ITERATIVE METHOD FOR SOLVING NONLINEAR EQUATIONS USING DECOMPOSITION METHOD

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ABSTRACT

In this paper, we suggest and analyze a new three-step method for solving nonlinear equations using the decomposition technique which is mainly due to Noor et al. [5]. We show that this new iterative method has fourth-order of convergence. To demonstrate the efficiency and performance of the new method, we tested several numerical examples and results are shown in the table-1.

KEYWORDS: Iterative methods, Two-step methods, Three-step methods, Nonlinear equations, Order of convergence

1. INTRODUCTION

In recent years, much attention has been given to develop several iterative methods for solving nonlinear equations, see [2-6] and the references therein. These methods can be classified as one-step, two-step and three-step methods. Two-step methods have been suggested by combining the well-known Newton method with other one-step implicit methods. In [2] Chun has proposed and studied several one-step and two-step iterative methods with higher-order convergence by using the decomposition technique of Adomian [1]. In the methods of Chun [2], higher-order differential derivatives are involved, which is a serious drawback. To overcome this drawback, we suggest and analyze a family of multi-step methods for solving nonlinear equations using a different type of decomposition, which does not involve the high-order differentials of the function. This new decomposition is mainly due to Noor et al. [5].

2. ITERATIVE METHODS

Consider the nonlinear equation
\[ f(x) = 0. \] (2.1)

We assume that \( \alpha \) is a simple root of (2.1) and \( \bar{\xi} \) is an initial guess sufficiently close to \( \alpha \). We can rewrite the nonlinear equation (2.1) as a coupled system:

\[ f(\gamma) + (x - \gamma) \frac{f'(\gamma) + f'(x)}{2} + g(x) = 0, \] (2.2)

\[ g(x) = f(x) - f(\gamma) - (x - \gamma) \frac{f'(\gamma) + f'(x)}{2}, \] (2.3)

where \( \gamma \) is the initial approximation for a zero of (2.1).

We can rewrite (2.2) in the following form:

\[ x = \gamma - \frac{2f(\gamma)}{f'(\gamma) + f'(x)} - \frac{2g(x)}{f'(\gamma) + f'(x)}. \] (2.4)

We can also rewrite (2.4) as:

\[ x = c + N(x), \] (2.5)

where

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\[ c = \gamma - \frac{2f(\gamma)}{f'(\gamma) + f''(x)}, \]  
(2.6)
and
\[ N(x) = -\frac{2g(x)}{f'(\gamma) + f''(x)}. \]  
(2.7)

Here \( N(x) \) is a linear operator.

We note that if \( x_0 \) is the initial guess, then from (2.2) and (2.3), we have
\[ f(x_0) = g(x_0). \]  
(2.8)

As in [6], the solution of (2.5) has the series form,
\[ x = \sum_{i=0}^{\infty} x_i, \]  
(2.9)

The nonlinear operator \( N(x) \) can be decomposed as
\[ N(\sum_{i=0}^{\infty} x_i) = N(x_0) + \sum_{i=1}^{\infty} \left\{ N\left( \sum_{j=0}^{i} x_j \right) - N\left( \sum_{j=0}^{i-1} x_j \right) \right\}. \]  
(2.10)

Combining (2.4), (2.7) and (2.10), we have
\[ x = \sum_{i=0}^{\infty} x_i = c + N(x_0) + \sum_{i=1}^{\infty} \left\{ N\left( \sum_{j=0}^{i} x_j \right) - N\left( \sum_{j=0}^{i-1} x_j \right) \right\}. \]  
(2.11)

Thus we have the following iterative scheme:
\[ x_0 = c, \]
\[ x_1 = N(x_0), \]
\[ x_2 = N(x_0 + x_1) - N(x_0), \]
\[ \vdots \]
\[ x_{n+1} = N(x_0 + x_1 + \ldots + x_n) - N(x_0 + x_1 + \ldots + x_{n-1}), \quad n = 1, 2, \ldots. \]

Then
\[ x_1 + x_2 + \ldots + x_{n+1} = N(x_0 + x_1 + \ldots + x_n), \quad n = 1, 2, \ldots, \]
and
\[ x = c + \sum_{i=1}^{\infty} x_i. \]  
(2.13)

It can be shown that the series \( \sum_{i=0}^{\infty} x_i \) converges absolutely and uniformly to a unique solution of equation (2.5).

From (2.6), (2.9), (2.10), (2.11) and (2.12), we have
\[ x_0 = c = \gamma - \frac{2f(\gamma)}{f'(\gamma) + f''(x)}, \]  
(2.14)
and
\[ x_1 = N(x_0) = -\frac{2g(x_0)}{f'(\gamma) + f''(x_0)} = -\frac{2f(x_0)}{f'(\gamma) + f''(x_0)}. \]  
(2.15)

This enables us to suggest the following method for solving the nonlinear equation (2.1).

Algorithm For the given \( x_o \) compute the approximate solution \( x_{n+1} \) by the iterative schemes:
\[ z_n = x_n - \frac{f(x_n)}{f'(x_n)}, \quad f'(x_n) \neq 0, \quad n = 0, 1, 2, \cdots, \] (2.16)

\[ x_{n+1} = x_n - \frac{2f(x_n)}{f'(x_n) + f''(z_n)}, \]

which is known as Weerakoon and Fernando method [7] and is cubically convergent.

Again by using (2.12), (2.13), (2.14) and (2.15), we conclude that

\[ x = c + x_1 = x_0 + N(x_0) \]

\[ = x_0 + N(x_0) + \gamma - \frac{2f(x)}{f'(x) + f'(x)} - \frac{2f(x_0)}{f'(x) + f'(x_0)}. \] (2.17)

Using (2.17), we can suggest the following three-step iterative method for solving nonlinear equation (2.1) as:

Algorithm: For the given \( x_0 \), compute the approximate solution \( x_{n+1} \) by the iterative schemes (AAU):

\[ z_n = x_n - \frac{f(x_n)}{f'(x_n)}, \quad f'(x_n) \neq 0, \]

\[ y_n = x_n - \frac{2f(x_n)}{f'(x_n) + f'(z_n)}, \]

\[ x_{n+1} = y_n - \frac{2f(y_n)}{f'(y_n) + f'(x_n)}. \]

Again, using (2.8) and (2.12), we can calculate

\[ x_2 = N(x_0 + x_1) = -\frac{2f(x_0 + x_1)}{f'(x) + f'(x_0 + x_1)}. \] (2.18)

From (2.11), (2.12), (2.13), (2.14), (2.15) and (2.18), we get

\[ x = c + x_1 + x_2 = x_0 + N(x_0) + N(x_0 + x_1) - N(x_0) \]

\[ = x_0 + N(x_0) + \gamma - \frac{2f(x)}{f'(x) + f'(x)} - \frac{2f(x_0 + x_1)}{f'(x) + f'(x_0 + x_1)}. \]

Using this, we can suggest and analyze the following four-step iterative method for solving nonlinear equation (2.1).

Algorithm: For a given \( x_0 \), compute the approximate solution by the iterative schemes:

\[ z_n = x_n - \frac{f(x_n)}{f'(x_n)}, \quad f'(x_n) \neq 0, \]

\[ y_n = x_n - \frac{2f(x_n)}{f'(x_n) + f'(z_n)}, \]

\[ s_n = -\frac{2f(y_n)}{f'(y_n) + f'(s_n)}, \]

\[ x_{n+1} = y_n - \frac{2f(y_n + s_n)}{f'(y_n + s_n)}. \]
3. ANALYSIS OF CONVERGENCE

Theorem: Assume that the function \( f : D \subset \mathbb{R} \to \mathbb{R} \) for an open interval \( D \) has a simple root \( \alpha \in D \).

Let \( f(x) \) be sufficiently smooth in the neighborhood of the root \( \alpha \), then the order of convergence of the method defined by Algorithm 2 is four.

Proof: The iterative scheme is given by

\[
z_n = x_n - \frac{f(x_n)}{f'(x_n)}, \quad f''(x_n) \neq 0,
\]

\[
y_n = x_n - \frac{2f(x_n)}{f'(x_n) + f''(z_n)},
\]

\[
x_{n+1} = y_n - \frac{2f(y_n)}{f'(x_n) + f'(y_n)}.
\]

Let \( z = \alpha \) be a simple zero of \( f \). By Taylor's expansion, we have,

\[
f(x_n) = f'(\alpha)[c_1 e_n + c_2 e_n^2 + c_3 e_n^3 + c_4 e_n^4 + c_5 e_n^5 + c_6 e_n^6 + O(e_n^7)]
\]

\[
f'(x_n) = f''(\alpha)[1 + 2c_2 c_4 e_n + 3c_3 c_4 e_n^2 + 4c_4 c_5 e_n^3 + 5c_5 c_6 e_n^4 + 6c_6 e_n^5 + O(e_n^6)]
\]

where

\[
c_k = \frac{1}{k!} \frac{f^{(k)}(\alpha)}{f''(\alpha)}, \quad k = 2, 3, \ldots, \text{ and } e_n = x_n - \alpha.
\]

From (3.4) and (3.5), we have

\[
\frac{f(x_n)}{f'(x_n)} = e_n - c_2 e_n^2 + (2c_2^2 - 2c_3)e_n^3 + (7c_2 c_3 - 4c_3^2 - 3c_4)e_n^4 + (2c_2 - c_3)e_n^5 + O(e_n^6).
\]

Using (3.1) and (3.6) we get

\[
z_n = \alpha + c_2 e_n^2 + (2c_3 - 2c_2^2)e_n^3 + (3c_4 - 7c_2 c_3 + 4c_2^2)e_n^4 + O(e_n^5).
\]

By Taylor's series, we have

\[
f(z_n) = f'(\alpha)[c_3 e_n^3 + (2c_3 - 2c_2^2)e_n^3 + (3c_4 - 7c_2 c_3 + 4c_2^2)e_n^4 + O(e_n^5)]
\]

and

\[
f'(z_n) = f''(\alpha)[1 + 2c_2^2 e_n + 3c_3 c_4 e_n^2 + 4c_4 c_5 e_n^3 + 5c_5 c_6 e_n^4 + 6c_6 e_n^5 + O(e_n^6)]
\]

From (3.2), (3.5) and (3.10) we get

\[
y_n = \alpha + \left(\frac{1}{2} c_3 + c_2^2\right)e_n^3 + \left(\frac{3}{2} c_2 c_3 - 3c_2^2\right)e_n^4 + \left(\frac{3}{2} c_3 + 2c_2 c_4 - 9c_2^2\right)e_n^5 + O(e_n^6),
\]

\[
+6c_2^2 - \frac{3}{4} c_3^2)e_n^5 + O(e_n^6),
\]

implies by Taylor's series,

\[
f(y_n) = f'(\alpha)[1 + 2c_2^2 e_n^3 + 3c_3 c_4 e_n^4 + 4c_4 c_5 e_n^5 + 5c_5 c_6 e_n^6 + 6c_6 e_n^7 + O(e_n^6)]
\]

\[
-9c_2 c_4 + 6c_4^2 - \frac{3}{4} c_3^2)e_n^5 + O(e_n^6),
\]

and
\[ f'(y_n) = f'(\alpha)[1 + (2c_2c_3 + 2c_2^3)e_n^3 + (2c_2c_4 + 3c_2c_2^2 - 6c_2^4)e_n^4 ] + (3c_2c_5 + 4c_2^2c_4 - 18c_2c_2^3 + 12c_2^5 - \frac{3}{2}c_2c_5^2)e_n^5 + O(e_n^6). \] (3.13)

Finally using (3.3), (3.5), (3.12) and (3.13), we obtain

\[ x_{n+1} = \alpha + \left( \frac{1}{2}c_2c_3 + c_2^3 \right)e_n^4 + \left( \frac{3}{4}c_3^2 + \frac{5}{2}c_3c_2^2 + c_2c_4 - 4c_2^4 \right)e_n^5 + O(e_n^6), \] (3.14)

implies

\[ e_{n+1} = \left( \frac{1}{2}c_2c_3 + c_2^3 \right)e_n^4 + \left( \frac{3}{4}c_3^2 + \frac{5}{2}c_3c_2^2 + c_2c_4 - 4c_2^4 \right)e_n^5 + O(e_n^6). \]

Hence proved.

4. Numerical Examples

In this section we consider some numerical examples to demonstrate the performance of the new developed iterative method. We compare Chun method [2] (CHU), Noor and Noor method [4] (NR3), Noor and Noor method [3] (NR1) and Noor and Noor method [5] (NR2) with the new developed method (AAU). All the computations for above mentioned methods, are performed using software Maple 9, precision 60 digits and \( \varepsilon = 10^{-18} \) as tolerance and also the following criteria is used for estimating the zero:

(i) \( \delta = |x_{n+1} - x_n| < \varepsilon. \)

(ii) \( |f(x_n)| < \varepsilon. \)

(iii) Maximum numbers of iterations \( = 500. \)

We use the following examples for numerical testing and results are given in the Tables of Results.

<table>
<thead>
<tr>
<th>Example</th>
<th>Exact Zero</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f_1 = \sin^2(x) - x^2 + 1, )</td>
<td>( \alpha = 1.4044916482153411 )</td>
</tr>
<tr>
<td>( f_2 = x^2 - e^x - 3x + 2, )</td>
<td>( \alpha = 0.2575302854398608 )</td>
</tr>
<tr>
<td>( f_3 = \cos(x) - x, )</td>
<td>( \alpha = 0.7390851332156067 )</td>
</tr>
<tr>
<td>( f_4 = (x - 1)^3 - 1, )</td>
<td>( \alpha = 2.0 )</td>
</tr>
<tr>
<td>( f_5 = x^3 - 10, )</td>
<td>( \alpha = 2.15443469003188372 )</td>
</tr>
<tr>
<td>( f_6 = xe^{x^2} - \sin^2(x) + 3 \cos(x) + 5, )</td>
<td>( \alpha = -1.2076478271309191 )</td>
</tr>
<tr>
<td>( f_7 = e^{x^2 + 7x - 30} - 1, )</td>
<td>( \alpha = 3 )</td>
</tr>
<tr>
<td>$f_{1}$, $x_0 = -1$</td>
<td>$n$</td>
</tr>
<tr>
<td>---------------------</td>
<td>-----</td>
</tr>
<tr>
<td>NW</td>
<td>7</td>
</tr>
<tr>
<td>NR1</td>
<td>7</td>
</tr>
<tr>
<td>NR2</td>
<td>6</td>
</tr>
<tr>
<td>NR3</td>
<td>8</td>
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<tr>
<td>CHU</td>
<td>15</td>
</tr>
<tr>
<td>AAU</td>
<td>4</td>
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<thead>
<tr>
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<th>$f(x_n)$</th>
<th>$\delta$</th>
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<td>6</td>
<td>2.9e-55</td>
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</tr>
<tr>
<td>NR1</td>
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<td>4.5e-25</td>
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<td>6</td>
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<td>1.0e-18</td>
</tr>
<tr>
<td>NR3</td>
<td>6</td>
<td>-1.2e-29</td>
<td>5.8e-15</td>
</tr>
<tr>
<td>CHU</td>
<td>5</td>
<td>1.5e-32</td>
<td>2.0e-16</td>
</tr>
<tr>
<td>AAU</td>
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<td>1.5e-31</td>
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<tr>
<th>$f_{3}$, $x_0 = 1.7$</th>
<th>$n$</th>
<th>$f(x_n)$</th>
<th>$\delta$</th>
</tr>
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<td>NW</td>
<td>5</td>
<td>-2.0e-32</td>
<td>2.3e-16</td>
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<tr>
<td>NR1</td>
<td>5</td>
<td>-2.1e-33</td>
<td>7.5e-17</td>
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<td>NR2</td>
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<td>3.5e-16</td>
</tr>
<tr>
<td>NR3</td>
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<td>6.2e-55</td>
<td>1.2e-27</td>
</tr>
<tr>
<td>CHU</td>
<td>5</td>
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<td>1.9e-17</td>
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<tr>
<td>AAU</td>
<td>4</td>
<td>0</td>
<td>1.0e-54</td>
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<th>$f(x_n)$</th>
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<td>NW</td>
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<td>NR1</td>
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<td>2.4e-15</td>
</tr>
<tr>
<td>NR2</td>
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<td>1.1e-27</td>
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<tr>
<td>NR3</td>
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<td>-6.0e-41</td>
<td>4.4e-21</td>
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<td>CHU</td>
<td>6</td>
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<td>8.0e-15</td>
</tr>
<tr>
<td>AAU</td>
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<td>0</td>
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<td>NR1</td>
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<td>1.4e-16</td>
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<td>CHU</td>
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\[ f_6, \ x_0 = -2 \]

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<td>8</td>
<td>9</td>
<td>7</td>
<td>7</td>
<td>5</td>
</tr>
<tr>
<td>[ x_0 ]</td>
<td>(-2.2e-40)</td>
<td>(-3.0e-9)</td>
<td>(-4.3e-42)</td>
<td>(1.8e-37)</td>
<td>(-6.0e-9)</td>
<td>(-1.3e-61)</td>
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<tr>
<td>[ y_0 ]</td>
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<td>(1.3e-17)</td>
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<td>(7.7e-20)</td>
<td>(6.4e-18)</td>
<td>(1.9e-16)</td>
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\[ f_7, \ x_0 \equiv 3.5 \]

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<th></th>
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<td>[ x_0 ]</td>
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<td>(1.0e-50)</td>
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<td>(4.8e-31)</td>
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<td>[ y_0 ]</td>
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<td>(7.3e-20)</td>
<td>(7.5e-17)</td>
<td>(2.2e-16)</td>
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</table>

5. CONCLUSION

In the Table-1, we observe that our iterative method (AAU) is comparable with all the cited methods and gives better results. The technique and idea of this paper can be developed to higher-order multi-step iterative methods for solving nonlinear equations, as well as a system of nonlinear equations.

REFERENCES