Some Numerical Methods for Ice Sheet Behaviour and Its Visualization

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ABSTRACT

The thermomechanical ice sheet modeling is used to simulate the behavior of ice sheets in the Antarctic region. This research investigates some parameters such as ice thickness, ice temperature and ice velocity. The numerical discretization to obtain a large sparse system of unconditionally stable is based on explicit and Crank Nicolson implicit methods. The numerical solver for solving the large sparse systems is Jacobi and Gauss Seidel methods. Matlab version 2011a has been chosen as the platform to support the numerical computations. The numerical results prove that the thermomechanical ice sheet modelling is well suited to simulate the ice sheet behavior in terms of thickness, temperature and velocity. The contribution of this paper is to successfully discretize the ice thickness model based on Finite Difference Method. The ice sheet model is considered a good prediction model based on its visualization using Comsol Multiphysics software.
KEYWORDS: Explicit Method, Crank-Nicolson Implicit Method, Thermomechanical Ice Sheet Model.

INTRODUCTION

An ice sheet in Antarctica region is a mass of ice that forms by snowfall in all season. It happens over thousands of years where the new snow and ice layers compress the older layers until the layer of snow becomes thick masses of ice. Greenland and Antarctica region cover most of the ice sheets on earth. The Antarctica ice sheet is the largest single mass of ice on earth and around 90 percent of fresh water on the earth's surface is held in the ice sheet [1]. According to the Intergovernmental Panel on Climate Change (IPCC), the Antarctica ice sheets have lost mass recently, it losses exceed accumulation of snowfall. IPCC also determined the sea level rise about 7 mm per year from the loss of Antarctic ice sheet mass. If all of the Antarctica ice melted, the seas would rise by over 60 meters [2]. In addition, sea levels also rise because of the global temperature has been increasing in Antarctica region. For instance, the increasing temperature would cause huge blocks of ice to slide into the ocean, causing the surrounding seas to accommodate new ice shelves. Once these shelves break up, it collapses to the warmer sea water. As the result, the ice shelves would melt and increase the water levels.

Several mathematical models have been developed to simulate different aspects of ice sheet behavior. Mathematical modeling and numerical simulation are some methodologies to study the various physical processes and future changes of the ice sheet. A vertical one-dimensional numerical model was developed to evaluate the heat transfer within the near-surface snow layer of the Antarctic ice sheet [3]. The combinations of some parameters in physical processes are simplified to monitor the radiation and ice behavior [3]. MacAyeal [4] governed one dimensional flowline models focus on the diffusion of mass-balance evolution equation. Grounded, frozen-bed ice sheet and temperature profile of the ice sheet have a strong influence on the speed of ice flow. Huybrechts and Oerlemans [5] proposed the most efficient of 3-dimensional ice sheet model. This model is a thermodynamic coupling temperature field. The Huybrechts model represents the time-dependent and 3-dimensional coupled thermomechanical equations of the ice sheet behavior [6].

In this paper we will govern a Glimmer model to simulate the ice sheet and the ice thickness behavior and compare the simulation result from the model to the actual data given by ANDRILL project. Glimmer model is generally used for Antarctic ice sheet model [7]. The model introduced by Payne and Baldwin [8] is used to govern the thermodynamic evolution of the physical processes of the ice sheet, in terms of restriction and observation with respect to time and space [7]. The problem is all the change parameters cannot be obtained directly based on the historical evolution and observation. Numerical simulation models are the precise technique could forecast the past, current and future behavior of these ice masses [9]. The knowledge of the ice thickness cover on Antartica could predict the future global sea level rise [10], thus, it could create awareness among human reaction based on the phenomena changes.

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MATHEMATICAL MODEL

This section explains the discretization of thermomechanical ice sheet model in terms of thickness. From the discretization, the system of linear equations is obtained in the form of tri-diagonal matrix.

2.1   Discretization for Thermomechanical Ice Sheet Model.

Let us consider $x_j = -L + (j-1)\Delta x$ where $j = 1,..., N_x + 1$ and $t_l = 0 + (l-1)\Delta t$ where $l = 1,..., M_t + 1$. Meanwhile let $h_{j,l}$ be $h(x_j, t_l)$ and $D_{j,l}$ be an approximation of $D(x_j, t_l)$.

$$\frac{\partial h}{\partial t} = B + \frac{\partial}{\partial x}\left(D \frac{\partial h}{\partial x}\right) \tag{1}$$

Explicit finite method is used to descretize the ice thickness in Equation (1) [11].

$$\frac{h_{j,l+1} - h_{j,l}}{\Delta t} = B + \frac{1}{\Delta x}\left(\bar{D}_{j+1/2,l}\left(\frac{h_{j+1,l} - h_{j,l}}{\Delta x}\right) - \bar{D}_{j-1/2,l}\left(\frac{h_{j,l} - h_{j-1,l}}{\Delta x}\right)\right)$$

By letting $\alpha = \dfrac{\Delta t}{2\left(\Delta x^2\right)}$, we get

$$h_{j,l+1} = B\Delta t + \alpha \bar{D}_{j+1/2,l}\left(h_{j+1,l} - h_{j,l}\right) - \alpha \bar{D}_{j-1/2,l}\left(h_{j,l} - h_{j-1,l}\right) + h_{j,l},$$

where $\bar{D}_{j-1/2,l} = \dfrac{\Gamma}{5}\left| \dfrac{h_{j,l} - h_{j-1,l}}{\Delta x}\right|^2 \left(\dfrac{h_{j,l} + h_{j-1,l}}{2}\right)^5$.

The above equation gives the system linear of equations in the form of $Ax = f_h$.

$$\begin{bmatrix} 1 & 0 & 0 & \cdots & \cdots & \cdots & 0 \\ 0 & 1 & 0 & \ddots & & & \vdots \\ 0 & 0 & 1 & 0 & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & 0 & 1 & 0 & 0 \\ \vdots & & & \ddots & 0 & 1 & 0 \\ 0 & \cdots & \cdots & \cdots & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} h_1^{l+1} \\ h_2^{l+1} \\ h_3^{l+1} \\ \vdots \\ h_{n-2}^{l+1} \\ h_{n-1}^{l+1} \\ h_n^{l+1} \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \\ f_3 \\ \vdots \\ f_{n-2} \\ f_{n-1} \\ f_n \end{bmatrix} + B\Delta t$$

where

$$f_n = \alpha \frac{\Gamma}{5}\left| \frac{h_{j+1,l} - h_{j,l}}{\Delta x}\right|^2 \left(\frac{h_{j+1,l} + h_{j,l}}{2}\right)^5 (h_{j-1,l} - h_{j,l}) - \alpha \frac{\Gamma}{5}\left| \frac{h_{j,l} - h_{j-1,l}}{\Delta x}\right|^2 \left(\frac{h_{j,l} + h_{j-1,l}}{2}\right)^5 (h_{j,l} - h_{j-1,l})$$

This system will be solved by using Jacobi (JB) and Gauss Seidel (GS) iterative method. Figure 1 shows the stencil of the method. Open circle indicates the new unknown values.
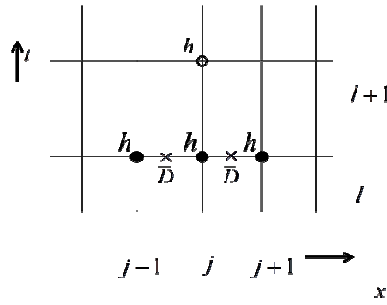
Figure 1. The stencil of the ice thickness using explicit method.

Crank-Nicolson method [12, 13] is implemented to discretize the ice thickness in Equations (2.1) [14].

$$\frac{h_{j,l+1} - h_{j,l}}{\Delta t} = B + \frac{1}{2\Delta x}\left(\overline{D}_{j+1/2,l}\left(\frac{h_{j+1,l} - h_{j,l}}{\Delta x}\right) - \overline{D}_{j-1/2,l}\left(\frac{h_{j,l} - h_{j-1,l}}{\Delta x}\right)\right) + \frac{1}{2\Delta x}\left(\overline{D}_{j+1/2,l+1}\left(\frac{h_{j+1,l+1} - h_{j,l+1}}{\Delta x}\right) - \overline{D}_{j-1/2,l+1}\left(\frac{h_{j,l+1} - h_{j-1,l+1}}{\Delta x}\right)\right)$$

Letting $\alpha = \dfrac{\Delta t}{2\left(\Delta x^2\right)}$, we get

$$h_{j,l+1} - \alpha\overline{D}_{j+1/2,l+1}\left(h_{j+1,l+1} - h_{j,l+1}\right) + \alpha\overline{D}_{j-1/2,l+1}\left(h_{j,l+1} - h_{j-1,l+1}\right) = B\Delta t + \alpha\overline{D}_{j+1/2,l}\left(h_{j+1,l} - h_{j,l}\right) - \alpha\overline{D}_{j-1/2,l}\left(h_{j,l} - h_{j-1,l}\right) + h_{j,l}$$

where

$$\overline{D}_{j+1/2,l} = \frac{\Gamma}{5}\left|\frac{h_{j+1,l} - h_{j,l}}{\Delta x}\right|^2\left(\frac{h_{j+1,l} + h_{j,l}}{2}\right)^5 \quad \text{and} \quad \overline{D}_{j-1/2,l} = \frac{\Gamma}{5}\left|\frac{h_{j,l} - h_{j-1,l}}{\Delta x}\right|^2\left(\frac{h_{j,l} + h_{j-1,l}}{2}\right)^5.$$

The above equation gives the system linear of equations in the form of $Ax = f_h$.

$$\begin{bmatrix} q_1 & p_1 & 0 & \cdots & \cdots & \cdots & 0 \\ p_1 & q_2 & p_2 & \ddots & & & \vdots \\ 0 & p_2 & q_3 & p_3 & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & p_{n-3} & q_{n-2} & p_{n-2} & 0 \\ \vdots & & & \ddots & p_{n-2} & q_{n-1} & p_{n-1} \\ 0 & \cdots & \cdots & \cdots & 0 & p_{n-1} & q_n \end{bmatrix}\begin{bmatrix} h_1^{l+1} \\ h_2^{l+1} \\ h_3^{l+1} \\ \vdots \\ h_{n-2}^{l+1} \\ h_{n-1}^{l+1} \\ h_n^{l+1} \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \\ f_3 \\ \vdots \\ f_{n-2} \\ f_{n-1} \\ f_n \end{bmatrix} + B\Delta t$$

where

$$p_n = -\alpha\left[\frac{\Gamma}{5}\left|\frac{h_{j+1,l+1} - h_{j,l+1}}{\Delta x}\right|^2\left(\frac{h_{j+1,l+1} + h_{j,l+1}}{2}\right)^5\right]$$

$$q_n = 1 + \alpha\left[\frac{\Gamma}{5}\left|\frac{h_{j+1,l+1} - h_{j,l+1}}{\Delta x}\right|^2\left(\frac{h_{j+1,l+1} + h_{j,l+1}}{2}\right)^5 + \frac{\Gamma}{5}\left|\frac{h_{j,l+1} - h_{j-1,l+1}}{\Delta x}\right|^2\left(\frac{h_{j,l+1} + h_{j-1,l+1}}{2}\right)^5\right]$$

$$f_n = \frac{\Gamma}{5}\left|\frac{h_{j+1,l} - h_{j,l}}{\Delta x}\right|^2\left(\frac{h_{j+1,l} + h_{j,l}}{2}\right)^5 h_{j-1,l} + \left(\begin{array}{c} 1 - \frac{\Gamma}{5}\left|\frac{h_{j+1,l} - h_{j,l}}{\Delta x}\right|^2\left(\frac{h_{j+1,l} + h_{j,l}}{2}\right)^5 \\ -\frac{\Gamma}{5}\left|\frac{h_{j,l} - h_{j-1,l}}{\Delta x}\right|^2\left(\frac{h_{j,l} + h_{j-1,l}}{2}\right)^5 \end{array}\right)$$

$$+\frac{\Gamma}{5}\left|\frac{h_{j+1,l} - h_{j,l}}{\Delta x}\right|^2\left(\frac{h_{j+1,l} + h_{j,l}}{2}\right)^5 h_{j+1,l}, \quad h_{0,l} = 0, h_{n+1,l} = 0.$$

This method has high accuracy and it is unconditionally stable. The nonlinear equation must be solved using Newton's method since D is calculated at the new time level, t_{l+1}. Then, the linear system of equations is solved by JB and GS iterative method. Figure 2 below shows the stencils of ice thickness.
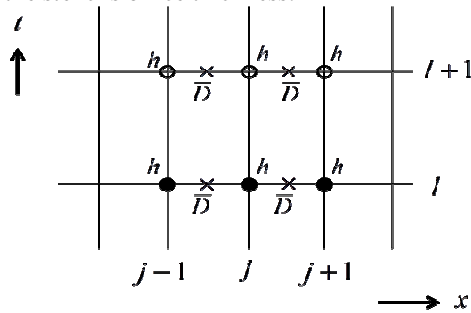

Figure 2. The stencil of the ice thickness using Crank-Nicolson method.

RESULTS

The simulation of ice sheet behaviour was carried out by implementing the thermomechanical ice sheet model. JB and GS methods are applied to solve the linear system equations derived from the ice thickness and diffusivity equations. The parameters and numerical results for the sequential algorithm are presented in Table 1 and 2 below. Table 1 presents the numerical results of the ice thickness while Table 2 presents the diffusivity of ice reaction.

The result from each table shows GS method converges faster than JB method. The iteration numbers of GS method are less than the JB method. The GS method needs 199 iterations while JB method needs 490 iterations which is shown in Table 1. The run time of GS method is only 218.0 seconds. Meanwhile JB method executes 698.5 seconds. In other words, GS method produced a faster simulation of ice sheet behavior compared to JB method. The value of root-mean-square error (RMSE) for the GS method is lower than the JB method. The value of RMSE also proved that the GS method converged faster than JB method. The table also shows the percentage of time execution, number of iterations and RMSE. From the ice thickness equation, the percentages of time execution, number of iterations and RMSE are 68.79 percent, 59.39 percent and 4.76 percent respectively. These percentages proved that GS method has better performance than JB method.

Table 1. Analysis of GS and JB method for ice thickness.

Parameters	GS	JB	$\left\lvert \dfrac{GS - JB}{JB} \right\rvert$
Time step	5	5	0
Convergence rate	1e-5	1e-5	0
Iteration	199	490	59.39%
Time execution	218.0 sec	698.5 sec	68.79%
RMSE	1.4985e-5	1.5734e-5	4.76%

Table 2. Analysis of GS and Jacobi method for diffusivity of ice.

Parameters	GS	JB	$\left\lvert \dfrac{GS - JB}{JB} \right\rvert \times 100\%$
Time step	5	5	0
Convergence rate	1e-5	1e-5	0
Iteration	199	490	59.39%
Time execution	218.0 sec	698.5 sec	68.79%
RMSE	4.5904e-5	5.090e-5	9.81%

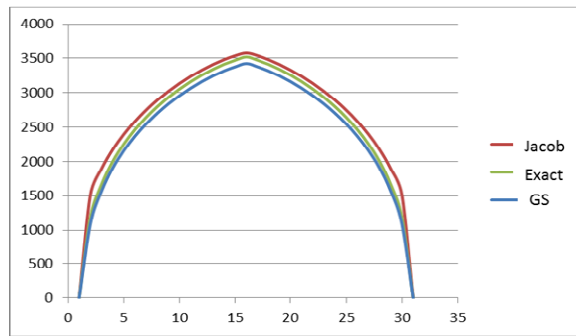Figure 3 and 4 shows the actual data and approximation data using JB and GS methods.

Figure 3. The ice thickness for exact solution and the numerical solution using JB and GS methods.
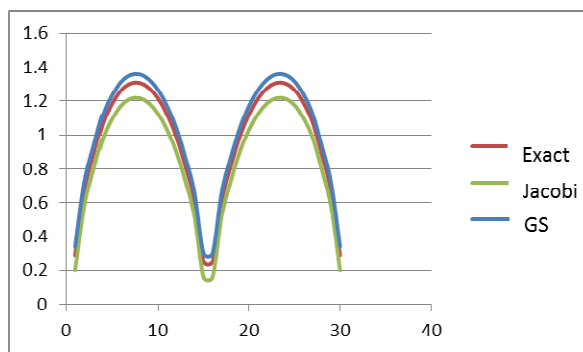

Figure 4. The diffusivity of ice for exact solution and the numerical solution using JB and and GS methods.

DISCUSSION

The melting process of ice involves several thermo-dynamical characteristics such as high density, specific heat and latent heat of fusion. The ice melting process is assumming in a rectangular enclosure. ComsolvMultiphysics software is used to solve the thermomechanical ice flow model numerically using finite element method. Firstly, develop the ice thickness equation by using general partial differential equation. Then, put the boundary flux with the value, $0.042 J/m^2 s$. Furthermore, to simulate the ice temperature, add physics of the heat transfer within the solid. Add the value of thermal conductivity of the ice with $2.31 W/mK$ and in the thermodynamic part, assigned the value of the density of ice is $918 kg/m^3$ whereas the heat capacity at constant pressure is $2052 J/kgK$.

The phase change from solid to liquid occurs when the energy is added to the solid. The energy alters the molecular structure when the temperature rises. The material used in this model is water or liquid. The material properties of water are available in the Material Library . The density and dynamic viscosity are functions of the temperature. Besides, put the mesh on that geometry in 'fine' level. It is important to approximate how fine should be the mesh needed to resolve temperature gradients.

The following is the steps to visualize the ice melting process using ComsolMultiphysics 4.3a:

Step 1:   Choose three dimensional for this model.
Step 2:   Choose 'General Partial Differential Equation' as the mathematical modeling of the ice melting.
Step 3:   Add physics characteristics to the model such as 'Heat Transfer in Fluids'.
Step 4:   Develop the 3D of ice melting by using the 'box' and 'work plane' toolbar and put the material to the model as 'water, liquid'.
Step 5:   For mesh settings, choose Extra fine.
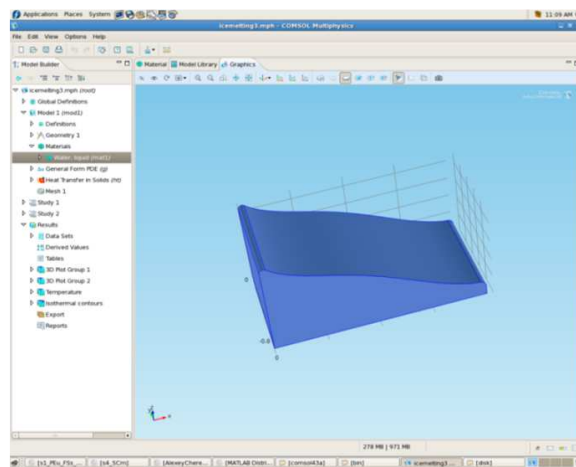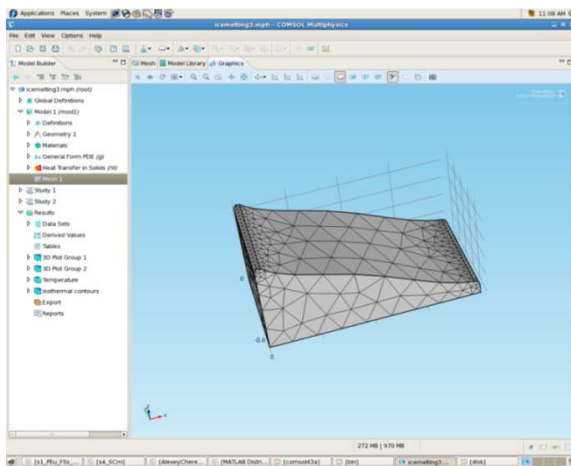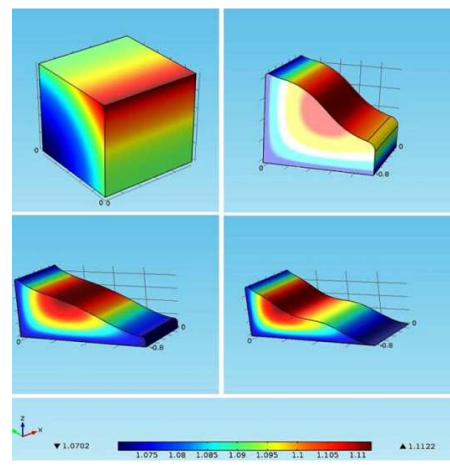Step 6:   Final result of ice melting process with different time.
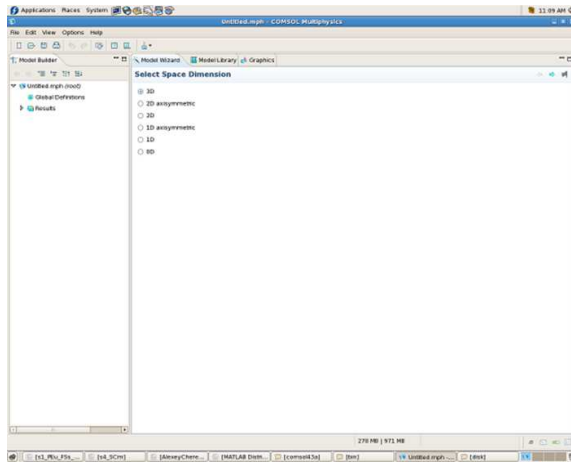
Step 1
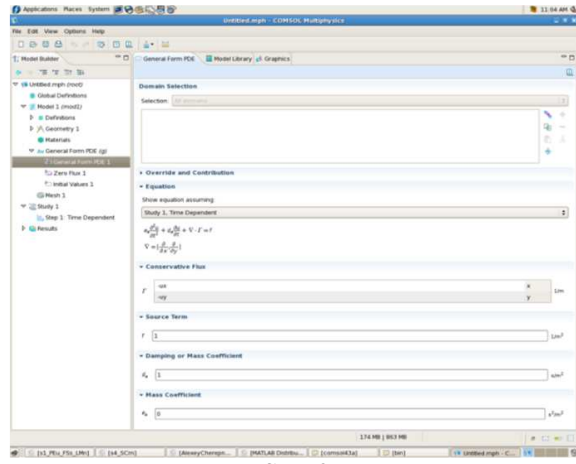
Step 2
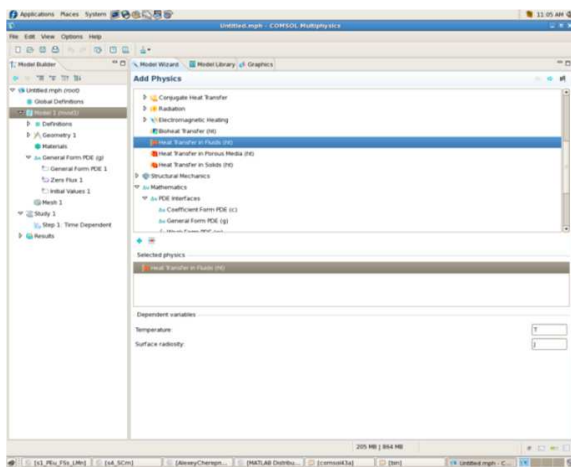
Step 3

Step 4

Step 5

Step 6

Figure 5. The visualization of ice melting process using Comsol Multiphysics software.

ComsolMultiphysics produced a good visualization of the ice melting process by showing the temperature distribution inside the ice. There are only several independent parameters that can manipulate inside the software so that it gives limited result. This is the pioneer way to develop the numerical simulation for observing the ice melting process behavior.

CONCLUSION

In this paper, thermomechanical ice sheet model has been implemented to simulate the ice sheet profile in terms of thickness, diffusion, temperature and velocity. The thermomechanical model is a simple and a suitable model for ice sheet simulation and the result is approximately similar to the analytical data. The finite difference method based on explicit and Crank-Nicholson method is used in order to perform the numerical implementations and simulation. This research has proved that the Crank-Nicholson method is more efficient and precise than the explicit method.

In the case of solving the system of linear equations for ice thickness and diffusion, the implementation of JB and GS method for solving the system linear of equations shows different results. The numerical results proved that the performance of GS method is better compared to the JB method in terms of time execution, number of iterations, convergence rate, and RMSE. The GS method is the alternative scheme to solve the tri-diagonal of linear system.

This research also successfully visualized the temperature behavior and the melting process of ice sheet by using ComsolMultiphysics. In the process of visualization, several steps have been facilitates such as defining the geometry of the thermomechanical model, determine the type of meshing, specifying the physical processes of ice sheet profile, discretizing the mathematical model by using finite element method and finally visualizing the results.

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