J. Appl. Environ. Biol. Sci., 5(10)222-225, 2015
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ISSN: 2090-4274
Journal of Applied Environmental and Biological Sciences
www.textroad.com

Cohen-Macaulay Ring Property of Finite Fuzzy Distributive Lattices

Ali Molkhasi[1]., Ebrahim Nazari[2]

[1,2] Department of Mathematics, Farhangyan University, Iran

Received: April 23, 2015
Accepted: August 8, 2015

ABSTRACT

In this paper relationship among the Cohen-Macaulay ring, finite fuzzy Priestley space, finite Fuzzy Distributive Lattices, fuzzy congruence, and **L**-fuzzy ideals of a fuzzy lattice are considered.

Subject Classification 2010: 06D72, 06B99.
KEYWORDS: Fuzzy Priestley space, Cohen-Macaulay ring, Fuzzy congruence.

1. INTRODUCTION

The study of fuzzy relations was started by Zadeh [16] in 1971. Since then many notions and results from the theory of ordered sets have been extended to the fuzzy ordered sets. In mathematics, a distributive lattice is a lattice in which the operations of join and meet distributive over each other. After that several researchers have applied the notion of fuzzy sets to the concept of congruence relation on general sets. In [15], Venugopalan introduced a definition of fuzzy ordered set (foset) (P,μ) and presented an example on the set of positive integers. He extended this concept to obtain a fuzzy lattice in which he defined a (fuzzy) relation as a generalization of equivalence. Yuan and Wu ([14]) introduced the concepts of fuzzy sublattices and fuzzy ideals of a alttice. In a series of papers, Priestley [8], [7] gave a theory of representation of istributive lattices. U. M. Swamy and others in [10], [11], [12], and [13] studied various fuzzy algebraic systems with truth values in abstract complete lattices satisfying the infinite meet distributivity of a set X, namely, $a\wedge(\sup X)=\sup\{a\wedge x \mid x\in X\}$ for all elements a and subsets X. Such lattices are called frames. N. Funayama and T. Nakayama proves that congruence relations on an arbitrary lattice have an interesting connection with distributive lattices: The paper K.R. Goodearl and F. Wehrung [5] is a rich source of information on connections between congruence lattice representation problems and ring theory. In this paper relationship among the Cohen-Macaulay ring, finite fuzzy Priestley space, finite fuzzy distributive Lattices, fuzzy congruence, and **L**-fuzzy ideals of lattice is considered. It shown that if R is Cohen-Macaulay ring and dual of a finite fuzzy Priestley space $\delta=(X,\tau,r)$ is $(L(\delta), \wedge, \vee, r_1)$, where $L(\delta)=\{Y\subseteq X \mid Y \text{ is increasing and } \tau\text{-clopen}\}$ and r_1 is a fuzzy order adequately chosen, then $R[L(\delta)][X_1,X_2,\dots]$ is WB-height-unmixed. In particular, if is the set of all **L**-fuzzy ideals of $L(\delta)$ and the set I(A) is ideals of $L(\delta)$, then $R[(FI)LL(\delta)][X_1,X_2,\dots]$ and $R[I(L(\delta))][X_1,X_2,\dots]$ are WB-height-unmixed.

2. Preliminaries

In this section, we recall some definitions and concepts that we shall need in the sequel. Let X be a non-empty set. A fuzzy set F on $X\times X$ (i.e., $X\times X$ \o $(\longrightarrow)$ $[0,1]$ mapping) is called a fuzzy binary relation on X. A fuzzy binary relation F on X is called reflexive, if $F(x,x)=1$, for all $x\in X$, antisymmetric, if $F(x,y)\wedge F(y,x)=0$ whenever $x\neq y$, for all $x,y\in X$, transitive, if $F(x,y)\wedge F(y,z)\leq F(x,z)$, for all $x,y,z\in X$.

A reflexive and transitive fuzzy relation is called a fuzzy preordering. A set equipped with a fuzzy order relation is called a fuzzy ordered set (foset). A fuzzy ordered space is a triplet (X,τ,F), where X is a non empty set, τ is a topology on X and F is a fuzzy order on X. A fuzzy lattice is a fuzzy order (A,F), where A is a non-empty crisp set, such that any two elements have a supremum and an infimum, it is denoted by $(A,\wedge,\vee,F)$, where the symbols $\wedge$ and $\vee$ stand for supremum and infimum, respectively. For $a,b\in A$, $a\vee b$ is the supremum of a and b with respect to the fuzzy order F, and $a\wedge b$ is the infimum of a and b with respect to the fuzzy order F. A fuzzy lattice A is called complete if every subset of A have a supremum and an infimum. A fuzzy lattice A is called fuzzy distributive lattice

***Corresponding Author:** Ali Molkhasi, Department of Mathematics, Farhangyan University, Iran
E-mail: molkhasi@cfu.ac.ir

(shortly, F-D-lattice) if for every $x,y,z\in A$, $x\vee(y\wedge z)=(x\vee y)\wedge(x\vee z)$ or $x\wedge(y\vee z)=(x\wedge y)\vee(x\wedge z)$. We have already encountered the fuzzy congruence on a lattice.

Definition 2.1. [9] Let X be a lattice and F be a fuzzy equivalence relation on X. Then F is join compatible if

$F(x_1 \vee x_2, y_1 \vee y_2) \geq F(x_1, y_1) \vee F(x_2, y_2)$ and F is meet compatible

$F(x_1 \wedge x_2, y_1 \wedge y_2) \geq F(x_1, y_1) \wedge F(x_2, y_2)$ if for all x_1,x_2,y_1,y_2 in X. If F is both join compatible and meet compatible, then F is a fuzzy congruence on X.

We know the fuzzy congruence on a lattice X is a fuzzy equivalence relation, it determines similarity classes. Let X/F denote the set of all similarity classes of X determined by the fuzzy congruence F. i.e., X/F={F_x/xÎX} where

F_x:$X \longrightarrow [0,1]$ is defined by . Define two binary operations and on X/F by

$$F_x \vee F_y = F_{x\vee y}, F_x \wedge F_y = F_{x\wedge y}$$

Definition 2.2. [9] Let X be a lattice and F a fuzzy congruence on X. The set X/F={F_x|xÎX} together with the binary operations $\vee$ and $\wedge$ is a lattice, which we call the factor lattice of X corresponding to the fuzzy congruence F on X.

3. Cohen-Macaulay Ring for Finite Fuzzy Distributive Lattices

In this section relationship among the Cohen-Macaulay Ring, Finite Fuzzy Distributive Lattices, and unmixed with respect to the set of weak Bourbaki associated primes is considered. Let us motivate our study of height of ideals.

Definition 3.1. A prime ideal P is an associated prime of I, if P=I:x for some $x\in R$.

Remember that the height of a prime ideal P is the maximum length of the chains of prime ideals of the following form,

$$P_1 Ì P_2 Ì \cdots Ì P_k = P.$$

We will denote the height of P by ht(P). An ideal I of R is said to be height-unmixed, if all the associated primes of I have equal height. That is ht(P)=ht(Q), for all $P,Q\in$Ass(I), where Ass(I) denotes the set of associated primes of I. An ideal I is said to be unmixed if there are no embedded primes among the associated primes of I. That is, $P\subseteq Q \Rightarrow P=Q$, for all $P,Q\in$Ass(I). We will say that an ideal is WB-height-unmixed, if it is height-unmixed with respect to the set of weak Bourbaki associated primes and an ideal is WB-unmixed if it is unmixed with respect to the set of weak Bourbaki associated primes. The set of weak Bourbaki associated primes of an ideal I is denoted by Ass_f(I). A prime ideal P is a weak Bourbaki associated prime of the ideal I if it is a minimal ideal of the form I:a, for some $a\in R$. For the proof of the following theorem, see [3].

Recall, throughout this section all fuzzy distributive lattices (F-D-lattices) are finite and homomorphisms preserve first (0) and last (1) elements. If $(A,\vee,\wedge,R)$ is a F-D-lattice, then its dual space is defined by: $T(A)=(X,\tau,F_1)$, where X is the set of 0-1 homomorphisms from A onto {0,1} , τ is the product topology induced by $\{0,1\}^A$ and F_1 is the fuzzy order on X. Indeed, F_1 is defined on F. The following theorem, shows dual of the finite fuzzy Priestley space is related polynomials ring coefficients in Cohen-Macaulay ring.

Theorem 3.2 If R is Cohen-Macaulay ring and dual of a finite fuzzy Priestley space $\delta=(X,\tau,r)$ is $(L(\delta), \wedge,\vee, r_1)$, where $L(\delta)=\{Y\subseteq X \mid Y$ is increasing and τ−clopen$\}$ and r_1 is a fuzzy order adequately chosen, then $R[L(\delta)][X_1,X_2,\ldots]$ is WB-height-unmixed.

Proof. Suppose that $h(r)\neq 0$. Then X is not an antichain, we can defined r_1, meet, join as follows:

$$E\vee F=E\cup F, \quad E\wedge F=E\cap F$$

For every E and F from L(δ), and we can defined:

It follows that $(L(\delta), \wedge,\vee, r_1)$, is a fuzzy distributive lattice. If $h(r)\neq 0$, then X is not an antichain, we choose:

$$m_0=\text{Ù}_x\text{Ù}_y\{\mu_r(x,y)|\,x\neq y \text{ and } \mu_r(x,y)\succ 0\}.$$

Then $m_0\neq 0$ we have:

$$r_1(E,F)=\begin{cases} 1 & \text{if} E=F \\ 0 & \text{othervise} \\ \text{Max}(m_0;\text{Ú}_{a\text{Î}E_{a\neq b};b\text{Î}F}r(a;b)) & E\text{Ì}F \end{cases}$$

and

$$E\vee F=E\cup F,\quad E\wedge F=E\cap F.$$

For every E and F from L(δ). Therefor (L(δ), $\wedge,\vee,r_1$) with fuzzy order r_1 and meet $\wedge$ and join $\vee$ form fuzzy distributive lattice.

Now if R satisfies GPIT (generalized principal ideal theorem), then the ring R is WB-height-unmixed if and only if R is WB-unmixed. In [1], it was shown that $R[X_1,X_2,\ldots]$ satisfies GPIT (if R is a Noetherian ring). The statement of this fact in [1] actually makes the assumption that R is a domain, however, the fact that R is a domain, is not necessary in the proof given in [1], so we will use the more general result. In [1], it is proved that, if R is a Cohen-Macaulay ring, then $R[X_1,X_2,\ldots]$ is WB-height-unmixed. In [3] we have if R is Cohen-Macaulay ring and if P is a distributive lattice, then R[P] is Cohen-Macaulay. Since R and L(δ) are Cohen-Macaulay ring and fuzzy distributive lattice, respectively, then R[L(δ)] is Cohen-Macaulay ring. Therefor, this shows that $R[L(\delta)][X_1,X_2,\ldots]$ is WB-height-unmixed (see [2] and [6]).

Recall, the congruences lattice of any lattice is distributive (Funayama and Nakayama) and Con(**A**) is an algebraic (Birkhoff and Frink). Recall that for an arbitrary L by denoted Con(L) for the lattice of all congruence relations of L. We have the following definition. As usual, we will say that an algebra **A**=(A, F) is said congruence distributive if the lattice Con(**A**) is distributive.

Theorem 3.3. Suppose that R be Cohen-Macaulay ring. Let L and K be lattices, F be a fuzzy congruences on L and S be a fuzzy congruence on K. Define the fuzzy relation F×S on L×K by $F\times S((a,b),(c,d))=F(a,c)\wedge S(b,d)$. Then $R[\text{Con}(F\times S)][X_1,X_2,\ldots]$ is WB-height-unmixed, where Con(F×S) is a fuzzy congruence lattice.

Proof. First we prove that F×S is fuzzy congruence on L×K. We have F, S are fuzzy congruence and

$$(F\times S)((a,b),(a,b))=F(a,a)\wedge S(b,b))=1$$

$$(F\times S)((a,b),(a,b))=F(a,c)\wedge S(b,d)$$

$$=(F\times S)((c,d),(a,b)$$

$$(F\times S)((a,b),(z_1,z_2))=F(a,z_1)\text{Ù}S(b,z_2)\geq$$

$$\sup_{c\text{Î}L}\{F(a,c)\text{Ù}F(c,z_1)\}\text{Ù}\sup_{d\text{Î}K}\{S(b,d)\text{Ù}S(d,z_2)\}$$

$$=\sup_{(c,d)\text{Î}L\times K}\{F(a,c)\text{Ù}F(c,z_1)\text{Ù}S(b,d)\text{Ù}S(d,z_2)\}$$

Thus F×S is reflexive, symmetric and transitive and hence a fuzzy equivalence relation. We know for an arbitrary lattice L, Con(L) is distributive lattice (we refer the reader to []). Therefor Con(F×S) is distributive lattice and F[Con(F×S)] is Cohen-Macaulay ring. Finally, $R[\text{Con}(F\times S)][X_1,X_2,\ldots]$ is WB-height-unmixed.

Recall that a nonempty subset I of a lattice $(A,\wedge,\vee,F)$ is called an ideal of A if $a,b\in I$ then $a\vee b\in I$ and $a\wedge x\in I$ for all $x\in A$. Clearly the set I(A) of ideals of A is a lattice in which, for any I and $J\in I(A)$, $I\wedge J=I\cap J$ and $I\vee J=\{x\leq A\,|\,x\leq a\vee b \text{ forsome } a\in I \text{ and } b\in J\}$.

Let $(A,\wedge,\vee,F)$ be a lattice with 0 and **L** a frame. An **L**-fuzzy subset B of A is called an **L**-fuzzy ideal of A if B(0)=1 and $B(x\vee y)=B(x)\wedge B(y)$ for all x and y in A. In this paper suppose that (FI)L(B) be the set of all **L**-fuzzy ideals of A.

Result: We know that if $(A,\wedge,\vee,F)$ is a lattice with 0 and **L** a frame, then (FI)L(B) is a distributive lattice if and only if I(A) is a distributive lattice if and only if A is a distributive lattice. By applying 3.1 we have if R is Cohen-Macaulay ring and δ=(X,τ,r) is a finite fuzzy Priestley space and its dual is defined by: (L(δ), $\wedge,\vee,r_1$), where

$L(\delta)=\{Y\subseteq X|Y$ is increasing and τ−clopen} and r_1 is a fuzzy order adequately chosen and **L** is a frame. Then $R[(FI)LL(\delta)][X_1,X_2,\ldots]$ and $R[I(L(\delta))][X_1,X_2,\ldots]$ are WB-height-unmixed.

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