J. Appl. Environ. Biol. Sci., 5(3)229-234, 2015
© 2015, TextRoad Publication

ISSN: 2090-4274
Journal of Applied Environmental and Biological Sciences
www.textroad.com

On the Entropy of Dynamical Systems on σ −mv-Algebras with State

M. Ebrahimi[1], U. Mohammadi[2,*], R. Ghasemkhani[2]

[1]Department of Mathematics, Faculty of Mathematics and Computer Shahid Bahonar University of Kerman, Kerman, Iran
[2]Department of Mathematics, Faculty of Science, University of Jiroft, Jiroft, Iran

Received: July 26, 2014
Accepted: January 25, 2015

ABSTRACT

One of the important concepts in physics and mathematics is entropy. The present paper deals with the theory of entropy using the Loomis-Sikorski representation of σ −MV -algebras. In this paper the conditional entropy of a finite partition ξ given N by use of conditional expectation on σ −MV-algebras when N is an arbitrary sub− σ −MV -algebra of the σ −MV –algebra M is defined. It is shown that the mean entropy of an MV-dynamical system is affine. Also the concept of a subsystem of an MV-dynamical system is introduced and the relation between their entropies is investigated.

KEYWORDS: σ −MV –algebra; Loomis-Sikorski theorem; Tribe; State; Partition; Entropy; Entropic distance.

INTRODUCTION

The concept of entropy originated in the physical and engineering sciences, but now plays a ubiquitous role in all areas of science. The term entropy was first used by the German physicist Rudolf Clausius in 1865 to denote a thermodynamic function, he introduced in 1854 which increase with time in all spontaneous natural processes. It is very often refered to as the degree of disorder or complexity of a dynamical system. Entropy in information theory was introduced by Claude Shannon [23] in 1948. In information theory, entropy is a measure of the uncertainty associated with a random variable. In 1958 Kolmogorov [11] introduced the concept of measure-theoretic entropy to ergodic theory. Kolmogorov's definition was improved by Sinai in 1959 [24]. The importance of entropy arises from its invariance under isomorphism. Therefore, systems with different entropies cannot be isomorphic. If one substitutes in the definition of entropy, the notion of a set partition by the notion of fuzzy partition, a larger class of invariants can be obtained [4, 5, 13]. A natural generalization of a tribe of functions is an MV-algebra. MV-algebras were introduced by Chang [2] in 1958, as the algebraic counterpart of propositional Lukasiewicz logic. At the same time their theory evolved in deep connection with the theory of lattice-ordered Abelian groups. The notion of a state (probability) on an MV-algebra was investigated by Mundici in [15]. In [8, 12] the conditional probability on MV-algebras is defined in a way that mimicks the classical (Boolean) approach of conditioning "an event by an event" or "an event by a subalgebra". On the MV- algebras with product there has been realized the construction of entropy in [18]. Also in 2003 Riecan [22] introduced the entropy of a dynamical system on an arbitrary MV-algebra. The aim of this paper is to define the conditional entropy of a finite partition given N by use of conditional expectation on σ −MV -algebras when N is an arbitrary sub- σ −MV -algebra of the σ −MV –algebra M. The paper is structured as follows. Basic notions and results are summarized in section 2, entropy of a finite partition and dynamical system is briefly in sections 3, 4, respectively. Section 5 contains the concept of subsystem of a σ −MV -dynamical system and the relation between their entropies.

Preliminaries

In this section we recall the basic notions on MV-algebras and probability theory on MV- algebras. See [7, 20] for further results on MV-algebras and [17, 21] for an exposition of probability

on MV-algebras. An MV - algebra $\mathrm{M} = (\mathrm{M}, 0, \perp, \oplus)$ is an algebra where $\oplus$ is an associative and commutative binary operation on M having 0 as the neutral element, a unary operation $\perp$ is involutive with $a \oplus 0^{\perp} = 0^{\perp}$ for all $a \in \mathrm{M}$, and moreover the identity

$$\mathrm{a}\oplus(\mathrm{a}\oplus\mathrm{b}^{\perp})^{\perp} = \mathrm{b}\oplus(\mathrm{b}\oplus\mathrm{a}^{\perp})^{\perp};$$

is satisfied for all a, b ∈ M.
An additional constant 1 and two binary operation $\odot$ is defined as follows

$$1 = 0^{\perp}, \mathrm{a}\odot\mathrm{b} = (\mathrm{a}^{\perp} \oplus \mathrm{b}^{\perp})^{\perp}$$

A partial order is defined on M by $\mathrm{a} \leq \mathrm{b}$ iff $\mathrm{a}^{\perp} \oplus \mathrm{b} = 1$. Two elements a, b in M are called orthogonal (denoted as $a \perp b$) if $\mathrm{a} \leq b^{\perp}$.
A sub-MV-algebra of M is a subset N of M containing the neutral element 0 of M, closed under the operations of M and endowed with the restriction of these operations to N.

*Corresponding Author: U. Mohammadi, Department of Mathematics, Faculty of Science, University of Jiroft, Jiroft, Iran

We also recall that an ideal of an MV -algebra M is a subset I of M such that $0 \in I, a \oplus b \in I$ whenever $a, b \in I$ and $a \leq b$ with $b \in I$ implies$a \in I$. An ideal I of M is called maximal if the only ideal strictly containing I is the improper ideal$M$. Let $I(M)$ and $M(M)$ denote the set of all ideals and the set of all maximal ideals of$M$, respectively. The space of all maximal ideals $M(M)$ is nonempty, and endowed with the so-called spectral topology formed by the sets of the form $0_J = \{K \epsilon M(M), J \not\subset K\}$ for all $J \in I(M)$, it becomes a compact Hausdorff space.
The following fact (a generalization of the Loomis-Sikorski theorem for σ_ MV -algebras) were proved in [6, 8, 14].

Theorem 2.1. Let M be a $\sigma -$MV -algebra and $M(M)$ be the maximal ideal space of M equipped with the spectral topology. Then there exists a tribe M* over $M(M)$ and a σ -homomorphism η of M* onto M. ■
A state m on a $\sigma -$complete MV-algebra M is a mapping $m : M \rightarrow [0;\ 1]$ such that for all $a, b, a_n \in M$:

$$(i) m(1) = 1$$

(ii)$a \odot b=0 \Longrightarrow m(a \oplus b)=m(a)+m(b)$

$$(iii) a_n \nearrow a \Longrightarrow m(a_n) \nearrow m(a)$$

The notation "$a_n \nearrow a$" stands for a_n is a no decreasing sequence [8]. The state m is faithful if $m(a) = 0$ implies $a = 0$ for any $a \in M$. We refer to [15] for basic notions and results regarding states.

Entropy of a finite partition
A finite subset $\xi = \{a_1, a_2, \ldots, a_n\}$ of elements of a σ_MV-algebra M is said to be ⊕-orthogonal if and only if

$$\oplus_{i=1}^{k} a_i \perp a_{k+1} \qquad \text{for } k = 1,2, \ldots, n-1.$$

Lemma 3.1. For any ⊕- orthogonal subset $\xi = \{a_1, a_2, \ldots, a_n\}$ and any state m of M, it holds that

$$m(\oplus_{i=1}^{k} a_i) = \sum_{i=1}^{n} m(a_i).$$

Proof: See [20**]**.

Definition 3.2. A finite collection $\xi = \{a_1, a_2, \ldots, a_n\}$ of elements of M is siad to be a partition of M if and only if:
(i) ξ is $\oplus -$ orthogonal subset;
(ii)$\oplus_{i=1}^{k} a_i = 1$.

Lemma 3.3. Let $\xi = \{a_1, a_2, \ldots, a_n\}$ be a partition of $M, b \in M.$ Then

$$m(b) = \sum_{i=1}^{n} m(b \odot a_i).$$

Proof: We first show that $\{b \odot a_1, b \odot a_2, \ldots, b \odot a_n\}$ is a $\oplus -$ orthogonal subset. Set $c_i = b \odot a_i ; i = 1,2, .. n$. *We need to prove that*

$$c_1 \perp c_2, c_1 \oplus c_2 \perp c_3, \ldots, c_1 \oplus c_2 \oplus \ldots . \oplus c_{n-1} \perp c_n$$

from the⊕ - orthogonally of the collection ξ we have $a_1 \leq a_2^{\perp}$. Using the monotonicity of operations $\oplus$ *and* $\odot$, we obtain
$b \odot a_1 \leq a_1 \leq a_2^{\perp} \leq b^{\perp} \oplus a_2^{\perp} = (b \odot a_2)^{\perp}$.
Hence $c_1 \perp c_2$. Similarly we can prove that $c_1 \oplus c_2 \perp c_3$, which is equivalent to $(b \odot a_1) \oplus (b \odot a_2) \leq (b \odot a_3)^{\perp}$. According to that $(b \odot a_3)^{\perp} = b^{\perp} \oplus a_3^{\perp}$ *and* $a_1 \oplus a_2 \leq a_3^{\perp}$ (ξ *is* $\oplus -$ *orthogonal*), *we can write*
$(b \odot a_1) \oplus (b \odot a_2) \leq a_1 \oplus a_2 \leq a_3^{\perp} \leq b^{\perp} + a_3^{\perp} = (b \odot a_3)^{\perp}$
So $c_1 \oplus c_2 \perp c_3$. And similarly $c_1 \oplus c_2 \oplus \ldots \oplus c_{i-1} \perp c_i$ for $i = 4, \ldots, n$.
Secondly, for the ⊕-orthogonal collection ξ, from Lemma 3.1 and definition of a state, we have

$$\sum_{i=1}^{n} m(b \odot a_i) = m(\oplus_{i=1}^{n} (b \odot a_i)). \blacksquare$$

Suppose $\xi = \{a_1, \ldots a_n\}$ *and* $\eta = \{b_1, \ldots b_p\}$ are two finite partitions of σ_ MV-algebra M.
Then we define $\xi < \eta$ (i.e $\eta$is a refinement of η) if there exists a partition *{I(1),..I(n)}* of the set *{1,...,p}* such that

$$a_i = \oplus_{j \in I(i)} b_j \qquad i = 1, \ldots n.$$

Note that if $\xi = \{a_1, \dots a_n\}$ *and* $\eta = \{c_1, \dots c_p\}$ *are two finite partitions of* M, *their join is*

$$\xi \nabla \eta = \{a_i \odot c_j; a_i \in \xi, c_j \in \eta\}.$$

We recall that if ξ is a finite partition of M with $\xi = \{a_1, \dots a_n\}$, then the entropy of ξ is the number

$$H(\xi) = -\sum_{i=1}^{n} \text{m(a\_i)}\, log(a_i)$$

Note that a non-negative real number $m(a|b)$ is a conditional state of *a* given *b* if $m(a|b)$ is any $m(b)m(a|b) = m(a \odot b)$.

Now let ξ, η be finite partitions of M, and $\xi = \{a_1, a_2, \dots a_3\}, \eta = \{c_1, c_2, \dots c_p\}$.

The entropy of ξ given η is the number

$$H(\xi\,|\eta) = -\sum_{j=1}^{p} m(c_j) \sum_{i=1}^{n} m\left((a_i)\middle|c_j\right) \log m(a_i|c_j) = -\sum_{i,j} m\,(a_i \odot c_j) log \frac{m(a_i \odot c_j)}{m(c_j)}$$

Remark 3.4. *Let* $\xi;\ \eta$ *and* ζ *be finite partitions of M. Then*

$$(i) H(\xi\,|\eta) \geq 0$$

$$(ii) H(\xi\ \nabla\eta) = H\,(\xi) + H(\eta\mid\xi\,);$$

$(iii) H(\xi) \leq H(\xi\ \nabla\eta);$

(iv) $H(\xi \mid \eta\nabla\zeta) \leq H\,(\xi|\eta)$.

Definition 3.5. *(* [8]) *An MV-conditional expectation in a state m, of a* $(a\ \in\ M)$ *given N is a* $\beta(N*)$ *−measurable function* $E(a|N):\ M(M)\ \longrightarrow [0;\ 1]$ *such that the following holds true for any* $b \in N$

$$\int_{M(M)} E(a|N)(w) dP^*{}_{b^*}\ (w) = \int_{M(M)} a^* d\,P^*{}_{b^*}(w).$$

Definition 3.6. *Let M be a* σ*_MV-algebra. If* ξ *is a finite partition of M with* $\xi = \{a_1, \dots a_n\}$ *and N is an arbitrary sub-*σ*_MV-algebra of M the entropy of* ξ *given N is the number*

$$H(\xi|N) = -\int_{M(M)} \sum_{i=1}^{n} E(a_i|N) \log E(a_i|N)\, dP^*.$$

Remark 3.7.

(i) Since $E(a_i|N)$ *is positive linear operator and we have* $0 \leq E(a_i|\,N) \leq 1\ P^* - a.e.$*, and therefore*

$$-\sum_{i=1}^{n} E(a_i|N) \log E(a_i|N)(w) \leq n\, max_{t\in[0,1]}(-t \log t) = n\, e.$$

Hence $H(\xi|N)$ *is finite.*

(ii). If $N = \{0,1\}$ then $H(\xi|N) = H(\xi)$. Since $N^* = \{0,1\}$ and the only maximal ideal of N is $\{0\}$. A version of conditional expectation $E(a \mid N)$ is P^*-a.e equal to $m(a)$. Indeed if $B \in \beta(N^*) = \{\emptyset, \{0\}\}$, then $P^*(B) \in \{0,1\}$ and

$$\int_B m(a) dP^* = m(a)p^*(B) = m(a)m^*(X_B) = m(a)m(X_B) = \int_{M(M)} E(a|N) dP_B^* = \int_B\ E(a|N) dP^*.$$

Where X_B is the characteristic function of B.

If *M* is a σ-MV-algebra, for any *a,b* ∈ *M* we set

$$a\Delta b = (a\ \odot\ b^{\perp}) \oplus (b\ \odot\ a^{\perp}).$$

Definition 3.8. *If* N, C *are (not necessarily finite) subsets of M where M is a* σ*_ MV-algebra and m is a state on M. We write* $N \leq_m C$ *if for every* $a\ \in\ N$ *ther exists* $c\ \in\ C$ *with*

$$m(a\Delta c) = 0$$

Theorem 3.9. *Suppose that* ξ *be a finite partition of* σ*-MV-algebra M and N be an arbitrary sub-* σ*-MV-algebra of M. Then*

$$(i) H(\xi|N) = 0\ iff\ \xi \leq_m N.$$

$(ii) H(\xi|N) = H(\xi\,)\ iff\ \xi, N$ *are independent.*

Proof: *Let* $\xi = \{a_1, a_2, \dots., a_n\}$.

(i).If $\xi \leq_m N$ then $E(a_i|N)(w)$*takes* only the values 0,1 so $H(\xi|N) = 0$. Conversely, if $0 = H(\xi|N) = -\int_{M(M)} \sum_{i=1}^{n} E(a_i|N) \log E(a_i|N) dP^*$

Then since $-E(a_i|N)(w) \log E(a_i|N)(w) \geq 0$. We have for each i, $E(a_i|N)$ takes only values 0, 1 therefore $\xi \leq_m N$.

(ii). Suppose $H(\xi|N) = H(\xi\,)$

Let $b \in N$ and D be the finite *sub-* σ*-MV-algebra of M consisting of the elements* $\{0, b, b^{\perp}, 1\}$. *Then*

$$H(\xi) \geq H(\xi|D) \geq H(\xi|N) = H(\xi).$$

Hence $H(\xi) \geq H(\xi|D)$ so $m(a \odot b) = m(a)m(b)$ for all $a\epsilon\xi$, by Theorem 3-8, therefore ξ, N are independent. Conversely, If ξ, N are independent then for each $a\epsilon\xi, E(a|N) = m(a)$.(because

$$\int_B E(a|N)d\, P^* = \int_{M(M)} E(a|N)dP_B^* = m\big(a \odot \eta\,(X_B)\big) = m(a)m(\eta\,(X_B)) = m(a)m(b)$$

for $B \in \beta(N^*)$and note if $B \in \beta(N^*)$then $\eta(X_B) = b \in \beta(N)$).. Therefore $H(\xi|N) = H(\xi\,)$. ∎

Entropy of MV-dynamical systems

By a dynamical system on a σ-MV-algebra we understand a triple (M, m, U), where $m : M \to [0;\ 1]$ is a state on M and $U : M \to M$ is a mapping satisfying the following conditions:

$$(i)U(a \oplus b) = U(a) \oplus U(b);$$
$$(ii)U(1) = 1;$$
$$(iii)m\big(U(a)\big) = m(a), a \in M.$$

The entropy of the partition $\xi = \{a_1, \dots, a_n\}$ is given by

$H(\xi, m) = -\sum_{i=1}^{n} m(a_i)log\big(m(a_i)\big);$

and we now define the mean entropy of U on ξ by

$H(\xi, m, U) = lom_{n\to\infty} \frac{1}{n} H\left(\nabla_{i=1}^{n-1} U^i(\xi)\right).$

It is of course necessary to establish that the above limit exists. But by proposition 5 of [18] this is a consequence of subadditivity. Furthermore the mean entropy of the dynamical system U is defined by

$$h(U, M) = sup_\xi H(\xi, m, U);$$

where the supermom is over all finite partitions *of σ-MV-algebra M*.

Theorem 4.1. *Suppose that ξ be a finite partition of σ-MV-algebra M. Then the mean entropy of U on ξ of the MV-dynamical system (M, m, U) is affine, i.e.*

$$H(\xi, \lambda m_1 + (1-\lambda)m_2, U) = \lambda H(\xi, m_1, U) + (1-\lambda)H(\xi, m_2, U)$$

for each pair $m_1$and $m_2$of states on M and $\lambda\epsilon$ [0*;* 1].

Proof : If m_1 and m_2 are two states and $\lambda\epsilon$ [0, 1] then

$H(\xi, \lambda m_1 + (1-\lambda)m_2) \geq \lambda H(\xi, m_1) + (1-\lambda)H(\xi, m_2).$ *(1)*

The 'concavity' inequality (1) is a direct consequence of the definition of $H(\xi, m)$ and the 'concavity' of the function *x*→ -*x*log *x*. Conversely, one has inequalities

$$-\log\big(\lambda m_1(a_i)\big) + (1-\lambda)m_2(a_i) \leq -\log(1-\lambda) - log(m_2(a_i))$$

and

$$-\log\big(\lambda m_1(a_i)\big) + (1-\lambda)m_2(a_i) \leq -log\,\lambda - log(m_2(a_i))$$

Because *x*→- log *x* is decreasing. Therefore one obtains the 'convexity' bound

$$H(\xi, \lambda m_1 + (1-\lambda)m_2) \leq \lambda H(\xi, m_1) + (1-\lambda)H(\xi, m_2) - \lambda log\lambda - (1-\lambda)log(1-\lambda). \quad (2)$$

Now replacing ξ by $\nabla_{i=1}^{n-1} U^i(\xi)$ in (1), dividing by n and taking the $lim_{n\to\infty}$, gives

$$H(\xi, \lambda m_1 + (1-\lambda)m_2) \geq \lambda H(\xi, m_1) + (1-\lambda)H(\xi, m_2).$$

Similarly from (2), since $\frac{-(\lambda \log\lambda + (1-\lambda)\log(1-\lambda)}{n} \to 0\ as\ n \to \infty$, one deduces the converse inequality

$$H(\xi, \lambda m_1 + (1-\lambda)m_2, U) \leq \lambda H(\xi, m_1, U) + (1-\lambda)H(\xi, m_2, U).$$

Hence one concludes the map *m* → *H*(*ξ*, *m*, *U*) is affine. This is a somewhat surprising and is of great significance in the application of mean entropy. ∎

Subsystem

The aim of this section is to state a condition that guarantees that the Kolomogorov-Sinai entropy on *σ-MV-algebra*s is indeed reached by considering this restricted superemum. In order to reach this goal, we introduce subsystem of an MV-dynamical system and entropic distance between partitions and show that, for computing the dynamical entropy, it is sufficient to consider only partitions that can approximate the sharp ones arbitrary well.

Definition 5.1. *Let (M, m, U) be an MV-dynamical system. A sub−σ-MV-algebra N of M is called invariant under U, if we have $U^{-1}(a) \in N$ whenever $a \in N$.*

Definition 5.2. *Suppose that* (*M*, *m*, *U*) *is an MV-dynamical system and N is a sub-σ-MV-algebra of M which is invariant under U, then* (*N*, *m*, *U*) *is called a subsystem of MV-dynamical system* (*M*, *m*, *U*).

Definition 5.3. *Let ξ and η be two partitions of M. The entropic distance $\rho_H(\xi, \eta)$of ξ with respect to η is*

$\rho_H(\xi,\eta) = H(\xi\nabla\eta,m) - H(\eta,m)$

Lemma 5.4. *Let ξ, η and ς be finite partitions of M. Then the entropic distance satisfies the following properties:*

$(i)\rho_H(\xi,\eta) \geq 0;$

$$(ii)\rho_H(\xi\nabla\eta,\varsigma) \leq \rho_H(\xi,\varsigma) + \rho_H(\eta,\varsigma);$$
$$(iii)\ \rho_H(\xi,\eta\nabla\varsigma) \leq \rho_H(\xi,\eta).$$

Proof : Statement (i) is actually the same as statement (iii) of Remark 3-6. In order to prove of (ii) we use the parts (ii) and (iv) of Remark 3.*4*

$$\rho_H(\xi\nabla\eta,\varsigma) = H(\xi\nabla\ \eta\nabla\varsigma,m) - H(\varsigma,m)$$
$$\leq H(\xi\nabla\varsigma,m) - H(\eta\nabla\varsigma,m) - 2H(\varsigma,m)$$
$$= \rho_H(\xi,\varsigma) + \rho_H(\eta,\varsigma).$$

(iii).

$$\rho_H(\xi,\eta\nabla\varsigma) = H(\xi\nabla\eta\nabla\varsigma,m) - H(\eta\nabla\varsigma,m)$$
$$= \rho_H(\xi\nabla\varsigma,\eta) + H(\eta,m) - \rho_H(\varsigma,\eta) - H(\eta,m)$$
$$\leq \rho_H(\xi,\eta) + \rho_H(\varsigma,\eta) - \rho_H(\varsigma,\eta) = \rho_H(\xi,\eta).\ \blacksquare$$

Definition 5.5. *A sub-σ-MV-algebra N of M is said to be H-dense in M if for any finite partition $\xi = \{a_1,\dots,a_n\}$ of M and for any $\epsilon > 0$ there exists a partition $\eta = \{c_1,\dots,c_p\}$ of N such that*

$\rho_H(\xi,\eta) \leq \epsilon$.

Theorem 5.6. *Let (M, m, U) be an MV-dynamical system. If a sub-σ-algebra N of M is invariant under U and H-dense, then*

$$h(U,M) = h(U,N).$$

Proof : Obviously, $h(U, N) \leq h(U, M)$.

As N is H-dense, we can find for each partition ξ of M and each $\epsilon > 0$ a partition η in N such that

$\rho_H(\xi,\eta) \leq \epsilon;$

we first use the property (iii) of Remark 3.4, which states that refining a partition increase the entropy

(3)
$$H(U^{n-1}(\xi)\nabla\dots\nabla U(\xi)\nabla\xi,m) \leq H(U^{n-1}(\eta)\nabla\dots\nabla U(\eta)\nabla\eta\nabla U^{n-1}(\xi)\nabla\dots\nabla U(\xi)\nabla\xi,m)$$

Next, by using of the definition of entropic distance we write

(4) $H(U^{n-1}(\eta)\nabla\dots\nabla U(\eta)\nabla\eta\nabla U^{n-1}(\xi)\nabla\dots\nabla U(\xi)\nabla\xi,m) =$
$\rho_H(U^{n-1}(\xi)\nabla\dots\nabla U(\xi)\nabla\xi, U^{n-1}(\eta)\nabla\dots\nabla U(\eta)\nabla\eta) \quad + H(U^{n-1})(\eta)\nabla\dots\nabla U(\eta)\nabla\eta,m)$

Suppose that we have obtained the upper bound

(5)
$$\rho_H(U^{n-1}(\xi)\nabla\dots\nabla U(\xi)\nabla\xi, U^{n-1}(\eta)\nabla\dots\nabla U(\eta)\nabla\eta) \leq n\rho_H(\xi,\eta)$$

for the first term in the sum in (4). Combining (3), (5) and dividing by *n*, we get

$$\frac{1}{n}H(U^{n-1}(\xi)\nabla\dots\nabla U(\xi)\nabla\xi,m) \leq \frac{1}{n}H\big((U^{n-1})(\eta)\nabla\dots\nabla U(\eta)\nabla\eta,m\big) + \rho_H(\xi,\eta)$$

Let *n* tend to infinity and using H-density, the theorem follows. It remains to prove (5)

$$\rho_H(U^{n-1}(\xi)\nabla\dots\nabla U(\xi)\nabla\xi, U^{n-1}(\eta)\nabla\dots\nabla U(\eta)\nabla\eta) \leq \sum_{j=o}^{n-1}\rho_H(U^j(\xi), U^{n-1}(\eta)\nabla\dots\nabla U(\eta)\nabla\eta)$$
$$\leq \sum_{j=0}^{n-1}\rho_H\left(U^j(\xi),U^j(\eta)\right) = n\rho_H(\xi,\eta).\ \blacksquare$$

The inequalities are an application of Remark 3.*4*.

Conclusions

We have defined the conditional entropy of a finite partition ξ given *N* by use of conditional expectation on σ-MV-algebras when *N* is an arbitrary sub-σ-MV-algebra of the σ-MV-algebra *M*. We also introduced the concept of a subsystem of an MV-dynamical system and we investigated the relation between their entropies. Methods developed in the present paper seem to be applicable also for other results on fuzzy ergodic theory of dynamical systems. The situation in MV-algebraic ergodic theory is however much more complicated as the operation $\oplus$ is not idempotent.

REFERENCES

[1] D. Butnariu, E. Klement, Triangular-norm based measures, in: E. pap (Ed), Handbook of Measure Theory, North-Holland, Amsterdam, 2002.

[2] C. C. Chang. Algebraic analysis of many valued logics. Transactions of the American Mathematical Society, 88 (1958) 467-490.

[3] R. Cignoli, I.M.L. DOttaviano, D. Mundici, Algebraic Foundation of Many-Valued-Reasoning, Kluwer Academic Publishers, Dordrecht, 2000.
[4] D. Dumetrescu, Entropy of a fuzzy process, Fuzzy Sets and Systems.55 (1993), 169-177.
[5] D. Dumetrescu, Entropy of fuzzy dynamical systems, Fuzzy Sets and Systems.70 (1995), 45-57.
[6] A. Dvurecenskij, Loomis-Sikorski theorem for _-complete MV-algebras and *l*-groups, J. Austral. Math. Soc. Ser. A 68 (2000) 261277.
[7] A. Dvurecenskij, S. Pulmannov, New Trends in Quantum Structures, Kluwer Academic Publishers, Dordrecht, Ister Science, Bratislava, 2000.
[8] A. Dvurecenskij, S. Pulmannova, Conditional probability on _-MV-algebras, Fuzzy Sets and Systems. 155 (2005) 102-118.
[9] M. Ebrahimi, U. Mohamadi, *m*-Generators of Fuzzy Dynamical Systems, Cankaya University Journal of Science and Engineering, 9 (2012), No. 2, 167-182.
[10] I. Leustean, Metric completions of MV-algebras with states: an approach to stochastic independence, Oxford University Press, Vol 21 No. 3, (2009) 493-508.
[11] A. N. Kolmogorov, New metric invariants of transitive dynamical systems and automorphisms of Lebesgue spaces, Dokl. Nauk. S.S.S.R, 119 (1958),861-864.
[12] T. Kroupa, Conditional probability on MV-algebras, Fuzzy Sets and Systems. 149 (2005) 369-381.
[13] P. Malicky, B. riecan, On the entropy of dynamical systems in: Proceedings Conference on Ergodic Theory and Related Topics II, Georgenthal, 1986, Teubner, Leipzig, 1987, 135-138.
[14] D. Mundici, Tensor products and the Loomis-Sirkorski theorem for MV-algebras, Adv.Appl. Math. 22 (1999) 227-248.
[15] D. Mundici, Averaging the truth-value in Lukasiewicz logic.Studia Logica, 55 (1995) 113-127.
[16] U. Mohamadi, Weighted information function of dynamical systems,Journal of mathematics and computer science, 10 (2014) 72-77.
[17] S. Nesheyev, E. Strmer, In dynamical entropy in operator algebras; Springer-Verlag:Berlin, 2006.
[18] J. Petrovicova, On the entropy of dynamical systems in product MV-algebras, Fuzzy Sets and Systems.121 (2001)347-351
[19] B. Riecan, D. Makechova, The entropy of fuzzy dynamical systems, general scheme and generators, Fuzzy Sets and Systems.96 (1998)191-199.
[20] B. Riecan, D. Mundici, Probability on MV-algebras, in: E. Pap(Ed.), Handbook of Measure Theory, Vol. II, Elsevier, Amsterdam, The Netherlands, 2002, pp. 869909.
[21] B. Riecan, T. Neubrunn, Integral, Measure and Ordering, Kluwer Academic Publishers, Dordrecht, 1997.
[22] B. Riecan, Kolmogorov-Sinaj entropy on MV-algebras. arxiv:math.DS/0304123 v2, 10pp, 2003.
[23] C.Shannon, A mathematical theory of communication. Bell Syst Tech Journal, 1948, 27:379-423.
[24] Ya. Sinai, On the notion of entropy of a dynamical system, Dokl. Akad. Nauk. S.S.S.R, 125 (1959), 768-771.
[25] P.Walters, An Introduction to Ergodic Theory, Springer Verlag, 1982.