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Unsteady Drainage of the Power Law Fluid Model Down a Vertical Cylinder

K. N. Memon[1*], A. M. Siddiqui[2], Syed Feroz Shah[1] and Salman Ahmad[3]

[1]Department of Basic Science, Mehran University of Engineering Technology Jamshoro, Pakistan
[2] Pennsylvania State University, York Campus, Edgecombe 17403, USA
[3] Department of Applied Mathematics & Statistics, Institute of Space Technology, Islamabad, Pakistan

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ABSTRACT

The unsteady problem of thin film flow of a Power law fluid on a vertical cylinder for a drainage problem has been studied. The nonlinear differential equation has been derived from the momentum equation by Jeffrey's approach (Gutfinger and Tallmadge 1964; Bagchi1965; Dutta 1973). Series solutions have been obtained by binomial series method. Expressions for velocity, flow rate, thickness of the fluid film, mean thickness, vorticity vector and force exerted by the fluid on the cylinder are calculated. The graphical results for velocity profile and thickness of the film are discussed and examined for different parameters of interest.

KEYWORDS: Thin film flow; power law fluid; Jeffrey's approach; Binomial series method.

1 INTRODUCTION

In recent years, the flow of non-Newtonian fluids have been intensively studied due to their industrial application in medical and engineering sciences. Tooth paste, greases, paints, drilling mud, blood, clay coatings, polymer melts etc., are the some examples of non-Newtonian fluids. In addition the conservation of mass and momentum, material constitutive equations are necessary for taking into account the memory effects of such fluids. It is very complicated to propose a particular model which display all properties of such fluids, appropriate to its mathematical complexity. Due to this cause numerous models have been projected to investigate the behavior of various kinds of non-Newtonian fluids (Deshpande and Barigou2001; Kemihaet al., 2006;Jieand Xi-Yun 2006; Mahomedet al., 2007).

In the category of non-Newtonian fluids the power law model have been extensively studied because of mathematical simplicity and wide spread industrial applications. During the last four decades significant progress has been made in the development of analytical solution and numerical algorithms of power law fluid flow problems (Yong-Li et al., 2009; Kapur 1963;Nejat et al.,. 2011; Ghoreishy and Razavi1998).

Study of thin film flow has received significant attention due to practical concentration in physical and biological sciences. Many researchers have grappled with the analysis these type of flows since their formulation. The non-Newtonian fluids have been used by researchers (Siddiqui et al., 2006; Hayat and Sajid 2007; Sajid and Hayat 2008) for thin film flow to investigate and solve them analytically and numerically.

In this paper we investigate the thin film flow down a vertical cylinder of a power law fluid using Jeffrey's approach(Jeffreys 1930; Van Rossum 1958; Gutfinger and Tallmadge 1964; Bagchi 1965; Dutta 1973)for drainage problem, two cases are discussed, Newtonian and power law fluid respectively. In Newtonian case we find the exact solution while in power law series solution is obtain. According to the best of our knowledge the solution of the problem has been not reported in the literature.

This letter is organized as follows. Section 2 contains the governing equations of the fluid model. In Section 3 the problem under consideration is formulated. In section 4, the governing equation of the problem is solved. Section 5 deals with the results and discussion. Concluding remarks are given in section 6.

2. BASIC EQUATIONS

The basic equations, governing the flow of incompressible power law fluid neglecting the thermal effects, are:

$$\nabla \cdot \mathbf{V} = 0, \quad (1)$$

*** Corresponding Author:** Mr. Kamran Nazir Memon, [1]Department of Basic Science, Mehran University of engineering technology jamshoro, Pakistan. kamrannazir025@yahoo.com

$$\rho \frac{D\mathbf{V}}{Dt} = \rho\mathbf{f} - \nabla p + div\, \mathbf{S}, \qquad (2)$$

where ρ is the constant density, $\mathbf{V}$ is the velocity vector, $\mathbf{f}$ is the body force, p is the pressure, $\mathbf{S}$ is the extra stress tensor and $\frac{D}{Dt}$ is denoting the material time derivative derivative. As discussed in(Bird et al., 1987), the stress tensor defining a power law fluid is given by:

$$\mathbf{S} = \mu \left| \sqrt{\frac{tr(\mathbf{A}_1^2)}{2}} \right|^{n-1} \mathbf{A}_1, \qquad (3)$$

where μ is the coefficient of viscosity and n is the power law index. The Rivilin-Ericksen tensor, $\mathbf{A_1}$ is defined by

$$\mathbf{A}_1 = (\nabla\mathbf{V}) + (\nabla\mathbf{V})^T. \quad (4)$$

Remark: On behalf of consequent model for $n < 1$ the fluid is "pseudoplastic" for model or "shear thinning" for $n > 1$ the fluid is "dilatant" or "shear-thickening" and for $n = 1$ the Newtonian fluid is recovered.

3. PROBLEM FORMULATION

Consider unsteady, laminar and parallel flow of an incompressible Power law fluid moving slowly down an infinite vertical cylinder. As a result, a thin fluid film of thickness *h* which varies with time adheres to the cylinder and drains down under the action of gravity. The geometry of the problem in *Figure* 1 shows that *rz*- coordinate system has been chosen such that *r*-axis is normal to the cylinder axis and *z*-axis along the cylinder in downward direction. For simplicity, we assume that the fluid is non-conducting and completely wets the cylinder. Further there is no applied (force) pressure driving the flow and body force is only due to gravity and therefore we shall look for a velocity and a stress field of the form:

$$\mathbf{V} = [0,0,w(r,t)], \qquad \mathbf{S} = S(r,t). \qquad (5)$$

Using equation (5), the continuity equation (1) is identically satisfied and the momentum equation (2) reduce to,

r – component:

$$0 = -\frac{\partial p}{\partial r}, \quad (6)$$

θ - component:

$$0 = -\frac{\partial p}{\partial \theta}, \qquad (7)$$

z – component:

$$\rho \frac{\partial w}{\partial t} + \frac{\partial p}{\partial z} = \frac{\eta}{r} \frac{\partial}{\partial r}\left(r \left| \frac{\partial w}{\partial r} \right|^{n-1} \frac{\partial w}{\partial r} \right) + \rho g. \qquad (8)$$

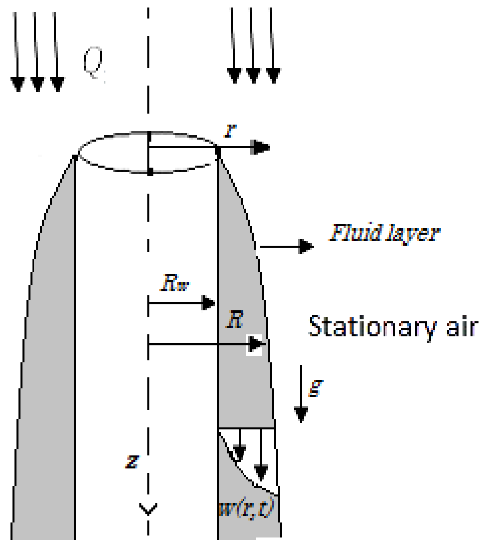

Figure1. Geometry of the thin film flow down a vertical cylinder.

Equations (6) and (7) implies that $p = p(z)$ only. Assume that pressure p is atmospheric pressure i.e., p is zero (gauge pressure) everywhere. As we are discussing the drainage flow problem, therefore, we take $\frac{\partial w}{\partial r}$ positive. Thus equation (8) reduces to,

$$\rho \frac{\partial w}{\partial t} = \frac{\eta}{r} \frac{\partial}{\partial r}\left(r\left(\frac{\partial w}{\partial r}\right)^n \right) + \rho g. \qquad (9)$$

Neglecting local acceleration term $\frac{\partial w}{\partial t}$ which is small compared to gravity except in the initial emptying of the vessel, we get,

$$\frac{1}{r} \frac{\partial}{\partial r}\left(r\left(\frac{\partial w}{\partial r}\right)^n \right) = -\frac{\rho g}{\eta}, \qquad (10)$$

which is non linear differential equation. The boundary conditions for this problem are:

$$\frac{\partial w}{\partial r} = 0 \quad at \quad r = R, \quad (11)$$

$$w = 0 \quad at \quad r = R_w, \quad (12)$$

4. Solution of the Problem

Integrating equation (10) with respect to r and using boundary condition (11), we obtain,

$$\frac{\partial w}{\partial r} = \left(\frac{\rho g}{2\eta}\right)^{\frac{1}{n}} \left(\frac{R^2}{r} - r\right)^{\frac{1}{n}}. \qquad (13)$$

which is a linear differential equation, here two cases arise:

Case-I: $n = 1$ (Newtonian fluid)

Case-II: $n \neq 1$ (Power law fluid)

Solution for the Newtonian Fluid

Velocity profile

For $n = 1$, the solution of equation (13) using boundary condition (12) is,

$$w = \frac{\rho g}{4\eta}\left[\left(R_w^2 - r^2\right) + 2R^2 \ln\left(\frac{r}{R_w}\right)\right]. \quad (14)$$

The z – component of the force exerted by the fluid on the cylinder surface is given by,

$$F_z = -\int_{R_w}^{R} \left(S_{rz}\right)_{r=R_w} dr. \quad (15)$$

Inserting the value of S_{rz} from equation (3) into equation (15), we get,

$$F_z = -\left(\frac{\rho g}{2}\right)(R - R_w)^2\left(1 + \frac{R}{R_w}\right). \quad (16)$$

The vorticity vector Ω is calculated as:

$$\Omega = \nabla \times \mathbf{V} = -\frac{\rho g}{2\eta}\left(\frac{R^2}{r} - r\right)\mathbf{j}, \quad (17)$$

where$\mathbf{j}$ is the unit vector in θ - direction. The negative sign indicates that vorticity decrease with the increase in r. We note that vorticity is zero at the free surface, while its magnitude is maximum at the cylinder given by $\frac{\rho g}{2\eta}\left(\frac{R^2}{R_w} - R_w\right)$.

Volume Flow Rate:

In dimention form, the flow rate Q, is given by,

$$Q = \int_0^{2\pi}\int_{R_w}^{R} rw(r)\,dr\,d\theta = 2\pi\int_{R_w}^{R} rw(r)\,dr. \quad (18)$$

By making use of equation (14) in (18), we obtain,

$$Q = -\frac{\rho g \pi}{8\eta}\left[\left(R^2 - R_w^2\right)^2 - 4R^4 \ln\left(\frac{R}{R_w}\right) + 2R^2\left(R^2 - R_w^2\right)\right]. \quad (19)$$

Thickness Of The Fluid Film:

Volume flow rate in term of continuity equation is given by,

$$-\frac{\partial Q}{\partial z} = 2\pi R\frac{\partial R}{\partial t}. \quad (20)$$

Substituting equation (19) in equation (20), after considerable simplification, we obtain,

$$\frac{\rho g}{\eta}\pi R\left[\left(R^2 - R_w^2\right) - 2R^2 \ln\left(\frac{R}{R_w}\right)\right]\frac{\partial R}{\partial z} = 2\pi R\frac{\partial R}{\partial t}. \quad (21)$$

After simplification, we get,

$$\frac{\partial z}{\partial t} = -\frac{\rho g}{2\eta}\left[\left(R^2 - R_w^2\right) - 2R^2 \ln\left(\frac{R}{R_w}\right)\right]. \quad (22)$$

Now integrating equation (22) with respect to t and using the boundary condition $R(0,t) = R_w$, we get the relation between film thickness z and t as,

$$z = -\frac{\rho g}{2\eta}t\left[\left(R^2 - R_w^2\right) - 2R^2 \ln\left(\frac{R}{R_w}\right)\right]. \quad (23)$$

Here two cases arise first for drainage on convex surface when $h = R - R_w$ and second for drainage on concave surface when $h = R_w - R$.

For convex surface by substitution $R = R_w + h$, we get,

$$z=\frac{\rho g}{2\eta}tR_w^2\left[\left(1-\left(1+\frac{h}{R_w}\right)^2\right)+2\left(1+\frac{h}{R_w}\right)^2\ln\left(1+\frac{h}{R_w}\right)\right], (24)$$

and for concave surface $R=R_w-h,$ we arrive at,

$$z=\frac{\rho g}{2\eta}tR_w^2\left[\left(1-\left(1-\frac{h}{R_w}\right)^2\right)+2\left(1-\frac{h}{R_w}\right)^2\ln\left(1-\frac{h}{R_w}\right)\right]. (25)$$

The amount of fluid adheres to the cylinder of length *z* is of great interest and can be obtained by defining mean thickness $\bar{h}$, which is,

$$\bar{h}=\frac{1}{z}\int_0^z h dz. \quad (26)$$

Differentiating equation (24) with respect to *h* and using the value of *dz* into equation (26) gives the mean thickness $\bar{h}$ for a cylinder of length z in terms of the fluid properties and the point thickness at z (Raghurman 1971).The mean thickness is given by,

$$\bar{h}=-\frac{\left(R_w h(2R_w-h)-\frac{h^3}{9}+2(2h^3+3R_w h^2-R_w^3)\ln\left(1+\frac{h}{R_w}\right)\right)}{3R_w^2\left[\left(1-\left(1+\frac{h}{R_w}\right)^2\right)+2\left(1+\frac{h}{R_w}\right)^2\ln\left(1+\frac{h}{R_w}\right)\right]}. \quad (27)$$

Solution for the Power Law Fluid
Velocity profile:

In case of $n\neq 1,$ from equation (13), we have,

$$\frac{\partial w}{\partial r}=\left(\frac{\rho g}{2\eta}\right)^{\frac{1}{n}}\left(\frac{R^2}{r}\right)^{\frac{1}{n}}\left(1-\frac{r^2}{R^2}\right)^{\frac{1}{n}}, \quad \text{where} \quad \left|\frac{r^2}{R^2}\right|<1. \qquad (28)$$

By the use of binomial series, it is simplified to:

$$\frac{\partial w}{\partial r}=\left(\frac{\rho g}{2\eta}\right)^{\frac{1}{n}}\left(\sum_{k=0}^{\infty}\binom{\frac{1}{n}}{k}(-1)^k R^{-2k+\frac{2}{n}} r^{2k-\frac{1}{n}}\right). (29)$$

The expression for velocity field is obtained by solving equation (29) corresponding to boundary condition (12) as,

$$w=\left(\frac{\rho g}{2\eta}\right)^{\frac{1}{n}}\left(\sum_{k=0}^{\infty}\binom{\frac{1}{n}}{k}\frac{(-1)^k R^{-2k+\frac{2}{n}}}{2k-\frac{1}{n}+1}\left(r^{2k-\frac{1}{n}+1}-R_w^{2k-\frac{1}{n}+1}\right)\right). \quad (30)$$

The vorticity vector Ω is calculated as:

$$\Omega=-\left(\frac{\rho g}{2\eta}\right)^{\frac{1}{n}}\left(\frac{R^2}{r}-r\right)^{\frac{1}{n}}\mathbf{j}, \quad (31)$$

where$\mathbf{j}$ is the unit vector in θ-direction. The negative sign indicates that vorticity decreases with the increase in *r*. We note that vorticity is zero at the free surface, while its magnitude is maximum, given by $\left(\frac{\rho g}{2\eta}\right)^{\frac{1}{n}}\left(\frac{R^2}{R_w}-R_w\right)^{\frac{1}{n}}$ at the cylinder.

Volume Flow Rate:
By making use of velocity field defined by equation (30) in (18), we obtain,

$$Q = 2\pi\left(\frac{\rho g}{2\eta}\right)^{\frac{1}{n}}\sum_{k=0}^{\infty}\binom{\frac{1}{n}}{k}\frac{(-1)^k R^{-2k+\frac{2}{n}}}{2k-\frac{1}{n}+1}\left[\frac{\left(R^{2k-\frac{1}{n}+3}-R_w^{2k-\frac{1}{n}+3}\right)}{2k-\frac{1}{n}+3}-\frac{R_w^{2k-\frac{1}{n}+1}}{2}\left(R^2-R_w^2\right)\right]. \quad (32)$$

Thickness Of The Fluid Film
Simplifying equation (32) after making use of equation (20), one obtains,

$$\frac{\partial z}{\partial t} = \left(\frac{\rho g}{2\eta}\right)^{\frac{1}{n}}\sum_{k=0}^{\infty}\binom{\frac{1}{n}}{k}\frac{(-1)^k R^{-2k+\frac{2}{n}-2}}{2k-\frac{1}{n}+1}\left[\frac{1}{2k-\frac{1}{n}+3}\left\{\left(\frac{1}{n}+3\right)R^{2k-\frac{1}{n}+3}-\left(-2k+\frac{2}{n}\right)R_w^{2k-\frac{1}{n}+3}\right\}\right.$$
$$\left.-\frac{R_w^{2k-\frac{1}{n}+1}}{2}\left\{\left(2+\frac{2}{n}-2k\right)R^2-\left(-2k+\frac{2}{n}\right)R_w^2\right\}\right]. \quad (33)$$

By integrating equation (33) with respect to t, and then using the boundary condition $R(0,t) = R_w$, we get the relation between film thickness z and t as,

$$z = \left(\frac{\rho g}{2\eta}\right)^{\frac{1}{n}}\sum_{k=0}^{\infty}\binom{\frac{1}{n}}{k}\frac{(-1)^k R^{-2k+\frac{2}{n}-2}}{2k-\frac{1}{n}+1}\left[\frac{1}{2k-\frac{1}{n}+3}\left\{\left(\frac{1}{n}+3\right)R^{2k-\frac{1}{n}+3}-\left(-2k+\frac{2}{n}\right)R_w^{2k-\frac{1}{n}+3}\right\}\right.$$
$$\left.-\frac{R_w^{2k-\frac{1}{n}+1}}{2}\left\{\left(2+\frac{2}{n}-2k\right)R^2-\left(-2k+\frac{2}{n}\right)R_w^2\right\}\right]t. \quad (34)$$

For drainage on a convex surface, we get,

$$z = \left(\frac{\rho g}{2\eta}\right)^{\frac{1}{n}}\sum_{k=0}^{\infty}\binom{\frac{1}{n}}{k}\frac{(-1)^k (R_w+h)^{-2k+\frac{2}{n}-2}}{2k-\frac{1}{n}+1}\left[\frac{1}{2k-\frac{1}{n}+3}\left\{\left(\frac{1}{n}+3\right)(R_w+h)^{2k-\frac{1}{n}+3}\right.\right.$$
$$\left.\left.-\left(-2k+\frac{2}{n}\right)R_w^{2k-\frac{1}{n}+3}\right\}-\frac{R_w^{2k-\frac{1}{n}+1}}{2}\left\{\left(2+\frac{2}{n}-2k\right)(R_w+h)^2-\left(-2k+\frac{2}{n}\right)R_w^2\right\}\right]t, \quad (35)$$

and for drainage on concave surface, we arrive at,

$$z = \left(\frac{\rho g}{2\eta}\right)^{\frac{1}{n}}\sum_{k=0}^{\infty}\binom{\frac{1}{n}}{k}\frac{(-1)^k (R_w-h)^{-2k+\frac{2}{n}-2}}{2k-\frac{1}{n}+1}\left[\frac{1}{2k-\frac{1}{n}+3}\left\{\left(\frac{1}{n}+3\right)(R_w-h)^{2k-\frac{1}{n}+3}\right.\right.$$
$$\left.\left.-\left(-2k+\frac{2}{n}\right)R_w^{2k-\frac{1}{n}+3}\right\}-\frac{R_w^{2k-\frac{1}{n}+1}}{2}\left\{\left(2+\frac{2}{n}-2k\right)(R_w-h)^2-\left(-2k+\frac{2}{n}\right)R_w^2\right\}\right]t. \quad (36)$$

The mean thickness is given by,

$$
\begin{aligned}
\bar{h} = {} & \frac{1}{K}\sum_{k=0}^{\infty}\binom{\frac{1}{n}}{k}\frac{(-1)^k}{2k-\frac{1}{n}+1}\Bigg[\frac{1}{2k-\frac{1}{n}+3}\Bigg\{h\left(\frac{1}{n}+3\right)(R_w+h)^{\frac{1}{n}+1}-h\left(-2k+\frac{2}{n}\right)R_w^{2k-\frac{1}{n}+3}(R_w+h)^{-2k+\frac{2}{n}-2} \\
& -\frac{(1+3n)}{(1+2n)}\left((R_w+h)^{\frac{1}{n}+2}-R_w^{\frac{1}{n}+2}\right)+\frac{(-2k+\frac{2}{n})R_w^{\frac{1}{n}+2}}{-2k+\frac{2}{n}-1}\left((R_w+h)^{-2k+\frac{2}{n}-1}R_w^{2k-\frac{2}{n}+1}-1\right)\Bigg\} \\
& -R_w^{2k-\frac{1}{n}+1}\Bigg\{h\left(1+\frac{1}{n}-k\right)(R_w+h)^{-2k+\frac{2}{n}}-h\left(-k+\frac{1}{n}\right)R_w^2(R_w+h)^{-2k+\frac{2}{n}-2}-\frac{\left(1+\frac{1}{n}-k\right)}{\left(-2k+\frac{2}{n}+1\right)}\left((R_w+h)^{-2k+\frac{2}{n}+1}-R_w^{-2k+\frac{2}{n}+1}\right) \\
& +\frac{\left(-k+\frac{1}{n}\right)}{-2k+\frac{2}{n}-1}R_w^{-2k+\frac{2}{n}+1}\left(R_w^{2k-\frac{2}{n}+1}(R_w+h)^{-2k+\frac{2}{n}-1}-1\right)\Bigg\}\Bigg],
\end{aligned}
\qquad (37)
$$

where,

$$
\begin{aligned}
K = {} & \sum_{k=0}^{\infty}\binom{\frac{1}{n}}{k}\frac{(-1)^k(R_w+h)^{-2k+\frac{2}{n}-2}}{2k-\frac{1}{n}+1}\Bigg[\frac{1}{2k-\frac{1}{n}+3}\Bigg\{\left(\frac{1}{n}+3\right)(R_w+h)^{2k-\frac{1}{n}+3} \\
& -\left(-2k+\frac{2}{n}\right)R_w^{2k-\frac{1}{n}+3}\Bigg\}-\frac{R_w^{2k-\frac{1}{n}+1}}{2}\Bigg\{\left(2+\frac{2}{n}-2k\right)(R_w+h)^2-\left(-2k+\frac{2}{n}\right)R_w^2\Bigg\}\Bigg].
\end{aligned}
$$

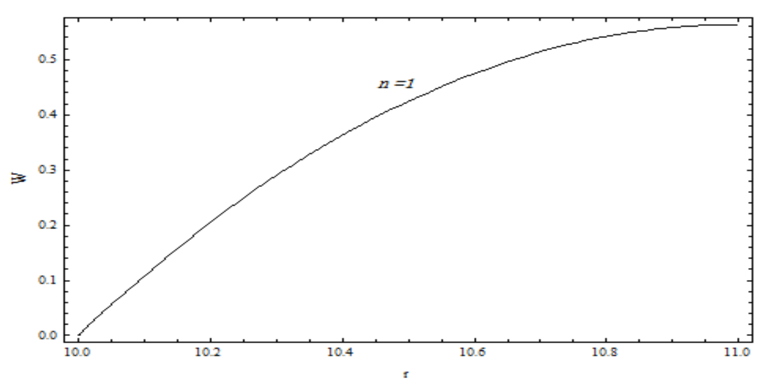

Figure2.**Velocity profile for Newtonian fluid for drainage in thin film flow, when η =7 *poise*, *R*=11 *cm*, R_w=10 *cm* and** $\rho = 0.78\, g/cm^3$**.**

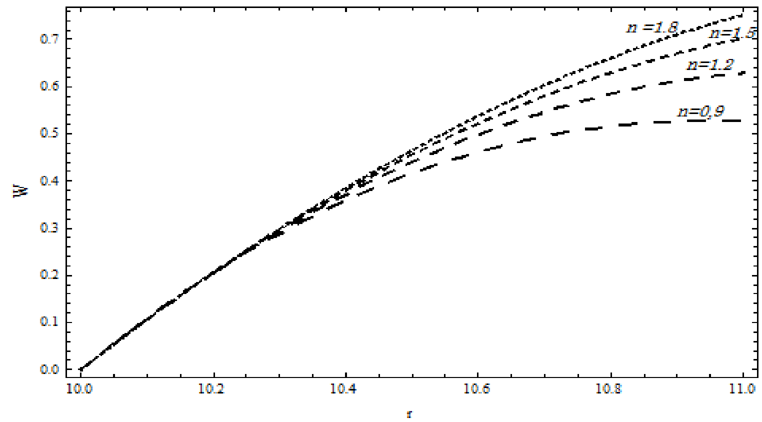

Figure3.**Velocity profile for Power law fluid for drainage in thin film flow for different values of** n**, when** η **=7** ***poise*****,** ***R*****=11** ***cm*****,** R_w**=10** ***cm*****and** $\rho = 0.78\, g / cm^3$**.**

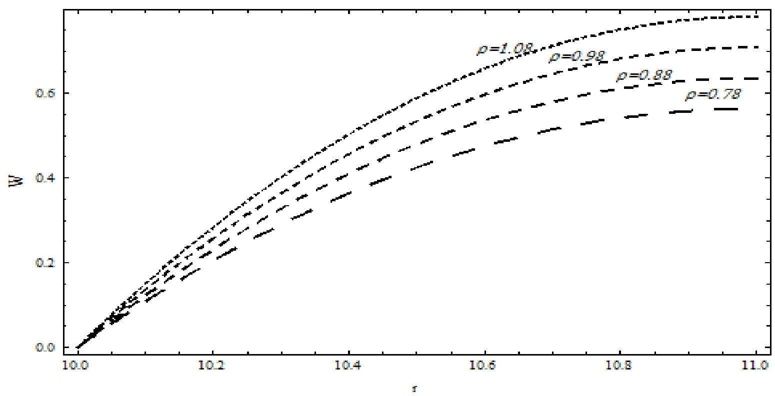

Figure. 4.**Velocity profile for Newtonian fluid for drainage in thin film flow for different values of** η **=7** ***poise*****, when** ***R*****=11** ***cm*****,** R_w**=*10 cm*.**

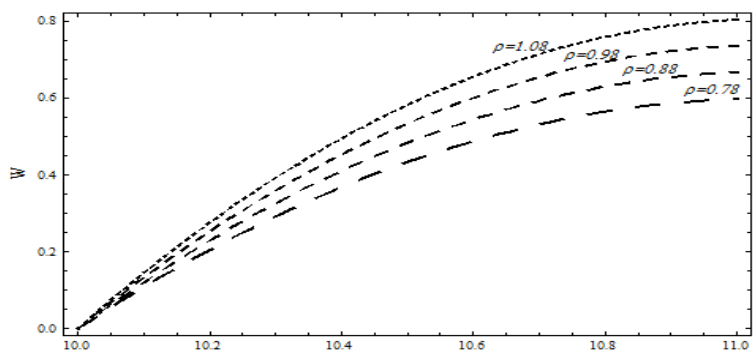

Figure5**.Velocity profile for Power law fluid for drainage in thin film flow for different values of** η **=7** ***poise*** **, when** ***R*****=11** ***cm*****,** R_w**=10** ***cm*** **and** $n = 1.1$.

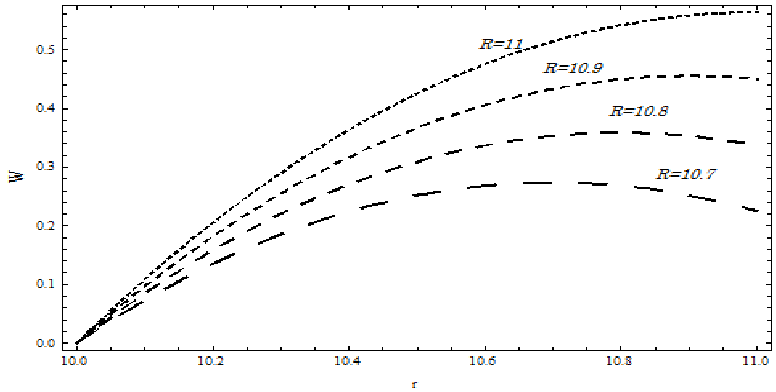

Figure6. Velocity profile for Newtonian fluid for drainage in thin film flow for different values of R measured in cm, when η =7 *poise*, R_w=10 *cm* and $\rho = 0.78\, g/cm^3$.

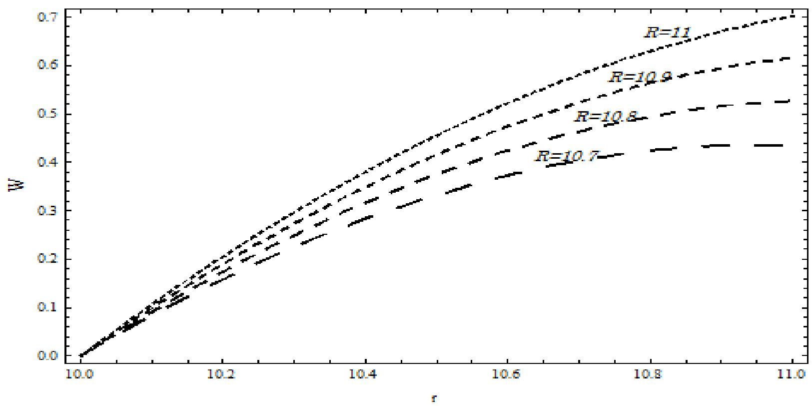

Figure7. Velocity profile for Power law fluids for drainage in thin film flow for different values of R when η =7 *poise*, R_w=10 *cm*, n=1.9 and $\rho = 0.78\, g/cm^3$.

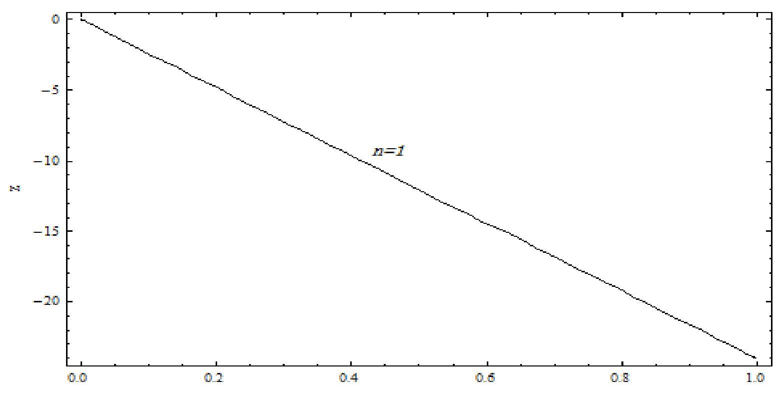

Figure8. Growth of film thickness with respect to time for Newtonian fluid, when R_w=10 *cm*, η =7 *poise*, h=1 *cm* and $\rho = 0.78\, g/cm^3$.

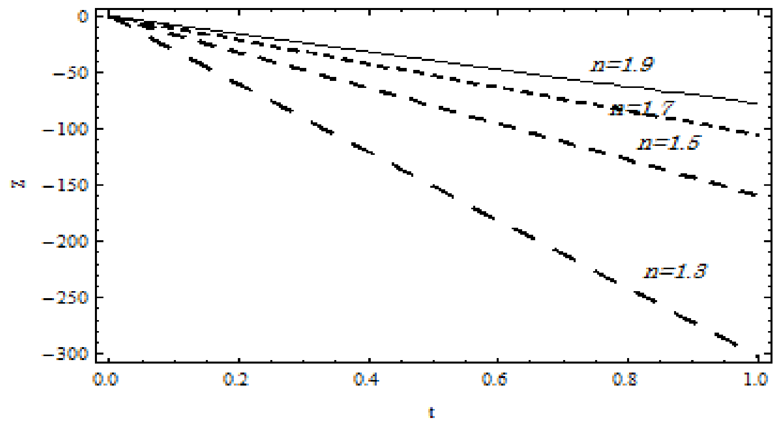

Figure9. Growth of film thickness with respect to time for Power law fluid for different value of *n*, when R_w=10 *cm*, η =7 *poise*, *h*=1*cm* and $\rho = 0.78\, g/cm^3$.

5. RESULTS AND DISCUSSION

The systematic investigation for effects of Power law index n, density ρ and R on velocity profile and growth of film thickness with respect to time are observed graphically in figures (2) – (9). The variation of axial velocity for n, ρ and R for both Newtonian and Power law fluid in case of drainage is displayed in figures (2) – (7). From figures (2) – (7), we observed that, with an increase in n, ρ and R, velocity profile increases. The difference of ρ and n for growth of film thickness with respect to time in figure (8) – (9) have been plotted, in which it is observed that thickness of fluid film increases for all *n*.

6. CONCLUDING REMARKS

We have presented results for the thin film flow field of a fluid called the Power law fluid, on a vertical cylinder for drainage problem. The resulting nonlinear differential equation has been solved by binomial series method, which is a suitable analytical method for the proposed problem. The velocity profile, vorticity vector, volume flow rate, growth of thickness of the fluid film, mean thickness and force exerted by the fluid on the cylinder have been derived for the title problem.

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