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Exact Solution of Two Thin Film Power Law Model Fluids on a Lubricant Vertical Belt

***Taza Gul[1], S.Islam[1], R. A. Shah [2]**

[1]Department of mathematics, Abdul Wali K.U Mardan, KP Pakistan
[2]Department of mathematics, U.E.T Peshawar, KP Pakistan

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ABSTRACT

In this article, the analytical study on the behavior of thin film double layer flows on a vertical moving belt with slip boundary conditions has been investigated. The exact solution has been found for the velocity fields and the interface velocity distributions on fully developed Laminar, Incompressible flow and the power law model of non-Newtonian fluids. The results include the profile of velocity, volume flux, average velocity, Vorticity function and skin friction. The characteristics of various Model parameters on the velocity fields and skin friction involve in the solution shown through graphs and has been discussed.

KEYWORDS: Thin film flows, Lifting, Drainage, Slip boundary conditions. non-Newtonian fluids and Power law model.

1. INTRODUCTION

The flow inside thin films is ubiquitous in Environmental sciences, Civil Engineering, mechanical Engineering, Geophysics, Biological sciences, and elsewhere. This is due to their rich applications such as reactor fluidization, wire and fiber coating, paper production, transpiration cooling, drilling mud's , oil wills, gaseous diffusion and fluid cells. The problem of chambers chemical and biological detection systems like fluid of many chemicals was countered regarding by Lavrik et al. [1]. Thin film unsteady flow with variable viscosity was investigated by Nadeem and Awais [2]. Munson and Young [3] discussed the thin-film flow of Newtonian fluids. Alam et al. investigated the thin-film flow of Johnson–Segalman fluids for lifting and drainage problems [4]. Thin film flow of Johnson-Segalman fluid with magneto hydrodynamics on a vertical belt with lifting and drainage problems has been discussed by Alam et al. [5]. They discussed the effect of the physical parameters like Stokes number, Weissenberg number and the applied magnetic field on velocity profile. They observed the effect of various physical parameters on the lift and drainage velocity profiles. In [6] they have investigated non-Newtonian thin film flows on a vertical cylinder. Farooq et al in [7] discussed Jeffrey fluid flow on a vertical belt with slip boundary conditions. They observed the effect of various parameters on the velocity profile. They also observed that magnetic field brings rigidity in the fluid.

One of the widely established models amongst non-Newtonian fluids is class of power law model fluids which has its constitutive equations based on strong theoretical foundations.

Furthermore the use of double layer in the Newtonian and non-Newtonian fluids is of particular interest like double layer wire and fiber coating, double layer paints in various chemical processing.

Relevant and attractive work may be found in the following published articles. The velocity of two immiscible and incompressible fluids between two parallel plates have been discussed by Bird et al in [8]. It was found in their studies that when the heights of two fluids are equal then the velocity of the less viscous fluid becomes maximum as compare to the more viscous fluid Kapur and Shukla [9] discussed the maximum velocity of both fluids at different points and their interface.

Later on Kapur and shukla [10] discussed the flow of n-immiscible fluids between two plates of different heights. They have shown that whatever the number of fluids and whatever their heights are, a unique velocity maximum always exists. Pinarbasi and Liakopoulos in [11] analyzed the linear stability at the interface of two non-Newtonian fluids with variable viscosity. Regarding effect of zeta –Potential on thin film, thickness and pressure have been discussed by Ping et at in [12]. Measure of the interfacial liquid film thickness between immiscible fluids of oil slug/droplet in a micro pipe have been investigated by Qiu et al in [13]. The flow of two immiscible fluids with uniform suction at the stationary plate was discussed by Sacheti in [14]. He notified that when suction is applied, an adverse pressure gradient causes back flow near the stationary plate. Recently Kim and Kwak [15]

*** Corresponding Author:** Taza Gul, Department of mathematics, Abdul Wali K.U Mardan, KP Pakistan
tazagulsafi@yahoo.com,

studied double layer coating liquid flows. Their approximations are based on Laminar flow and Power Law Model of non-Newtonian fluids. They discussed the coating liquid flow of immiscible resin in model of capillary annuls, where the surface of glass fiber moves at high fiber drawing speed. The power-law model comparatively gives a simple constitutive Equation. Therefore, many mathematical problems are related with this model. Power law model represents inelastic time-independent non-Newtonian liquids which shows shear-thinning or shear-thickening behavior. The work under various configurations on the thin film flows have been discussed by Mildinova et al in [16]. They have shown in Power Law Model that the non-linear interaction in a falling film of a non-Newtonian liquid exhibits a tendency towards permanent two dimensional waves as in Newtonian liquids. Astarita discussed boundary-layer flows in [17]. Taza Gul et al. [18,19] investigated thin film flow of third grade fluids for lifting and drainage problems in the presence of constant and variable viscosity. Where they studied the effect of various parameters on the lift and drainage velocity profiles. Taza Gul et al. [20] discussed exact solution of two thin film non-Newtonian immiscible fluids on a vertical belt with no slip conditions. They analyzed the effect of various physical parameters. The relevant work can also be seen in [21-23].The main aim of the present work is to study thin film fluid flows of two immiscible non-Newtonian power law model fluids on a vertical belt with slip boundary conditions.

2. BASIC EQUATIONS

Consider thin film double layer of Power Law Model fluids over a vertical belt.
The governing equations are:

$$\nabla.\boldsymbol{U_j} = 0, j = 1,2 \quad (1)$$

$$\rho_j \frac{D\boldsymbol{U_j}}{Dt} = \nabla.\boldsymbol{T_j} + \rho_j \boldsymbol{g}, j = 1,2 \quad (2)$$

The subscripts $j = 1,2$ represent primary and secondary layer liquids,

where $\rho_j, j = 1,2$ are densities of two non-Newtonian fluids, $\boldsymbol{g}$ denotes gravity, $\boldsymbol{u}_j, j = 1,2$ are primary and secondary velocity vectors of the fluids, $\frac{D}{Dt} = \frac{\partial}{\partial t} + (\boldsymbol{u}.\nabla)$ denotes material time derivative and $\boldsymbol{T_j}$ are the Cauchy stresses.

3. Formulation of the lift problem:

Consider two immiscible thin film and incompressible fluid layers. Assume that a wide lubricant flat belt moves vertically upward at constant speed U through a large bath of Power law model liquid. We assume the total thickness of the two layers is δ .The belt carries with itself a layer of liquid of constant thickness δ_1 taking as first thin liquid layer. The second layer at the interface h is of thickness $\delta - \delta_1$. For analysis Cartesian coordinate is used, in which the y-axis is taken parallel to the surface of the belt and x-axis, perpendicular to the belt.

Assuming the shear stress is negligible small at the interface between the first and the second fluid layers ,the flow is steady, Laminar and that it satisfies the constitutive equation of Power Law Model.
Velocity fields for both fluid layers:

$$\boldsymbol{U_j} = (0, u_j(x), 0), j = 1,2 \quad (3)$$

Boundary conditions are:

$$u_1 = U - \gamma\tau_{1xy}, \text{at } x = 0, \quad (4)$$

$$\frac{du_2}{dt} = 0, \text{ at } x = \delta, \quad (5)$$

$$u_1 = u_2 \text{ at } x = h, \quad (6)$$

$$k_1\left(\frac{du_1}{dx}\right)^n = k_2\left(\frac{du_2}{dx}\right)^n, \text{ at } x = h, \quad (7)$$

Here h shows interface between thin films, U is the velocity of belt, δ is the uniform total thickness of films. $\frac{du_1}{dx}$ and $\frac{du_2}{dx}$ are the shear rates of both thin fluid films. n_1 and n_2 are the flow behavior indexes. k_1 and k_2 are known as flow consistency indexes.

Inserting, the velocity field given from equation (3) in equations (1) and (2), the continuity equation (1) satisfies identically and equation (2) reduces to:

$$\frac{d}{dx}\left(\frac{du_j}{dx}\right)^n = \frac{1}{\mu_j}\left(\frac{\partial p}{\partial y} + \rho_j g\right), j = 1,2. \quad (8)$$

suppose constant pressure gradient $\frac{\partial p}{\partial y} = \lambda.$

The shear stresses of both fluid layers are:

$$k_1\left(\frac{du_1}{dx}\right)^n = k_2\frac{(-\lambda + \rho_2 g)}{\mu_2}(\delta - h) - k_1\frac{(-\lambda + \rho_1 g)}{\mu_1}(x - h), \quad (9)$$

$$k_2\left(\frac{du_2}{dx}\right)^n = k_2\frac{(-\lambda + \rho_2 g)}{\mu_2}(\delta - h), \quad (10)$$

Integrating Eq. (8) twice and using boundary conditions from Eqs.(4-7), the velocity profiles of both fluid layers are :

$$u_1(x) = U + \frac{n\mu_1 k_1^{\frac{-1}{n}}}{(-\lambda + \rho_1 g)(1+n)}\left[\left(\frac{(-\lambda + \rho_2 g)}{\mu_2}(\delta - h) + \frac{(-\lambda + \rho_1 g)h}{\mu_1}\right)^{\frac{1+n}{n}} - \left(\frac{(-\lambda + \rho_2 g)}{\mu_2}(\delta - h) + (h - x)\frac{(-\lambda + \rho_1 g)}{\mu_1}\right)^{\frac{1+n}{n}}\right] - \gamma\left(\frac{(-\lambda + \rho_2 g)}{\mu_2}(\delta - h) - \frac{(-\lambda + \rho_1 g)}{\mu_1}(h - x)\right), \quad (11)$$

$$u_2(x) = U + \frac{n\mu_2 k_2^{\frac{-1}{n}}}{(-\lambda + \rho_2 g)(1+n)}\left[\left(\frac{(-\lambda + \rho_2 g)}{\mu_2}(\delta - h)\right)^{\frac{1+n}{n}} + \left((\delta - x)\frac{(-\lambda + \rho_2 g)h}{\mu_2}\right)^{\frac{1+n}{n}}\right] + \frac{n\mu_1 k_1^{\frac{-1}{n}}}{(-\lambda + \rho_1 g)(1+n)}\left[\left(\frac{(-\lambda + \rho_2 g)}{\mu_2}(\delta - h) + \frac{(-\lambda + \rho_1 g)h}{\mu_1}\right)^{\frac{1+n}{n}} - \left(\frac{(-\lambda + \rho_2 g)}{\mu_2}(\delta - h)\right)^{\frac{1+n}{n}}\right] - \gamma\left(\frac{(-\lambda + \rho_2 g)}{\mu_2}(\delta - h)\right), \quad (12)$$

$u_1(x)$ is the velocity of fluid layer with moving belt and $u_2(x)$ is the velocity of fluid layer with interface.

3.1.1 Shear stress and Skin friction:

The shear stress of first layer is:

$$\tau_{1xy} = k_1\left(\frac{du_1}{dx}\right)^n, \quad (13)$$

Inserting Eq.(9) in Eq. (13) at $x = 0$ the shear stress of first layer become:

$$\tau_{1xy}\,|_{x=0} = \frac{(\delta - h)k_2(-\lambda + \rho_2 g)}{\mu_2} + \frac{hk_1(-\lambda + \rho_1 g)}{\mu_1}, \quad (14)$$

The coefficient of skin friction is defined as**:**

$$C_f(0) = \frac{\tau_{1xy}\,|_{x=0}}{\frac{1}{2}\rho U^2}, \quad (15)$$

$$C_f(0) = \frac{2}{R_e^2}\left(\frac{(h-\delta)k_2(\lambda - \rho_2 g)}{\mu_2} - \frac{hk_1(\lambda - \rho_1 g)}{\mu_1}\right), \quad (16)$$

3.1.2 Flow rate and average velocity of second layer:
The flow rate per unit width is given by the formula:

$$Q = \int_0^{\delta} u_2(x)dx, \quad (17)$$

Inserting Eq. (12) in Eq. (17) and integrating we obtain.

$$Q = U\delta - \frac{\delta\gamma(h - \delta)(\lambda - \rho_2 g)}{\mu_2} - \frac{n(h - \delta)\delta k_2^{\frac{-1}{n}}}{(1+n)}\left(\frac{(h - \delta)(\lambda - \rho_2 g)}{\mu_2}\right)^{\frac{1}{n}} -$$

$$\frac{n^2\delta^2 k_2^{\frac{-1}{n}}}{(1+n)(1+2n)}\left(\frac{\delta(-\lambda + \rho_2 g)}{\mu_2}\right)^{\frac{1}{n}} + \frac{n\delta k_1^{\frac{-1}{n}}\mu_1}{(1+n)(-\lambda + \rho_1 g)}\left[\left(\frac{(h - \delta)(-\lambda + \rho_2 g)}{\mu_2}\right)^{1+\frac{1}{n}}\right.$$

$$\left.\left(\frac{h(-\lambda + \rho_1 g)}{\mu_1} - \frac{(h - \delta)(-\lambda + \rho_2 g)}{\mu_2}\right)^{1+\frac{1}{n}}\right],$$

The average velocity $\overline{u}$ is given by:

$$Q = \overline{u}, \quad (18)$$

3.1.3 Vorticity of the lifting flow:
The vorticity vector) of the flow is given by:

$$\vec{\omega} = \nabla \times U = \frac{du_2}{dx}, \quad (19)$$

$$\vec{\omega} = k_2^{\frac{-1}{n}}\left(\frac{(\delta - x)(-\lambda + \rho_2 g)}{\mu_2}\right)^{\frac{1}{n}},$$

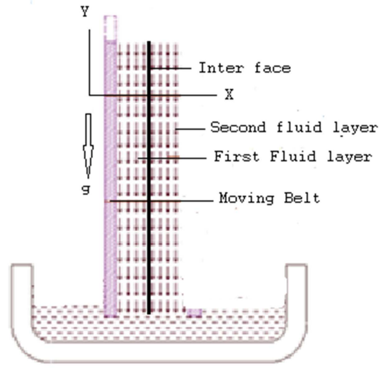

Fig.1 Geometry of Lifting problem.

4. Drainage problem for non-Newtonian Power Law Model:

For drainage, we consider two immiscible thin film layer of Power Law Model fluids. One fluid layer falls on the stationary lubricant vertical belt while the other fluid layer falls on the inter surface. The gravity caused the fluid motion so g is mentioned negative in Eq. (8). We assume that the flows are Incompressible, uniform, Steady and Laminar. The thicknesses are same as in previous problem. The coordinate axis are same as in previous problem.

The boundary conditions for drainage problem are:

$$u_1 = -\tau_{1xy} \text{ at } x = 0, \tag{20}$$

$$\frac{du_2}{dx} = 0, \text{ at } x = \delta, \tag{21}$$

$$u_1 = u_2 \text{ at } x = h, \tag{22}$$

$$k_1\left(\frac{du_1}{dx}\right)^n = k_2\left(\frac{du_2}{dx}\right)^n, \text{ at } x = h, \tag{23}$$

Using boundary conditions (19-23) in the governing Eq. (8), the velocity profiles of both fluids are:

$$u_1(x) = \frac{-n\mu_1 k_1^{\frac{-1}{n}}}{(1+n)(\lambda+\rho_1 g)}\left[\left(\frac{(\lambda+\rho_2 g)}{\mu_2}(h-\delta) - \frac{h(\lambda+\rho_1 g)}{\mu_1}\right)^{\frac{1+n}{n}} - \left(\frac{(\lambda+\rho_2 g)}{\mu_2}(h-\delta) - \frac{(\lambda+\rho_1 g)}{\mu_1}(h-x)\right)^{\frac{1+n}{n}}\right] - \gamma\left(\frac{(\lambda+\rho_2 g)}{\mu_2}(h-\delta) + \frac{(\lambda+\rho_1 g)}{\mu_1}(h-x)\right), \tag{24}$$

$$u_2(x) = \frac{-n\mu_2 k_2^{\frac{-1}{n}}}{(1+n)(\lambda+\rho_2 g)}\left[\left(\frac{(\lambda+\rho_2 g)}{\mu_2}(h-\delta)\right)^{\frac{1+n}{n}} + \left((\delta-x)\frac{(\lambda+\rho_2 g)}{\mu_2}\right)^{\frac{1+n}{n}}\right] - \frac{n\mu_1 k_1^{\frac{-1}{n}}}{(1+n)(\lambda+\rho_1 g)}\left[\left(\frac{(\lambda+\rho_2 g)}{\mu_2}(h-\delta) - \frac{(\lambda+\rho_1 g)h}{\mu_1}\right)^{\frac{1+n}{n}} - \left((h-\delta)\frac{(\lambda+\rho_2 g)}{\mu_2}\right)^{\frac{1+n}{n}}\right] - \gamma\left(\frac{(\lambda+\rho_2 g)}{\mu_2}(h-\delta)\right), \tag{25}$$

u_1 is the velocity of fluid with stationary belt and u_2 is the velocity of fluid with free surface.

4.1.1 Flow rate and average velocity of second layer:
The flow rate per unit width is given by the formula:

$$Q = \int_0^{\delta} u_2(x)dx, \tag{26}$$

inserting Eq. (25) in Eq. (26) and integrating we obtain:

$$Q = \frac{\delta\gamma(h-\delta)(\lambda+\rho_2 g)}{\mu_2} - \frac{n(h-\delta)\delta k_2^{\frac{-1}{n}}}{(1+n)}\left(\frac{(\delta-h)(\lambda+\rho_2 g)}{\mu_2}\right)^{\frac{1}{n}} - \frac{n^2\delta^2 k_2^{\frac{-1}{n}}}{(1+n)(1+2n)}\left(\frac{-\delta(\lambda+\rho_2 g)}{\mu_2}\right)^{\frac{1}{n}}$$
$$-\frac{n\delta k_1^{\frac{-1}{n}}\mu_1}{(1+n)(\lambda+\rho_1 g)}\left[\left(\frac{(\delta-h)(\lambda+\rho_2 g)}{\mu_2}\right)^{1+\frac{1}{n}} + \left(\frac{(h-\delta)(\lambda+\rho_2 g)}{\mu_2} - \frac{h(\lambda+\rho_1 g)}{\mu_1}\right)^{1+\frac{1}{n}}\right],$$

The average velocity $\bar{u}$ is given by:

$$Q = \bar{u}, \tag{27}$$

Vorticity of the lifting and drainage velocity profiles are same mentioned in Eq. (19)

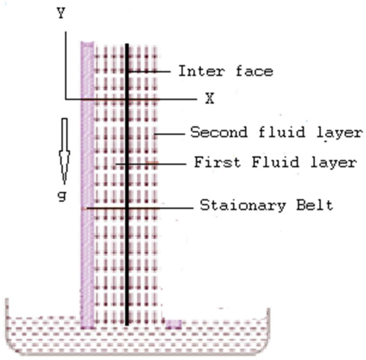

Fig.2 Geometry of Drainage problem.

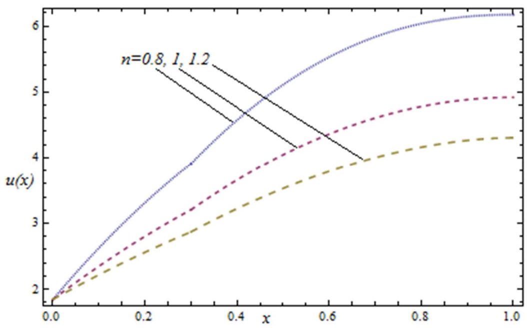

Fig.3. Effect of power index n for both fluid layers in lift problem when
$\delta = 1, \mu_1 = 0.6, \mu_2 = 0.4, \rho_1 = 0.5, \rho_2 = 0.4, g = 4$
$\lambda = 0.2, U = 2, k_1 = 0.6, k_2 = 0.5, h = 0.3, \gamma = 0.1.$

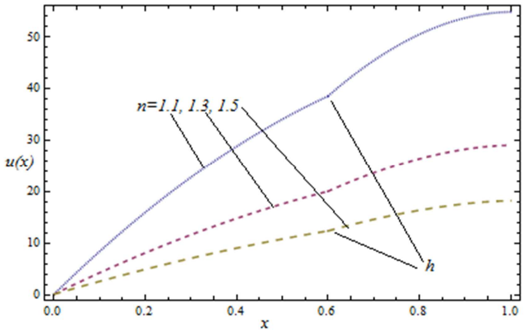

Fig.4 Effect of flow behavior index n on drainage velocity when
$\delta = 1, \lambda = 0.3, \mu_1 = 0.7, \mu_2 = 0.5, \rho_1 = 0.5, \rho_2 = 0.4,$
$g = 3, U = 2, k_1 = 0.02, k_2 = 0.01, h = 0.6, \gamma = 0.3.$

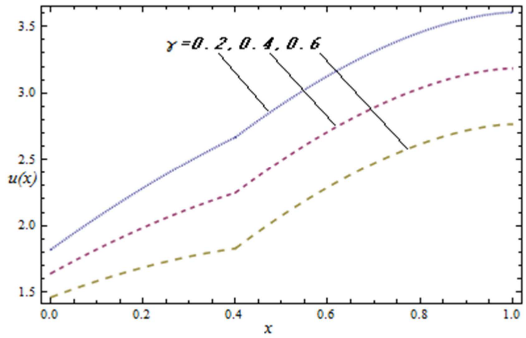

Fig.5 Effect of slip parameter on lift velocity when
$\delta = 1, n = 1.3, \mu_1 = 0.6, \mu_2 = 0.4, g = 3, \lambda = 0.9,$
$U = 2, k_1 = 0.02, k_2 = 0.01, h = 0.6, \gamma = 0.3$

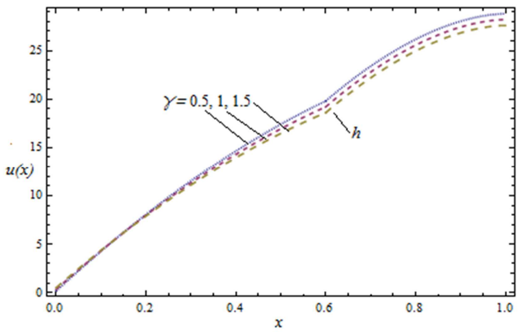

Fig.6 Effect of slip parameter on drainage velocity when
$\delta = 1, n = 1.3, \rho_1 = 0.5, \rho_2 = 0.4, \mu_1 = 0.7, \mu_2 = 0.5,$
$g = 3, \lambda = 0.3, U = 2, k_1 = 0.02, k_2 = 0.01, h = 0.6,$

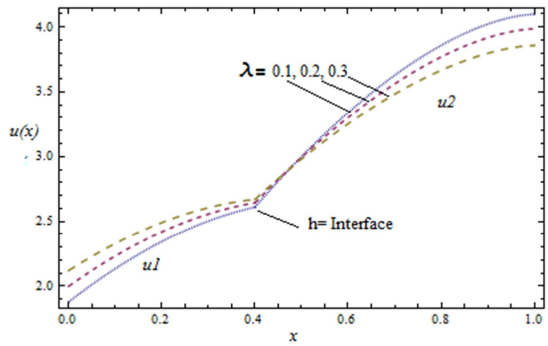

Fig.7 Effect of Pressure gradient on the lift velocity when
$\delta = 1, n = 1.5, \mu_1 = 0.2, \mu_2 = 0.1, \rho_1 = 0.5, \rho_2 = 0.2,$
$U = 2, k_1 = 0.6, k_2 = 0.5, h = 0.4, \gamma = 0.3, g = 4.$

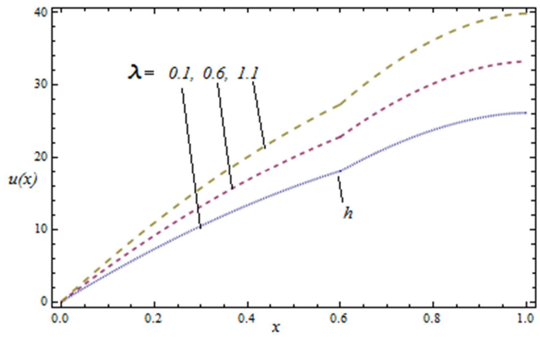

Fig.8 Effect of pressure gradient on drainage velocity when
$\delta = 1, n = 1.3, \mu_1 = 0.7, \mu_2 = 0.5, \rho_1 = 0.5, \rho_2 = 0.4,$
$g = 3, U = 2, k_1 = 0.0.02, k_2 = 0.02, h = 0.6, \gamma = 0.3$

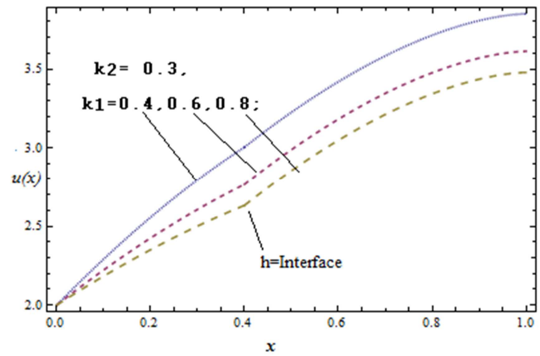

Fig.9 Effect of the viscosity parameters k_1 and k_2 for both fluid layers in lift problem when
$\delta = 1, n = 1.5, \rho_1 = 0.5, \rho_2 = 0.4, g = 4, U = 2,$
$\lambda = 0.2, h = 0.4,$

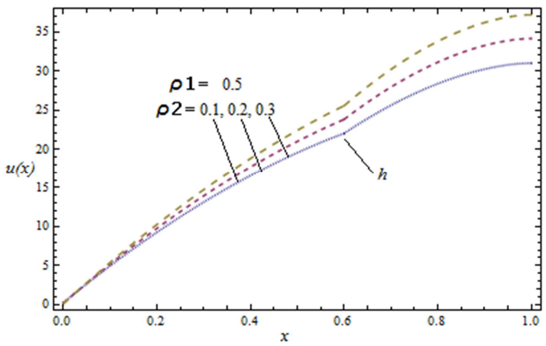

Fig.10 Effect of density parameters ρ_1 and ρ_2 for both fluid layers in drainage problem.

$$\delta = 1, n = 1.3, \mu_1 = 0.7, \mu_2 = 0.5, g = 4, h = 0.6$$
$$\lambda = 0.2, U = 2, k_1 = 0.02, k_2 = 0.01, \gamma = 0.3,$$

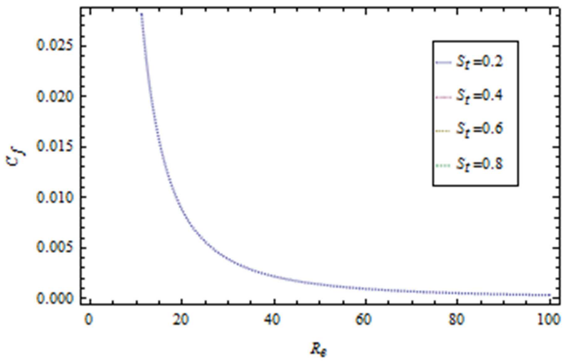

Fig.11 Effect of skin friction verses Reynolds number

$$\delta = 1, \mu_1 = 0.6, \mu_2 = 0.4, \rho_1 = 0.5, \rho_2 = 0.4, g = 4$$
$$\lambda = 0.2, U = 2, k_1 = 0.6, k_2 = 0.5, h = 0.3, \gamma = 0.1$$
$$n = 1.3$$

5. RESULTS AND DISCUSSION

In this article, the effect of flow behavior index n ,slip parameter γ ,density parameter ρ, pressure gradient parameter λ and slip parameter γ of the fluid layers u_1, at the belt and u_2, at the inter face h have been shown for both lift and drainage problems in Figs. 3-11. Geometry of lift and drainage fluid layers has been shown in Figs 1 and 2 respectively. The effect of flow behavior index n for lift and drainage velocity profiles have been shown in Fig. 3 and 4. There are three categories for various effects of this index. In first type, when $n < 1$ fluid layers are shear thinning (pseudo plastic). In other words the apparent viscosity (Ostwald–de Waele power law) decreases with increasing shear rate like blood, latex paint and ice etc. are the examples of such fluid layers. In second type, when $n = 1$ at low shear rate means viscosity is independent of shear rate or in other words zero shear viscosity in this category the fluid layers become Newtonian (like melts and solutions). In third type, when $n > 1$, then the apparent viscosity increases with increasing shear rate, shows the behavior of shear thickening also known as dilatant fluid layers . This effect is generally encountered in suspension. This class is similar to shear thinning systems. The only difference is that the increase in shear rate increase the apparent viscosity. Common examples of such fluids are Corn flour in water and thick suspension. In both lift and drainage velocity profiles, increase in n, decreases velocity profiles of both fluid layers. Effect of the slip parameter γ have been shown in Figs 5 and 6 for lift and drainage velocity profiles respectively. Increase in slip parameter γ decreases velocity of both fluid layers.

Where decrease in lift velocity is rapid than drainage velocity. The reason is that the friction force decreases gradually towards the surface of the fluid when slip parameter increases. Effect of the pressure gradient parameter λ have been shown in Figs 7 and 8 for lift as well as drainage velocity profiles. Increase in parameter λ increases the velocity of first fluid layer and the velocity of second fluid reducing at a small distance after interface. In drainage velocity profile the layer velocity is strictly decreasing. Because lubrication reduces friction force near the belt due to which the velocity of fluid layer increases near the belt and gradually decreases at the surface. Effect of the viscosity parameters on the lift velocity has been shown in Fig 9. Increasing viscosity decreasing velocity of both layers. Increasing density of fluid layers increasing drainage velocity profile has been shown in Fig 10. Fig. 11 shows the effect of local Reynolds number verses skin friction. It shows that Reynolds number decreases the skin friction. For large values of Reynolds number the skin friction vanishes.

6.CONCLUSION

The constitutive equation governing the flow of a power law model for lifting and drainage of double layer, has been solved exactly. It is concluded that for small values of $n_1 = 1, n_2 = 1$ the velocity profile tends to Newtonian one. However, when $n_1 < 1, n_2 < 1$ these profiles become more flattened showing the shear-thinning effect. But when $n_1 > 1, n_2 > 1$ then it can be seen that the speed of boundary layers are relatively small to that of shear thinning and Newtonian fluids. The study shows that Reynolds number decreases the skin friction. It has also been found that the skin friction vanishes when large values of Reynold number are taken. According to the best of our knowledge there is no previous literature about discussed problem, this is our first attempt to handle this problem.

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