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Observational Banach Manifolds

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Abstract

In this paper, the concept of selective real manifolds is extended. It is proved that the product of two selective Banach manifolds is a selective Banach manifold. The notion of the α–level differentiation of the mappings between selective Banach manifolds is presented. Basic properties of (r, α)–differentiable maps are studied. Tangent space at a point of a selective Banach manifold is considered.
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1. INTRODUCTION

Banach spaces are named after the Polish mathematician Stefan Banach, who introduced and made a systematic study of them in 1920–1922 along with Hans Hahn and Eduard Helly. Banach spaces originally grew out of the study of function spaces by Hilbert, Fréchet, and Riesz earlier in the century. Banach spaces play a central role in functional analysis. In other areas of analysis, the spaces under study are often Banach spaces. Linear differential equations are often fruitfully studied via teqniques from linear functional analysis, including Banach spaces [1, 4]. In contrast, the proper setting for an important class of nonlinear partial differential equations is a nonlinear version of functional analysis, that is based on infinite dimensional manifolds modeled on Banach spaces [5, 9].

The problem of finding a mathematical model that can change the signature of a metric on a manifold is an interesting topic for both physicists and mathematicians. It was remained unsolved for a long time [6, 14] and it has been considered from observer viewpoint in [12]. The mathematical model of one dimensional observer has been considered first in 2004 [2, 10]. In [7, 11], the notion of multi-dimensional observers introduced, and it is employed to prove a version of Tychonoff Theorem and new concept of topological entropy [11].

The notion of selective manifolds on R^n, as the suitable finite space for the problem of unity has been presented in 2011 [12].

In 2009, relative smooth topological spaces were introduced and the properties of continuous relative smooth mappings were studied [3]. Moreover, the representation of a relative smooth topology was considered and the notion of relative smooth compactness was represented [3].

In 2014, the concept of U-equivalence space was introduced and its characteristics are studied.
Specifically, the notions of U-equivalence continuity and quotient U-equivalence space were considered [13].

In this paper, we put forward the notion of a selective manifold over a Banach space [8] and we study its characteristics. In this respect, we introduce the idea of a selective manifold on a Banach space in section 3. We prove that the product of two selective Banach manifolds is a selective Banach manifold. Next, we present the concept of α–level differentiation of the mappings between selective manifolds and we prove a new version of chain rule theorem for the mappings between selective Banach manifolds in section 4. Then, we study the notion of the tangent space of a selective Banach manifold at a given point in section 5. In the next section, we provide some preliminaries on observers.

2. PRELIMINARIES

Let M be a set. By an observer of dimension m on M, we mean a mapping $\mu : M \to \prod_{j\in J} I_j$, where $I_j = [0,1]$ for every $j \in J$ and $CardJ = m$. From a physical point of view, an observer

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considers finitely many physical parameters like speed, energy etc. Thus, the product of $[0,1]$ is used in order to define an observer mathematically. The observational models are widely applied in biology, dynamical systems, and geometry.

Let $\mu : M \to \prod_{j\in J} I_j$ and $\eta : M \to \prod_{j\in J} I_j$ be two observers of dimension m, where $I_j = [0,1]$ for all $j \in J$. By $\eta \subseteq \mu$, we mean that $\eta_j(x) \le \mu_j(x)$ for all $x \in M$ and $j \in J$. We write $\eta \subset \mu$ if $\eta \subseteq \mu$ and for given $x \in M$, there is $j \in J$ $\eta_j(x) < \mu_j(x)$.

Let μ and η be two observers of dimension m on the set M. We define the two m dimensional observers $\mu \cup \eta$ and $\mu \cap \eta$ by:

$$(\mu \cup \eta)_j(x) = \sup\{\mu_j(x), \eta_j(x)\};$$
$$(\mu \cap \eta)_j(x) = \inf\{\mu_j(x), \eta_j(x)\}.$$

Let τ_μ be a collection of subsets of μ for which the following conditions are satisfied:

a1) $\mu \in \tau_\mu$ and $\chi_\phi^m \in \tau_\mu$, where $\chi_\phi^m(x) = \prod_{j\in J} 0$;

a2) $\lambda \bigcap \eta \in \tau_\mu$, whenever $\lambda, \eta \in \tau_\mu$;

a3) if $\{\eta_\alpha\}_{\alpha\in\Lambda}$ is any collection of τ_μ, then $\bigcup_{\alpha\in\Lambda} \in \tau_\mu$, where Λ is an index set.

The collection τ_μ is called the μ - topology on M.

3. THE μ - SELECTIVE BANACH MANIFOLDS

Let λ be an observer of dimension m on M. λ is called the constant observer if there exists $j \in J$ such that $\lambda_j(x)$ is constant for all $x \in M$. The collection of all constant observers is denoted by C. Suppose that r and s be two constant observers with $r < s$. If $\lambda \in \tau_\mu$, then we define

$$\lambda^{-1}(r,s) \text{ by } \{x \in M \mid r_j(x) < \lambda_j(x) < s_j(x) \quad \forall j \in J\}.$$

We use the notation $(\eta_\mu)_{\mathbf{r}}^s$ to denote the topology of M is generated by $\{\lambda^{-1}(r,s) \mid \lambda \in \tau_\mu\}$ over M.

Definition 2.1. Let $K_1 \subseteq \bigcup_{r,s\in C} (\tau_\mu)_r^s$ and $K_1 \neq \phi$. Put

$$K = \{U \mid U = U_1 \bigcap \mu^{-1}(r,s) \quad \& \quad r,s \in C \quad \& \quad U_1 \in K_1\}.$$

By a chrat for M, we mean a pair (U,ϕ), where $U \in K$ and ϕ is a one to one mapping over U such that $\phi(U)$ is a C^n Banach manifold. Put

$$D = \{(U,\phi) \mid (U,\phi) \text{ is a chart for } M\}.$$

D is called a C^n μ - structure if

b1) $\mu(M) = \mu(\bigcup_{U\in K} U)$;

b2) if $(U,\phi) \in D$ and $(V,\psi) \in D$ for which $\mu(U) = \mu(V)$, then there exist a one to one and onto map $h_0 : U \to V$ and a C^n Banach diffeomorphism $h : \phi(U) \to \psi(V)$ such that $ho\phi = \psi oh_0$, where the restriction of h_0 to $U \bigcap V$ is the identity map.

Definition. 2.2. Let E be a Banach space. (M,D) is called a μ–selective Banach manifold modeled over E if D is a C^n μ–structure.

Example. 2.3. Let M be a compact smooth manifold of dimension m. Put

$$N=\{(f,p)\mid f:M\to M \text{ is smooth and } p \text{ is a hyperbolic fixed point for } f\}.$$

We define

$$\mu: N\to[0,1]\times[0,1]$$
$$\mu(f,p)=\left(\frac{1}{\sigma_p(f)+1},\frac{1}{\delta_p(f)+1}\right);$$

where

$\sigma_p(f)=Card(\{\lambda\mid\lambda$ is an eigenvalue of f at the hyperbolic fixed point p & $\|\lambda\|>1\})$;

and

$\delta_p(f)=Card(\{\lambda\mid\lambda$ is an eigenvalue of f at the hyperbolic fixed point p & $\|\lambda\|<1\})$.

Suppose that r and s be two arbitrary constant observers on $N, r<s$, and $\Upsilon=\{\lambda^{-1}(a,b)\mid\lambda\in\tau_\mu\}$. Let $(\tau_\mu)_r^s$ denotes the topology generated by Υ over N. Put

$$K=\{U\mid U=U_0\cap\mu^{-1}(r,s) \quad \& \quad r,s\in C \quad \& \quad U_0\in K_1\};$$

where K_1 is as Definition 2.1. We define

$$\phi: U\to L(R^m,R^m)$$
$$(f,p)\mapsto df(p);$$

where p is a hyperbolic fixed point for f. Then, $D=\{(U,\phi)\mid U\in K\}$ is a smooth μ–structure for N, and (N,D) is a μ–selective Banach manifold.

Let (M,D_i) be C^n μ_i–selective Banach manifold for $i\in\{1,2\}$. We write $D_1\le D_2$ if there exists a one to one map $f:\bigcup_{(U,\phi)\in D_1}U\to\bigcup_{(V,\psi)\in D_2}V$ such that $\mu_1(U)=\mu_2(f(U))$ and $(f(U),\phi of^{-1})\in D_2$, where $(U,\phi)\in D_1$. D_1 and D_2 are called equivalent if $D_1\le D_2$ and $D_2\le D_1$, where D_i is C^n μ_i–structure, and $i=1,2$.

Theorem. 2.4. Let (M,D) be a C^n μ - selective Banach manifold. There exists a C^n μ- structure, A, on M such that every C^n μ–structure, D_0, on M with $A\le D_0$ is equivalent to A.

Proof. If Λ is the set of C^n μ–structures of M, and $\{D_1,D_2,\ldots\}$ is a chain of elements of Λ, then $H=\bigcup_{i\in|N}D_i$ is a C^n μ- structure.

We only prove the second axiom: if $(U,\phi),(V,\psi)\in H$ and $\mu(U)=\mu(V)$, then there exist $i\le j$ such that $(U,\phi)\in D_i$, and $(V,\psi)\in D_j$. Moreover, there is an injective mapping f such that $(f(U),\phi of^{-1})\in D_j$. Since $\mu(f(U))=\mu(V)$, there exists an injective mapping η_0 from $f(U)$ to V and a C^n difeomorphism η from $\psi(U)$ to $\phi(V)$ such that $\phi o\eta_0 of=\eta o\phi$. These properties of $\eta_0 of$ and η imply that axiom (b_2) is valid for (U,ϕ) and (V,ψ). Thus, each chain has a maximal element. Therefore, Zorn's Lemma implies that Λ has at least one maximal element that we call it A.

Any C^n μ–structure, A, that satisfies the conditions of Theorem 2.4 is called a C^n maximal μ–structure.

Definition. 2.5. (M, A, μ) is called C^n selective Banach manifold if A is a C^n maximal μ -structure on M and $\mu(M \setminus \bigcup_{(U,\phi)\in A} U) = \{0\}$. In this case, A is called a C^n μ - atlas.

Theorem. 2.6. Let (M_i, D_i) be C^n μ_i- selective Banach manifold for $i \in \{1,2\}$. Then, $(M_1 \times M_2, \Omega)$ is a C^n μ - selective Banach manifold, where

$$\Omega = \{(U_\alpha \times V_{\beta,} \phi_\alpha \times \psi_\beta) \mid (U_\alpha, \phi_\alpha) \in D_1 \quad \& \quad (V_\beta, \psi_\beta) \in D_2\};$$

whenever

$$\phi_\alpha \times \psi_\beta : U_\alpha \times V_\beta \to (\phi_\alpha \times \psi_\beta)(U_\alpha \times V_\beta)$$
$$(\phi_\alpha \times \psi_\beta)(x, y) = (\phi_\alpha(x), \psi_\beta(y));$$

and

$$\mu : M_1 \times M_2 \to [0,1]\times[0,1]$$
$$\mu(x, y) = (\mu_1(x), \mu_2(y)).$$

Proof. First, we prove

$$\mu(M_1 \times M_2) = \mu(\bigcup_{\alpha,\beta\in\Lambda} (U_\alpha \times V_\beta)).$$

Let $(x, y) \in M_1 \times M_2$ be arbitrary. We have $\mu(x, y) = (\mu_1(x), \mu_2(y))$. Since

$$\mu_1(x) \in \mu_1(\bigcup_{\alpha\in\Lambda} U_\alpha) \quad \& \quad \mu_2(y) \in \mu_2(\bigcup_{\beta\in\Lambda} V_\beta).$$

Then

$$\mu(x, y) \in (\bigcup_{\alpha\in\Lambda} (U_\alpha \times V_\beta)).$$

Thus,

$$\mu(M_1 \times M_2) \subseteq \mu(\bigcup_{\alpha,\beta\in\Lambda} (U_\alpha \times V_\beta)). \qquad \text{(A)}$$

On the other hand, we have

$$\mu(\bigcup_{\alpha,\beta\in\Lambda} (U_\alpha \times V_\beta)) \subseteq \mu(M_1 \times M_2). \qquad \text{(B)}$$

Thus,

$$\mu(M_1 \times M_2) = \mu(\bigcup_{\alpha,\beta\in\Lambda} (U_\alpha \times V_\beta)).$$

Now, we show the second axiom is satisfied. Let $(U_\alpha \times V_\beta, \phi_\alpha \times \psi_\beta), (U_\sigma \times V_\delta, \phi_\sigma \times \psi_\delta) \in \Omega$ for which we have

$$\mu(U_\alpha \times V_\beta) = \mu(U_\sigma \times V_\delta);$$

then, we obtain

$$(\mu_1(U_\alpha), \mu_2(V_\beta)) = (\mu_1(U_\sigma), \mu_2(V_\delta)).$$

Since $\mu_1(U_\alpha) = \mu_1(U_\sigma)$, there exists a one to one and onto mapping $\xi_0 : U_\alpha \to U_\sigma$ and a C^n diffeomorphism $\xi : \phi_\alpha(U_\alpha) \to \phi_\sigma(U_\sigma)$ such that

$$\xi o \phi_\alpha = \phi_\sigma o \xi_0. \qquad \text{(C)}$$

Since $\mu_2(V_\beta)=\mu_2(V_\delta)$, then

$$\eta o \psi_\beta = \psi_\delta o \eta_0; \tag{D}$$

for some one to one and onto mapping $\eta_0 : V_\beta \to V_\delta$ and some C^n diffeomorphism $\eta : \psi_\beta(V_\beta) \to \psi_\delta(V_\delta)$. Now, we define

$$\xi_0 \times \eta_0 : U_\alpha \times V_\beta \to U_\sigma \times V_\delta$$
$$(\xi_0 \times \eta_0)(u,v) = (\xi_0(u), \eta_0(v)).$$

The bijectivity of ξ_0 and η_0 implies the bijectivity of $\xi_0 \times \eta_0$.
Also, we define

$$\xi \times \eta : \phi_\alpha(U_\alpha) \times \psi_\beta(V_\beta) \to \phi_\sigma(U_\sigma) \times \psi_\delta(V_\delta)$$
$$(\xi \times \eta)(x,y) = (\xi(x), \eta(y)).$$

Notice that $\xi \times \eta$ is a diffeomorphism since ξ and η are diffeomorphisms.

Finally, we show that the following diagram commutes.

$$\begin{array}{ccc} U_\alpha \times V_\beta & \xrightarrow{\xi_0 \times \eta_0} & U_\sigma \times V_\delta \\ \phi_0 \times \psi_\beta \downarrow & & \downarrow \phi_\sigma \times \psi_\delta \\ \phi_\alpha(U_\alpha) \times \psi_\beta(V_\beta) & \xrightarrow[\xi \times \eta]{} & \phi_\sigma(U_\sigma) \times \psi_\delta(V_\delta) \end{array}$$

In fact, for arbitrary $(u,v) \in U_\alpha \times V_\beta$, we have

$$\begin{aligned} [(\xi \times \eta) o (\phi_\alpha \times \psi_\beta)](u,v) &= (\xi \times \eta)(\phi_\alpha(u), \psi_\beta(v)) \\ &= (\xi(\phi_\alpha(u)), \eta(\psi_\beta(v))) \\ &= ((\phi_\sigma o \xi_0)(u), (\psi_\delta o \eta_0)(v)) \\ &= (\phi_\sigma \times \psi_\delta)(\xi_0(u), \eta_0(v)) \\ &= [(\phi_\sigma \times \psi_\delta) o (\xi_0 \times \eta_0)](u,v). \end{aligned}$$ □

By use of Theorem 2.6. and induction, we deduce the following theorem:

Theorem. 2.7. Let (M_i, D_i) be C^n μ_i- selective Banach manifold for $i \in \{1,\ldots,m\}$. Then $(\prod_{j=1}^m M_j, \Omega)$ is a C^n μ- selective Bnach Manifold, where

$$\Omega = \{(\prod_{j=1}^m U_{\alpha j}, \prod_{j=1}^m \phi_{\alpha j}) | (U_{\alpha j}, \phi_{\alpha j}) \in D_j, \qquad j=1,\ldots,m\};$$

$$\prod_{j=1}^m \phi_{\alpha j} : \prod_{j=1}^m U_{\alpha j} \to (\prod_{j=1}^m \phi_{\alpha j})(\prod_{j=1}^m U_{\alpha j}) \quad \text{with} \quad (\prod_{j=1}^m \phi_{\alpha j})(x_{\alpha 1},\ldots,x_{\alpha m}) = \prod_{j=1}^m \phi_\alpha \ (x_{\alpha j});$$

and

$$\mu : \prod_{j=1}^m M_j \to \prod_{j=1}^m I_j \quad \text{with} \quad \mu(x_1,\ldots,x_m) = \prod_{j=1}^m \mu_j(x_j);$$

whenever $I_j = [0,1]$ for all $j \in \{1,\ldots,m\}$.

4. THE α - LEVEL DIFFERENTIATION OF MAPS BETWEEN SELECTIVE BANACH MANIFOLDS

In this section, the concept of the (r,α) - differentiation of the mappings between selective Banach manifolds is presented, and its characteristics are studied. Specifically, a new version of chain rule theorem for the composition of mappings between selective Banach manifolds is proved.

Let (M_i, D_{M_i}) be a C^n μ_i - selective Banach manifold that is modeled over E_i for $i \in \{1,2\}$. For every $\alpha \in [0,1]$, and every mapping $f : M_1 \to M_2$, put

$K_{f,\alpha} = \{(U,\phi;V,\psi) \mid (U,\phi) \in D_{M_1} \;\; \& \;\; (V,\psi) \in D_{M_2} \;\; \& \;\; f(U) \subseteq V \;\; \& \;\; \mu(U) = \gamma(V) = \alpha\};$

where D_{M_i} is a C^n μ_i - structure on M_i, and E_i is a Banach space for $i \in \{1,2\}$.

Definition 3.1. Let the mapping $f : M_1 \to M_2$ be as above, $p \in M_1$, and $r \geq 0$. We say that f is (r,α) - differentiable at p if the following conditions are satisfied:

c1) $K_{f,\alpha} \neq \phi$;

c2) $\mu_2 of = \mu_1$;

c3) if $(V,\psi) \in D_{M_2}$ and $\mu_2(V) = \alpha$, then $\gamma(V) = \mu_1(f^{-1}(V))$.

c4) If $(U,\phi;V,\psi) \in K_{f,\alpha}$, then the mapping $\psi of o \phi^{-1} : \phi(U) \to \psi(V)$ is a C^r - map in a neighborhood of $\phi(p)$.

Condition (c4) of Definition 3.1. implies that the definition of the (r,α) - differentiable map is independent of the choice of the charts.

Definition 3.2. We say that the mapping f is continous at p if $\psi of o \phi^{-1}$ is a continuous map at $\phi(p)$.

Remark. 3.3. If f is $(r+1,\alpha)$ - differentiable, then f is (r,α) - differentiable.

Definition. 3.4. We say that the mapping f is (r,α) –differentiable on M if f is (r,α) –differentiable at every $p \in M$. If $r = \infty$, then we say that f is α - smooth .

Theorem. 3.5. Let (M, D_M) be a C^n μ - selective Banach manifold. Then, the identity mapping $id : M \to M$ is (r,α) - differentiable for all $\alpha \in \text{Im}(\mu)$, where $id(m) = m$ for all $m \in M$.

Proof. Let $\alpha \in \text{Im}(\mu)$. Suppose that $(U,\phi) \in D_M$ for all $m \in M$, where $m \in U$, and $\mu(U) = \alpha$, then $id(U) = U$. Thus $K_{id,\alpha} \neq \phi$ and $\mu o(id) = \mu$. Let $(V,\psi) \in D_M$, and $\mu(V) = \alpha$. Since $(id)^{-1}(V) = V$, then $\mu(V) = \mu(id)^{-1}(V)$. Finally, the mapping $\phi o(id) o \phi^{-1} : \phi(U) \to \phi(U)$ is differentiable since it is the identity mapping on a Banach space.

Theorem. 3.**6.** Let (M_i, D_{M_i}) is a C^n μ_i - selective Banach manifold for $i \in \{1,2\}$, and $f : M_1 \to M_2$ is a map. If $(V,\psi) \in D_{M_2}$, then there exists $(U,\phi) \in D_{M_1}$ such that $f^{-1}(V) = U$.

Proof. Let $(V,\psi) \in D_{M_2}$, then $V = V_1 \cap \mu_2^{-1}(r,s)$, where r,s are two constant observers on M_2, and $V_1 = \lambda^{-1}(r_0, s_0)$, whenever r_0 and s_0 are two constant observers on M_2, and $\lambda \in \tau_{\mu 2}$. Thus,

$$\begin{aligned} f^{-1}(V) &= f^{-1}(V_1) \cap f^{-1}(\mu_2^{-1}(r,s)) \\ &= f^{-1}(V_1) \cap (\mu_2 of)^{-1}(r',s') \\ &= f^{-1}(V_1) \cap (\mu_1)^{-1}(r',s'); \end{aligned}$$

where

$$\begin{aligned} f^{-1}(\mu_2^{-1}(r,s)) &= \{f^{-1}(x) \mid r(x) < \mu_2(x) < s(x) \quad \& \quad x \in M_2\} \\ &= \{f^{-1}(x) \mid (rofof^{-1})(x) < (\mu_2 ofof^{-1})(x) < (sofof^{-1})(x) \quad \& \quad x \in M_2\}. \end{aligned}$$

Put $rof = r'$, and $sof = s'$. Then, r' and s' are two constant observers on M_1.

We claim that $f^{-1}(V_1) = U_1$, where $U_1 = \beta^{-1}(m_0, n_0), \beta \in \tau_{\mu_1}$, and m_0 and n_0 are constant observers on M_1. Notice that

$$f^{-1}(V_1) = f^{-1}(\lambda^{-1}(r_0, s_0)) = (\lambda o f)^{-1}(m_0, n_0);$$

where

$$f^{-1}(\lambda^{-1}(r_0, s_0)) = \{ f^{-1}(x) \mid r_0(x) < \lambda(x) < s_0(x) \quad \& \quad x \in M_2 \}$$

$$= \{ f^{-1}(x) \mid (r_0 o f o f^{-1})(x) < (\lambda o f o f^{-1})(x) < (s_0 o f o f^{-1})(x) \quad \& \quad x \in M_2 \}.$$

Put $\beta = \lambda o f, m_0 = r_0 o f$, and $n_0 = s_0 o f$. Then, for constant observers m_0 and n_0 on M_1, we have

$$f^{-1}(V_1) = \beta^{-1}(m_0, n_0).$$

Since $\lambda \subseteq \mu_2$, we get $\lambda o f \subset \mu_0 o f$. Thus, $\beta \in \tau_{\mu_1}$. □

Here, we establish chain rule Theorem for the composition of the (r, α) - differentiable maps between μ_i - selective Banach manifolds.

Theorem. 3.7. Let (M_i, D_{M_i}) be a C^n μ_i - selective Banach manifold over a Banach space E_i for $i \in \{1,2,3\}$, and $f : M_1 \to M_2$ and $g : M_2 \to M_3$ be two (r, α) - differentiable maps. Then, the mapping $gof : M_1 \to M_3$ is (r, α) - differentiable.

Proof. Let $p \in M_1$ be given. Since $g : M_1 \to M_3$ is (r, α) - differentiable, we can choose the charts $(V, \tilde{\phi}) \in D_{M_2}$ and $(W, \tilde{\tilde{\phi}}) \in D_{M_2}$ such that $g(f(p)) \in W,\ f(p) \in V,\ g(V) \subseteq W$, and

$$\mu_2(V) = \mu_3(W) = \alpha. \qquad \text{(E)}$$

It follows from Theorem 3.6 that for $(V, \tilde{\phi}) \in D_{M_2}$, there exists $(U, \phi) \in D_{M_1}$ such that $f^{-1}(V) = U$. Since $f : M_1 \to M_2$ is (r, α) - differentiale at p, we have $\alpha = \mu_2(V) = \mu_1(f^{-1}(V))$. Thus, $\alpha = \mu_2(V) = \mu_1(U)$. Now, Equation (E) implies that $\alpha = \mu_1(U) = \mu_3(W)$.

Also, we have

$$(gof)(U) = g(f(U)) = g(V) \in W;$$

$$\mu_1((gof)^{-1}(W)) = \mu_1(f^{-1}(g^{-1}(W)) = \mu_1(f^{-1}(V))$$

$$= \mu_2(V) = \mu_2(g^{-1}(W)) = \mu_3(W);$$

and

$$\mu_3 o(gof) = (\mu_3 og)of = \mu_2 of = \mu_1.$$

Finally, the mapping

$$\tilde{\tilde{\phi}} o(gof) o\phi^{-1} = (\tilde{\tilde{\phi}} ogo\tilde{\phi}^{-1}) o(\tilde{\phi} ofo\phi^{-1}) : \phi(U) \to \tilde{\tilde{\phi}}(W)$$

is a C^r - map at $\phi(p)$, since $\tilde{\tilde{\phi}} ogo\tilde{\phi}^{-1}$ and $\tilde{\phi} ofo\phi^{-1}$ are C^r - maps.

Let (M_i, D_{M_i}) be a C^n μ_i - selective Banach manifold over a Banach space E_i for $i \in \{1,2\}$.

Definition. 3.8. Suppose that $f : M_1 \to M_2$ is an (r, α) - differentiable map. f is called (r, α) - diffeomorphism if f is bijective, and $f^{-1} : M_1 \to M_2$ is also (r, α) - differentiable.

Theorem. 3.9. Let (M_i, D_{M_i}) be a C^n μ_i - selective Banach manifold over a Banach space E_i for $i \in \{1,2,3\}$, $f : M_1 \to M_2$ be a an α - smooth diffeomorphism, and $g : M_2 \to M_3$ be also any map. Then g is α - smooth if and only if gof is α - smooth.

Proof. If $g : M_2 \to M_3$ is α - smooth, then gof is α - smooth by Theorem 3.7.

Now, suppose that $gof : M_1 \to M_3$ to be α- smooth, and q be any point in M_2. Since $f : M_1 \to M_2$ is surjective, there exists $p \in M_1$ such that $f(p) = q$. Put $g(q) = m$. It follows from α- smoothness of gof that $K_{gof,\alpha} \neq \phi$. Thus, there exists $(U, \phi : W, \eta) \in K_{gof,\alpha}$ such that $(U, \phi) \in D_{M_1}$ around p, and $(W, \eta) \in D_{M_3}$ around m, where $(gof)(U) \subseteq W$, and $\mu_1(U) = \mu_3(W) = \alpha$.

Considering Theorem 3.6, for the mapping $f^{-1} : M_2 \to M_1$, and for the chart $(U, \phi) \in D_{M_1}$, $(f^{-1})^{-1}(U) = V$ i.e. $f(U) = V$ for some chart $(V, \psi) \in D_{M_2}$ around $q \in M_2$. Thus,

$$g(V) = g(f(U)) = (gof)(U) \subseteq W.$$

Since $f^{-1} : M_2 \to M_1$ is α-smooth, for the chart $(U, \phi) \in D_{M_1}$ with $\mu_1(U) = \alpha$, we have

$$\mu_1(U) = \mu_2((f^{-1})^{-1}(U)) = \mu_2(f(U)) = \mu_2(V).$$

Therefore, $K_{g,\alpha} \neq \phi$.

The α- smoothness of $f : M_1 \to M_2$ and $gof : M_1 \to M_3$ result in $\mu_2 of = \mu_1$ and $\mu_3 o(gof) = \mu_1$, respectively. Thus, $\mu_3 o(gof) = \mu_2 of$. Since f is invertible, $\mu_3 og = \mu_2$. Therefore, Definition 3.1 (c2) is satisfied by $g : M_2 \to M_3$.

Let $(W, \eta) \in D_{M_3}$, and $\mu_3(W) = \alpha$. Considering $f : M_1 \to M_2$ is α-smooth, we get

$$\mu_2 of = \mu_1. \qquad \text{(F)}$$

It follows from invertibility of f and Equation (F) that

$$\mu_2 = \mu_1 of^{-1}. \qquad \text{(G)}$$

Now, considering $gof : M_1 \to M_3$ is α-smooth, and using Equation (G), we obtain

$$\alpha = \mu_3(W) = \mu_1((gof)^{-1}(W)) = \mu_1(f^{-1}(g^{-1}(W))) = \mu_2(g^{-1}(W)).$$

Therefore, Definition 3.1 (c3) is satisfied by $g : M_2 \to M_3$.

Finally, we inspect whether the Definition 3.1 (c4) is satisfied by $g : M_2 \to M_3$.

$$\begin{aligned} \eta ogo \psi^{-1} &= \eta ogo(fof^{-1}) o \psi^{-1} \\ &= (\eta ogof) o(\phi^{-1} o \phi) o(f^{-1} o \psi^{-1}) \\ &= [\eta o(gof) o \phi^{-1}] o(\phi of^{-1} o \psi^{-1}). \end{aligned}$$

The mapping $\eta o(gof) o \phi^{-1}$ and $\phi of^{-1} o \psi^{-1}$ are smooth since gof and f^{-1} are α- smooth. Therefore, the mapping $\eta ogo \psi^{-1}$ is smooth.

5. THE TANGENT SPACE OF SELECTIVE BANACH MANIFOLDS

In this section, we present the notion of the tangent space to a μ - selective Banach manifold at a given point, and study its properties. Throughout this section, we assume that every vector space is defined over the field F , where $\mathrm{F} = \mathbb{R}$ *or* $F = \mathbb{C}$.

Assume that (M, A, μ) is a C^n selective Banach manifold modeled over the Banach space E, and $n \geq 1$. For every $p \in M$ and $\alpha \in \operatorname{Im}\mu$, we define an (r, α)- differentiable multi-path through p by a multi-function `$\gamma : (-1,1) \to M$' satisfying the following conditions.

d1) $\gamma(0) = p;$

d2) $\phi o \gamma : (-1,1) \to E$ is a C^r map for all $(U,\phi) \in A$, where $U \bigcap \gamma((-1,1)) \neq \phi$, $\mu(U) = \alpha$, and $r \geq 1$.

We denote the set of all (r,α)–differentiable multi-paths through p by $W^{p,\alpha}$ and define the relation '~' by

$$\gamma \sim \beta \Leftrightarrow \frac{d(\phi o \gamma)}{dt}(0) = \frac{d(\phi o \beta)}{dt}(0);$$

where $U \bigcap \gamma((-1,1)) \neq \theta$, $U \bigcap \beta((-1,1)) \neq \theta$, and $\mu(U) = \alpha$. This relation is an equivalence relation.

Definition. 4.1. We denote $\frac{W^{p,\alpha}}{\sim}$ by $T_p^{\mu,\alpha}(M)$ and call it the tangent space of level α to M at p.

Let γ be an (r,α)- differentiable multi-path through p, $(U,\phi) \in A$, $U \cap \gamma((-1,1)) \neq \phi$, and $\mu(U) = \alpha$. We define the j th component of γ by

$$\gamma_j^U : (-1,1) \to U \quad \text{with} \quad \gamma_j^U(t) = \gamma(t) \cap U;$$

where $t \in \gamma^{-1}(U)$, $j \in J$, and $CardJ < \infty$.

Let $r \geq 1$. The mapping γ_j^U is a C^r map for every $j \in J$.

Restricting ~ to U, we have $[\gamma_j^U] \in T_p(U)$. Put

$$A^{p,\alpha} = \{(U,\phi) \mid (U,\phi) \in A \quad \& \quad \mu(U) = \alpha \quad \& \quad p \in U\}.$$

Now, define: $(U_1,\phi_1) \sim' (U_2,\phi_2)$ if the following conditions are satisfied:

e1) $\phi_i(U_1 \cap U_2)$ is Banach embedded sub-manifold of $\phi_i(U_i)$;

e2) there exists a Banach diffeomorphism $f_i : \phi_i(U_1 \cap U_2) \to \phi_i(U_i)$;

for $i \in \{1,2\}$.

Theorem. 4.2. There exists a one to one correspondence $f : T_p^{\mu,\alpha}(M) \to \prod_{[U,\phi]} T_P(U)$ that gives a vector space structure to $T_p^{\mu,\alpha}(M)$ by transferring the structure of $\prod_{[U,\phi]} T_P(U)$ to $T_P^{\mu,\alpha}(M)$.

Proof. Define the mapping f by

$$f : T_P^{\mu,\alpha}(M) \to \prod_{[U,\phi]} T_P(U)$$
$$f(\{\ \gamma\ \}) = \prod_{[U,\phi]} [\gamma_j^U].$$

Clearly, f is one to one and onto. Let $w, v \in T_P^{\mu,\alpha}(M)$ be given. Then $T_p^{\mu,\alpha}(M)$ endowed with the operations `+' and `.'

$$v + w := f^{-1}((f(v) + f(w));$$
$$c.v := f^{-1}(c.f(v));$$

is a vector space.

In fact, for $(v,w),(v',w') \in T_p^{\mu,\alpha}(M) \times T_p^{\mu,\alpha}(M)$ with $v = v'$ and $w = w'$, we have

$$v + w = f^{-1}(f(v) + f(w)) = f^{-1}(f(v') + f(w')) = v' + w';$$
$$c.v := f^{-1}(c.f(v)) = f^{-1}(c.f(v')) = c.v'.$$

Thus, the operations `+' and `.' are well-defined. It is easy to check that the other conditions are held.

Definition. 4.3. Let (M_i, D_{M_i}) be a C^n μ_i - selective Banach manifold for $\alpha \in \operatorname{Im}\mu_i$ where $i \in \{1,2\}$ and $f : M_1 \to M_2$ is an (r,α) - differentiable mapping at $p \in M_1$. We denote the α - level differential of f at p by $d_p^\alpha f$, and define it as the linear mapping

$$d_p^\alpha(f) : T\mu_p^{\mu_1\alpha}(M_1) \to T_{f(p)}^{\mu_2,\alpha}(M_2)$$
$$\{\ \gamma\ \} \mapsto \{\ fo\gamma\ \};$$

where $\gamma : (-1,1) \to M_1$, and $\gamma(0) = p$.

The observers have an essential role in (r,α) - differentiability of mappings between μ -selective Banach manifolds so that a mapping that is (r,α) - differentiable with respect to a given observer may not be (r,α) - differentiable with respect to another choice of observer. As an example, for the constant mapping $f \equiv c$, consider the observer for which $K_{f,\alpha} = \phi$. In this case, f is not (r,α) - differentiable.

Theorem. 4.4. Let (M_i, D_{M_i}) be a C^n μ_i - selective Banach manifold for $i \in \{1,2\}$, and $f : M_1 \to M_2$ is a constant mapping. If f is (r,α) - differentiable, then $d_p^\alpha f = 0$ for all $p \in M_1$.

Proof. Since f is (r,α) - differentiable, then $K_{f,\alpha} \neq \phi$. Thus, $d_{\phi(p)}(\psi o f o \phi^{-1}) = 0$ for every $(U,\phi;V,\psi) \in K_{f,\alpha}$. Therefore for all $p \in M_1, d_p^\alpha f = 0$.

6. CONCLUSION

In this paper, the concept of the μ –selective Banach manifold is introduced, and its fundamental properties are studied. Specifically, it is proved that the product of two selective Banach manifolds is a selective Banach manifold. Next, the concept of the α –level differentiation of the mappings between selective Banach manifolds is presented and a version of chain rule theorem is given for the mappings between μ –selective Banach manifolds. Moreover, the notion of the tangent space of a selective Banach manifold at a given point is studied.

It will be interesting to establish a new version of Inverse Function Theorem for the μ –selective Banach manifolds for further research.

REFERENCES

1. Nicolas Bourbaki, 1981. Espaces vectoriels topologiques, Analysis. Nasson-Dunod, PP: 1-3.
2. Fatehi Nia, M. and V. Jafari, 2014. Attractors in relative semi-dynamical systems. J. Dyn. Syst. Geom. Theor., **12**(1):1-10.
3. Ghazanfari, B., 2009. Relative smooth topological spaces. Hindawi Publishing Corporation Advances in Fuzzy qqqSystems, Volume 2009, Article ID 172917, 5 Page, doi: 10.1155/2009/172917.
4. Yuri E. Gliklikh, 2011. Global and stochastic analysis with applications to mathematical Physics, Analysis. Springer, pp: 3-6.
5. Daniel Grieser, Stefan Teufel, Andras Vasy, 2013. Microlocal methods in mathematical Physics and global analysis, Analysis. ---------Springer, pp: 7-9.
6. Khadekar, G.S. and G. Kondwar, 2006. Spherically symmetric static fluids in Rosen's bimetric theory of ----------- gravitation. J. Dyn. Syst. Geom. Theor., **4**(1): 95-102.
7. Khalisa, K., 2011. Folding of ruled manifold and its deformations. J. Basic APP. Sci. Res., 1(5): 339-344.
8. Kriegl, A. and P. Michor, 1997. The convenient setting global analysis. Mathematical surveys and monographs --------(American Mathematical Society).
9. Albert C.J. Luo, 2012. Global analysis of nonlinear dynamics, Dynamical systems. Springer, PP: 2-3.

10. Molaei, M.R., 2004. Relative semi–dynamical systems. International Journal of Uncertainty, Fuzziness and ---------------Knowledge-based Systems, 12(2): 237-243.
11. Molaei, M.R., 2009. Observational modeling of topological spaces. Chaos solitons & Fractals, 42: 615–619, doi:10.1016/j.chaos.2009.01.035.
12. Molaei, M.R., 2011. Selective manifolds. Journal of Advanced Research in Applied Mathematics, **3**(2): 1-7, doi:10.5373/jaram.617.103010.
13. Omidi, F. and M.R. Molaei, 2014. U-equivalence spaces. J. APPL. Environ. Biol. Sci. 4(2): 299-305.
14. Santilli, R.M., 2003. Iso-, Geno-, Hyper-mechanics for matter, their isoduals for antimatter and their novel --------applica- tions in physics, chemistry and biology. J. Dyn. Syst. Geom. Theor., **1**(2): 121-193.