4 f}{\partial x^2\partial y2}\right]_{0,j}\\ &+\beta[(1-\frac{k^2h^2}{6})g_y+\frac{h^2}{6}(1-\frac{k^2h^2}{20})(\frac{\partial f}{\partial x})_{0,j}+\frac{h^4}{120}(\frac{\partial^3 u}{\partial x^3})_{0,j}+\frac{h^4}{72}(\frac{\partial^3 f}{\partial x\partial y})_{0,j}],\end{aligned} \tag{37}$$

where $\beta=\frac{a_2}{L_1^*}=\frac{a_3}{L_2^*}=\frac{120}{120-k^4h^4}$ to eliminate all the elements with $i=-1$. Equation (37) yields that for Neumann boundary points the stencil is of 5 points for 2 dimension. The matrix is inverted including the extra line $i=-1$. Applying Neumann boundary conditions accuracy and CPU timing remain the same in all examples.

Different methods are used to solve equation (22) such as finite difference scheme [6], fourth order compact finite difference scheme [5], LU decomposition etc. We are solving this equation by multigrid method with sixth order compact finite difference scheme using Krylov subspace method as a smoother.

4 Multigrid Method

The results obtained from 6th-order accurate CFDM, 4th-order CFDM and 2nd-order CDM are in sparse linear systems, can be solved efficiently by multigrid methods. In order to remove high frequency error, some relaxation methods are utilized in mgm. Multigrid method makes the use of coarse grid correction in order to smooth the error. Efficient multigrid method are used in [2, 10] for solving two dimensional poisson equation using fourth-order compact difference scheme. It is obvious from their results that multigrid method with sixth-order CFDM is most tiresome than the 4th-order CFM and the corresponding 2nd-order CDM. The standard multigrid method with a point Guass-Seidel type relaxation and standard mesh coarsening strategy does not work very well with unequal mesh sizes discretized poisson equation [9].

The strategy used is the line relaxation to replace point relaxation. That is the line Guass-Seidel relaxation can be shown to be very efficient in removing high frequency errors. The coefficient matrix of the sixth-order compact finite difference with this ordering can be written in block tri-diagonal matrix of the block order $(N-1)$. The order of the coefficient matrix U is of $(N-1)\times(N-1)$, where $U=diag[U_1,U_0,U_1]$, where $U_0=diag[a_2,a_1,a_2]$, $U_1=diag[a_3,a_2,a_3]$,

are both symmetric tri-diagonal sub-matrices of order $(N-1)$. They represent the sub matrix of of each grid line along one direction. Where u_j is the part of the solution vector representing the grid points on each jth line, and f_j is the corresponding part on the right-hand side vector. On each level U_0 needs only one factorization. The factorization cost of U_0 is negligible because matrix B has constant blocks which does not change from one grid line to another grid line.

In multigrid method with LU-decompositon or Gauss-Seidel relaxations, we use bilinear interpolation to transfer correction from a coarse grid to a fine grid, and we also use a full-weighting scheme to update the residual on a coarse grid.

Multigrid Algorithm:

Algorithm 1 *Assuming that we set up these parameters in multigrid :*

v_1 *is the no: of pre smoothing iterations.*

v_2 *is the no: of post smoothing iterations.*

$\gamma=1$ in our case for V-cylcel.

FAS Multigrid Cycle

$$\varphi^h \leftarrow FASCYC^h(\varphi^h, f^h, v_1, v_2, \gamma)$$

1. if Ω^h is the coarsest grid, then solve equation (18) using a time marching technique and then stop.
Else do the pre-smoothing step:

$$\varphi^h \leftarrow Smoother(\varphi^h, f^h, v_1, tol), \quad (Pre-Smoothig)$$

2. ***Restiction:***

$$\varphi^{2h}=I_h^{2h}\varphi^h, \varphi^{2h}=\varphi^{2h},$$

$$f^{2h}=I_h^{2h}(f^h-N^h\varphi^h)+N^{2h}\varphi^{2h},$$

$$\varphi^{2h} \leftarrow FASCYC^{2h}(\varphi^{2h}, f^{2h}, v_1, v_2).$$

3. ***Interpolation:***

$$\varphi^h \leftarrow \varphi^h + I_{2h}^h(\varphi^{2h}-\varphi^{2h}).$$

4.

$$\varphi^h \leftarrow Smoother^{v_2}(\varphi^h, f^h, v_2). \quad (Post-Smoothing)$$

Here the restriction operator I_h^{2h} is by full weighting and the interpolation I_{2h}^h by bilinear operator.

5 Krylov subspace as an accelerated multigrid

Krylov subspace iteration methods are an important techniques based on projection used to solve large linear systems. Krylov subspace technique is that technique in which the subspace K_m is spanned by the vectors of the krylov sequence

$$K_m(A,r_0)=span\{r-0, Ar_0, A^2r_0, \ldots A^{m-1}r_0\},$$

where K_m is the m-th Krylov subspace and $r_0=b-Ax_0$. In order to solve Helmholtz equation the standard multigrid method is modified in the following manner .

1. Krylov subspace iteration is used such as Generalized Minimum Residual Method (GMRES) as a smoother which provide a maximum reduction of oscillatory components.
2. Multigrid method is used as a preconditioner for a Generalized Minimum Residual Method iteration for the original equation $Ax=b$, to handle modes with eigen values that are closed to the origin on all grids.

We use V-cycle multigrid method and Generalized Minimum Residual Method iteration preconditioned by the same V-cycle multigrid method. The outer iteration is run until the stopping criterion

$$\frac{|r_m|}{|r_0|}<10^{-6}$$

is satisfied. Where r_m is the residual of m-th iteration.

6 Numerical Calculations

In order to check the efficiency and applicability of the multigrid method, we give solution of a 2D Helmholtz equation in [0,1]×[0,1] for different source functions $f(x,y)$ and homogeneous Dirichlet BC.

Example 1

$$u_{xx}(\mathrm{x},\mathrm{y})+u_{yy}(\mathrm{x},\mathrm{y})+k^2u(\mathrm{x},\mathrm{y})=(\mathrm{k}^2-2\pi^2)\sin(\pi\,\mathrm{x})\sin(\pi\,\mathrm{y}),\ \ \mathrm{x}\in[0,1],\mathrm{y}\in[0,1], \tag{38}$$

Exact solution is $u(x,y)=\sin(\pi x)\sin(\pi y)$.

In comparison sixth-order compact finite difference scheme is taken against the standard 4th and 2nd-order compact difference scheme, in terms of solution of accuracy, multigid convergence rate and CPU timing. All multigrid methods use V-cycle or W-cycle algorithm, the coarsest grid is the one with at least one dimension being the coarsest possible. One pre-smoothing iteration and one post smoothing iteration are applied at each level. The iteration stop when the 2-norm of the residual vector reduced by 10^{-14}. The maximum absolute error between the exact solution and the computed solution over the fine grid reported the maximum absolute error. We use l_2-norm for comparison the numerical solution and the exact solution and defined as:

$$|e_2|=\frac{1}{N}\sqrt{\sum_{i,j=0}^{N}e_{i,j}^2}\ , \tag{39}$$

where the error vector $e_{ij}=u_{ij}-v_{ij}$ and the residual $r=(f-Av)=Ae$, N is the number of nodes, k is the wave number, also M_1 is the MGM with 2nd-order, M_2 is MGM with 4th-order, M_3 is MGM with 6th-order and M_4 is MGM with sixth order accelerated by Krylov subspace. The l_2-norm of the error for k=10 and different values of N, are presented in the following table.

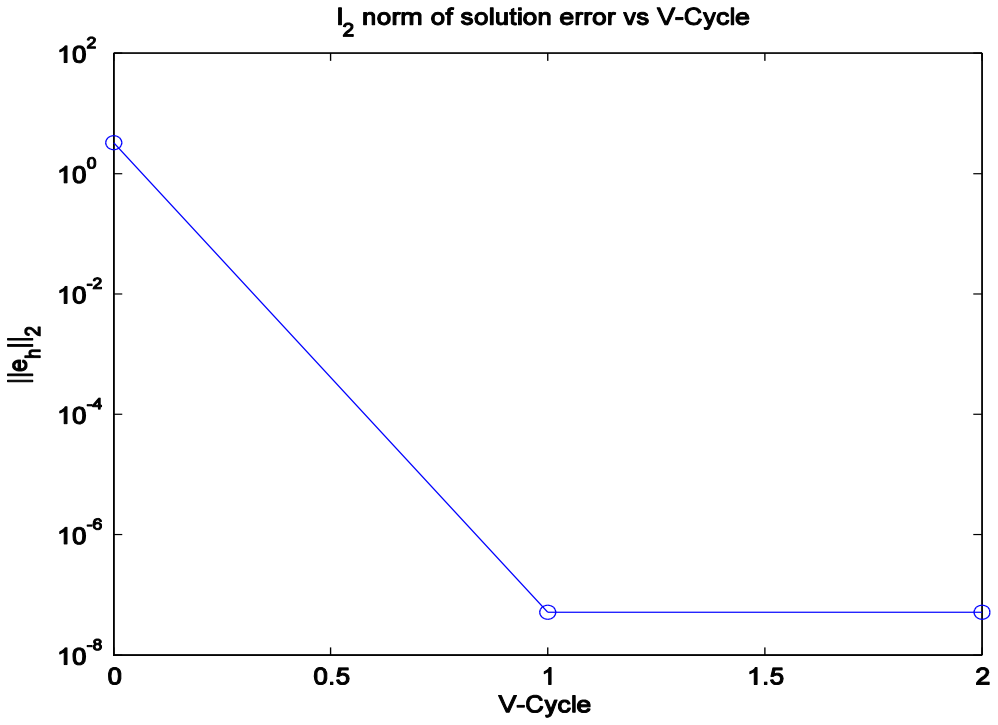

Table 1. Comparison of maximum absolute errors and CPU(seconds) for a multigrid method with different discretization schemes, for example 1, $|e_2|$, k=10, N=4,8,16,32,64,128

N	$M_1(\|e_2\|)$	CPU	V	$M_2(\|e_2\|)$	CPU	V	$M_3(\|e_2\|)$	CPU	V
4	$1.0000e^{0}$	0.105	2	$2.2284e^{-4}$	0.042	2	$6.2252e^{-7}$	0.042	2
8	$3.1345e^{-3}$	0.140	2	$9.7200e^{-5}$	0.055	2	$3.2845e^{-7}$	0.054	2
16	$7.8851e^{-4}$	0.780	2	$1.9170e^{-6}$	0.061	2	$4.5025e^{-9}$	0.060	2
32	$1.9741e^{-4}$	0.810	2	$6.9331e^{-8}$	0.088	2	$6.6001e^{-11}$	0.087	2
64	$4.9377e^{-5}$	0.940	2	$3.5772e^{-9}$	0.322	2	$9.9922e^{-13}$	0.313	2
128	$1.2346e^{-5}$	1.137	2	$2.1063e^{-10}$	2.233	2	$1.5369e^{-14}$	2.224	2

We have examine the performance of scheme for different values of k. When k=0, the Helmholtz equation reduces to poisson's equation. Error is reducing versus k increasing, the scheme is sensitive to the value when $4 \le k \le 5$. If we increase the value of k up to 100, the error is reducing vary fast. And if we increase k more, then the error reducing process is stopped and the residuals is still reducing. We also observed that using krylov subspace as a smoother the error is reduced and the residuals decreases versus the number of restarts. We have also examine the behavior of scheme for different values of N.

The data in table 1, 2 indicates the behavior of N and k for different values. The data in table 3, 4 and 5 indicates the behavior of error and residuals where Krylov subspace is applied. In Figures 1 and 2 the left figure is the error graph, the middle figure is the exact solution and the right figure is the approximate solution of both problems. The restarted GMRES convergence rate is indicated in Figure 3.

Example 2

$$u_{xx}(x,y)+u_{yy}(x,y)+k^2u(x,y)=-2[(1-6x^2)(y^2-y^4)+(1-6y^2)(x^2-x^4)]$$

$$+\,k^2(x^2-x^4)(y^2-y^4),(x,y)\in[0,1]. \qquad (40)$$

With the same boundary conditions mentioned above, i.e The exact solution for example 2 is $u(x,y)=(x^2-x^4)(y^4-y^2)$

Table 2. Comparison of maximum absolute errors and CPU(seconds) for a multigrid method with different discretization schemes for example 2,$|e_2|$, k=100, and the same values of N

N	$M_1(\|e_2\|)$	CPU	V	$M_2(\|e_2\|)$	CPU	V	$M_3(\|e_2\|)$	CPU	V
4	$6.0562e^{-2}$	0.062	2	$6.3220e^{-6}$	0.042	2	$5.6567e^{-7}$	0.042	2
8	$1.2872e^{-4}$	0.074	2	$8.1311e^{-7}$	0.059	2	$7.5902e^{-8}$	0.057	2
16	$3.2304e^{-5}$	0.080	2	$3.2134e^{-7}$	0.080	2	$6.2834e^{-8}$	0.063	2
32	$8.1531e^{-6}$	0.093	2	$9.8314e^{-8}$	0.187	2	$5.5698e^{-8}$	0.181	2
64	$2.0389e^{-6}$	0.118	2	$7.0389e^{-8}$	0.329	2	$5.2102e^{-8}$	0.313	2
128	$5.0970e^{-7}$	0.290	2	$6.9061e^{-8}$	2.269	2	$5.1442e^{-8}$	2.252	2

Table 3. Comparison of maximum absolute errors, convergence rate and Residuals for a multigrid method with Krylov subspace, for example 1, $|e_2|$, k=10 and the same values of N

N	$M_3(\|e_2\|)$	Res	Conv-rate	$M_4(\|e_2\|)$	Res	Conv-rate
4	$6.2252e^{-7}$	$1.6889e^{-13}$	7.0856	$2.2284e^{-7}$	$2.7661e^{-16}$	6.9530
8	$3.2845e^{-7}$	$3.1793e^{-14}$	7.0856	$5.5413e^{-8}$	$4.5025e^{-15}$	9.1802
16	$4.5025e^{-9}$	$4.2357e^{-14}$	9.2496	$3.3681e^{-9}$	$4.9474e^{-15}$	11.3443
32	$6.6001e^{-11}$	$6.5354e^{-14}$	11.3840	$3.6209e^{-11}$	$4.5025e^{-9}$	13.4792
64	$9.9922e^{-13}$	$1.3362e^{-13}$	13.5055	$5.3078e^{-13}$	$1.0692e^{-14}$	15.6001
128	$1.5369e^{-14}$	$2.7153e^{-13}$	15.6195	$1.2354e^{-15}$	$2.0906e^{-14}$	17.7144

Table 4. Comparison of maximum absolute errors, Residuals and convergence rate for a multigrid method with Klylov subspace, for example 2, $|e_2|$, k=100 and the same values of N

N	$M_3(\|e_2\|)$	Res	Conv-rate	$M_4(\|e_2\|)$	Res	Conv-rate
4	$5.6567e^{-7}$	$7.3562e^{-7}$	2.4223	$3.2845e^{-7}$	$2.2346e^{-7}$	3.2545
8	$7.5902e^{-8}$	$3.7529e^{-7}$	3.0570	$6.2834e^{-8}$	$2.6743e^{-7}$	3.5201
16	$6.2834e^{-8}$	$5.6561e^{-7}$	3.0632	$5.6437e^{-8}$	$4.5321e^{-7}$	3.5431
32	$5.5698e^{-8}$	$5.8652e^{-7}$	3.1810	$4.9156e^{-8}$	$4.2264e^{-5}$	4.2422
64	$5.2102e^{-8}$	$2.6240e^{-2}$	3.8313	$2.2351e^{-8}$	$2.2854e^{-5}$	4.2872
128	$5.1442e^{-8}$	$2.7584e^{-2}$	3.9003	$8.9830e^{-9}$	$5.6567e^{-7}$	4.7341

Example 3

$$u_{xx}(x,y)+u_{yy}(x,y)+k^2u(x,y)=[(k^2+x^2+y^2-2\pi^2)e^{xy}\sin(\pi x)\sin(\pi y)]$$
$$+(2\pi y)e^{xy}\cos(\pi x)\sin(\pi y)+(2\pi x)e^{xy}\cos(\pi y)\sin(\pi x) \qquad (41)$$

The exact solution for this example is $u(x,y)=e^{xy}\sin(\pi x)\sin(\pi y)$.

Table 5. Comparison of maximum absolute errors, Residuals and convergence rate for a multigrid method with Klylov subspace for example (3), $|e_2|$, k=10 and the same values of N

N	$M_3(\|e_2\|)$	Res	Conv-rate	$M_4(\|e_2\|)$	Res	Conv-rate
4	$1.6168e^{-5}$	$1.5320e^{-10}$	2.6042	$1.5262e^{-5}$	$1.5200e^{-10}$	2.6043
8	$1.9511e^{-6}$	$1.4272e^{-12}$	3.3056	$1.1951e^{-6}$	$9.0949e^{-13}$	3.3206
16	$3.2287e^{-7}$	$4.3529e^{-13}$	3.7561	$3.2162e^{-7}$	$4.8712e^{-14}$	3.8503
32	$1.0946e^{-7}$	$1.5326e^{-7}$	4.0139	$1.0854e^{-7}$	$1.0169e^{-8}$	4.0156
64	$6.2242e^{-8}$	$4.1465e^{-6}$	4.0158	$5.8491e^{-8}$	$4.5457e^{-6}$	4.1774
128	$3.5523e^{-8}$	$1.6593e^{-2}$	4.6861	$1.1055e^{-8}$	$1.6266e^{-6}$	4.9396

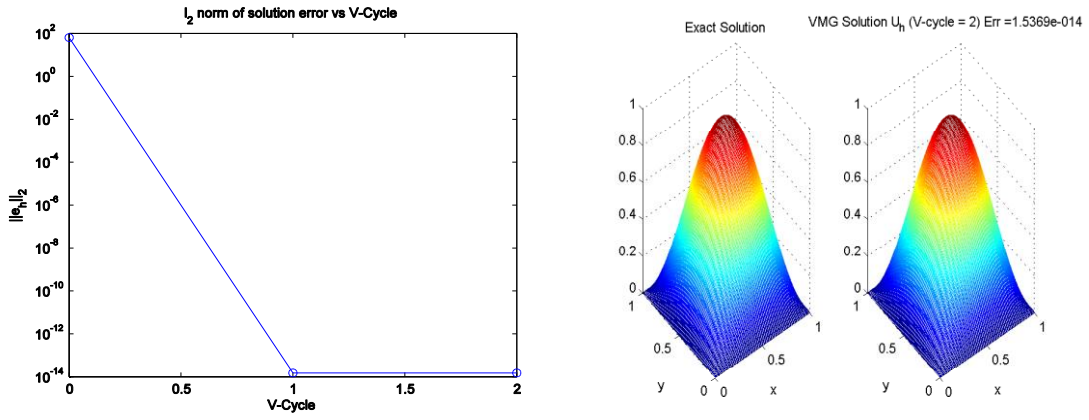

Figure 1: Left: shows the error graph. Middle: exact solution. Right: the approximation solution. The error vector $e_{ij}=u_{ij}-v_{ij}$ and N=128 are the number of nodes and k=10 for example 1.

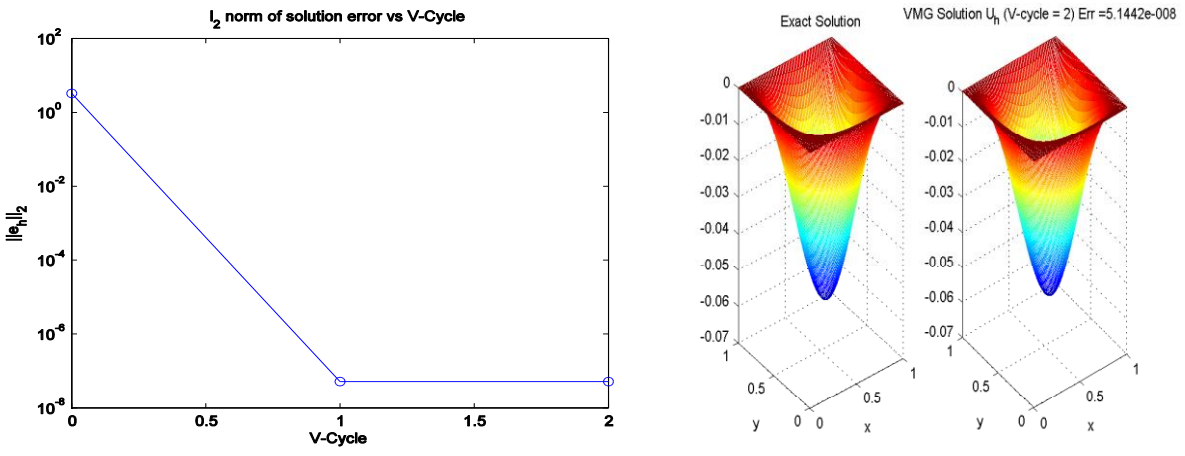

Figure 2: Left: shows the error graph. Middle: exact solution. Right: the approximation solution. The error vector $e_{ij}=u_{ij}-v_{ij}$ and N=128 are the number of nodes and k=100 for example 2.

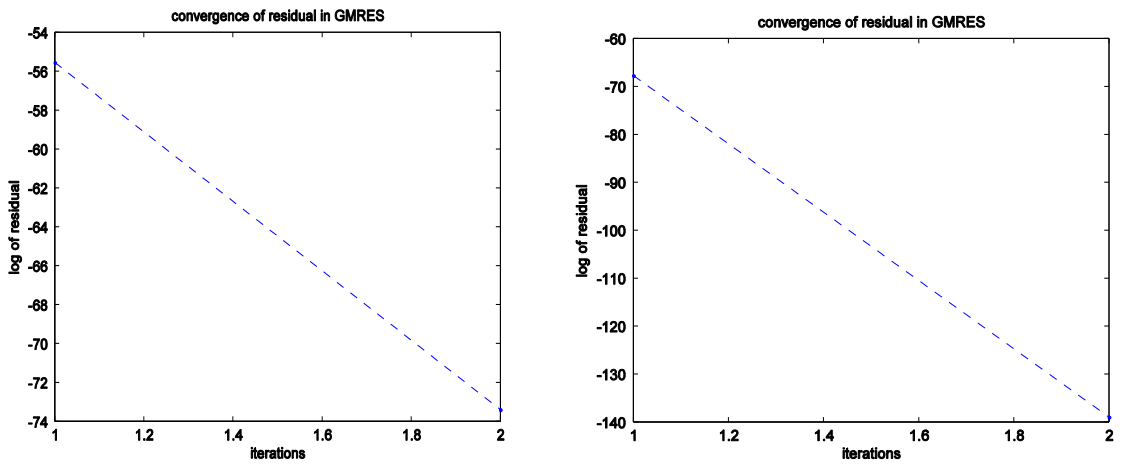

Figure3 : Left: when m=128×128, the restarted GMRES converges in 2.1 restarts for example 1. Right: when m=128×128, the restarted GMRES converges in 2 restarts for example 2.

7 Conclusion

We have studied a 6th-order compact finite difference scheme with equal mesh sizes for discretizing a 2D Helmholtz equation. We have used special multigrid methods which solve the resulting system efficiently. It is observed that multigrid method with the Gauss-Seidel relaxation and LU-decomposition by Gaussian elimination work very well in solving the 6th-order CFDM-discretized 2D Helmholtz equation, however using GMRES as a smoother the residuls decreases very fast as m increases. Numerical results shows that multigrid method accelerated by krylove subspace on 6th-order CFDM has the required accuracy and more faster than the 2nd-order FDM and 4th-order FDM. It is also obvious from the results that an increasing in the wave number k, overall error does not decreases.

REFERENCES

[1] R. F. Boisvert, A fourth-order accurate Fourier method for the Helmholtz equation in three dimensions. ACM Trans. Math. Softw. 13(1987): 221–234.

[2] M. M. Gupta, J. Kouatchou and J. Zhang, Comparision of second and fourth order discretizations for multigrid poisson solver. Journal of Computatiional physics. (1997): 132–226.

[3] I. Harari, T. J. R. Hughes, Finite element methods for the Helmholtz equation in an exterior domain model problem. Computer Methods in applied Mechanics and Engineering. 87(1991): 59–96.

[4] O. Mehdizadeh, M. Paraschiviou, Investigation of a two-dimentional spectral method for helmholtz's equation. Journal of Computational physics. 189(2003): 111–129.

[5] M. Nabavi, M. H. K. Siddique, J. Dargahi, Sixth-order accurate compact finite-difference method for the Helmholtz equation. Journal of sound and Vibration. 307(2007): 972–982.

[6] I. Singer, E. Turkel, High-order finite difference method for the helmholtz equation. Computer Methods in applied Mechanics and Engineering. 163(1998): 343–358.

[7] G. Sutmann, Compact finite difference scheme of sixth order for the Helmholtz equation. Journal of Computational and Applied Math. 293(1987): 15–31.

[8] S. Tsynkov and E. Turkel, A Cartesian Perfectly Matchedayer for the Helholtz Equation. Artificial Boundary Conditions with Applications to CEM. Loic Tourvete, ovascience Publishin. (2001): 279–309.

[9] P. Wesseling, An introduction to multigrid Methods. Wily, Chichester, Uk, 1992.

[10] J. Zhang, Fast and high accuracy multigrid solution for three dimension poisson equation. Journal of Computational physics. (1998): 143–449.